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CRITICAL CASE OF NONLINEAR SCHRÖDINGER EQUATIONS WITH INVERSE-SQUARE POTENTIALS ON BOUNDED DOMAINS

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Abstract. Nonlinear Schrödinger equations $(\mathrm{NLS})_a$ with strongly singular potential $a|x|^{-2}$ on a bounded domain Ω are considered. If $\Omega = \mathbb{R}^N$ and $a > -(N-2)^2/4$, then the global existence of weak solutions is confirmed by applying the energy methods established by N. Okazawa, T. Suzuki, T. Yokota (2012). Here $a = -(N-2)^2/4$ is excluded because $D(P_{a(N)}^{1/2})$ is not equal to $H^1(\mathbb{R}^N)$, where $P_{a(N)} := -\Delta - (N-2)^2/(4|x|^2)$ is nonnegative and selfadjoint in $L^2(\mathbb{R}^N)$. On the other hand, if Ω is a smooth and bounded domain with $0 \in \Omega$, the Hardy-Poincaré inequality is proved in J. L. Vazquez, E. Zuazua (2000). Hence we can see that $H_0^1(\Omega) \subset D(P_{a(N)}^{1/2}) \subset H^s(\Omega)$ $(s < 1)$. Therefore we can construct global weak solutions to $(\mathrm{NLS})_a$ on Ω by the energy methods.

Keywords: energy method; nonlinear Schrödinger equation; inverse-square potential; Hardy-Poincare inequality

MSC 2010: 35Q55, 35Q40, 81Q15

1. Introduction

Let Ω be a bounded domain with smooth boundary and $0 \in \Omega$. In this paper we consider the following initial-boundary value problem for nonlinear Schrödinger equations with inverse-square potential:

$$(\mathrm{NLS})_a \qquad \begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + \lambda|u|^{p-1}u & \text{in } \mathbb{R}\times\Omega,\\ u(0,x) = u_0(x) & \text{on } \Omega,\\ u(t,x) = 0 & \text{on } \mathbb{R}\times\partial\Omega, \end{cases}$$

where $\mathrm{i} = \sqrt{-1}$, $N \geqslant 3$, $\lambda \in \mathbb{R}$, $p \geqslant 1$ and

$$a \geqslant -\frac{(N-2)^2}{4}. \tag{1.1}$$

The condition (1.1) is related to the nonnegative selfadjointness of $P_a := -\Delta + a|x|^{-2}$ in $L^2(\Omega)$. If $\Omega = \mathbb{R}^N$, then the solvability of $(\mathrm{NLS})_0$ has been proved by many authors, for example, Ginibre-Velo [5] and Kato [6]. Their ideas are based on the contraction principle. They transform $(\mathrm{NLS})_0$ into the integral equation the right-hand side of which is later interpreted as the contraction mapping. On the other hand, if $\Omega = \mathbb{R}^N$ and $a \neq 0$, then the solvability of $(\mathrm{NLS})_a$ was proved by Okazawa-Suzuki-Yokota [8], [9] for $a > -(N-2)^2/4$, excluding $a = -(N-2)^2/4$ (see also Suzuki [10] for nonlocal nonlinearities). In [8] the solvability depends on the Strichartz estimates established by Burq, Planchon, Stalker and Tahvildar-Zadeh [1]:

$$\begin{aligned} \|\mathrm{e}^{-\mathrm{i}tP_a}\varphi\|_{L^\tau(\mathbb{R};\,L^\varrho(\mathbb{R}^N))} &\leqslant C\|\varphi\|_{L^2(\mathbb{R}^N)}, \quad \varphi \in L^2(\mathbb{R}^N), \\ \|\nabla \mathrm{e}^{-\mathrm{i}tP_a}\varphi\|_{L^\tau(\mathbb{R};\,L^\varrho(\mathbb{R}^N))} &\leqslant C\|\nabla\varphi\|_{L^2(\mathbb{R}^N)}, \quad \varphi \in H^1(\mathbb{R}^N),\ \tau > 2/\nu_0, \end{aligned} \tag{1.2}$$

where $\nu_0 := \sqrt{a + (N-2)^2/4} > 0$. Note that the dispersive estimates (that is, L^p-L^q type estimates) for $\mathrm{e}^{-\mathrm{i}tP_a}$ are not yet known but local smoothing estimates are satisfied and they are the key of proving the Strichartz estimates (see [1], [2] for details). By virtue of (1.2), Kato's method can be applied under a further restriction on a. The restriction is removed by [9]. In [9] the authors established the energy methods for the abstract Cauchy problem for nonlinear Schrödinger equations (see also Section 2). The methods need neither the dispersive estimates nor the Strichartz estimates to construct a weak solution.

If Ω is a bounded domain, then the Strichartz estimate does not hold in general even when $a = 0$ (see [4], Remark 2.7.3). Thus the contraction principle as in [6] does not work well. It is worth noticing that Cazenave [3] developed his method which can be applied to $(\mathrm{NLS})_0$ on a bounded domain Ω. However, his method of approximation for nonlinear terms depends essentially on the m-accretivity of $-\Delta$ in $L^q(\Omega)$, $(2N/(N+2) < q < 2N/(N-2))$. Unfortunately, P_a is not expected to be m-accretive in $L^q(\Omega)$ $(q \neq 2)$ if a is near to $-(N-2)^2/4$ (see e.g. Okazawa [7]). Thus Cazenave's method does not seem to work well for $(\mathrm{NLS})_a$ when $a \neq 0$ on bounded domains. As is done in [9], they have succeeded in replacing the m-accretivity of P_a in $L^q(\Omega)$ by the nonnegative selfadjointness in $H^{-1}(\Omega)$. Therefore the energy methods are suitable for the Cauchy-Dirichlet problem for nonlinear Schrödinger equations $(\mathrm{NLS})_a$.

Now we turn our eyes to the case $a = -(N-2)^2/4$. If $a > -(N-2)^2/4$, then we have the equivalence between $H_0^1(\Omega)$ and $D(P_a^{1/2})$ via the Hardy inequality:

$$c_1(a)\|\nabla\varphi\|_{L^2(\Omega)}^2 \leqslant \|\nabla\varphi\|_{L^2}^2 + a\Big\|\frac{\varphi}{|x|}\Big\|_{L^2(\Omega)}^2 \leqslant c_2(a)\|\nabla\varphi\|_{L^2(\Omega)}^2,$$

where

$$c_1(a) := 1 - \frac{4a_-}{(N-2)^2}, \quad c_2(a) := 1 + \frac{4a_+}{(N-2)^2}.$$

But the equivalence breaks down when $a = -(N-2)^2/4$ ($c_1(-(N-2)^2/4) = 0$). On the other hand, Vazquez-Zuazua [12], Corollary 2.3, proved the *Hardy-Poincaré inequality*:

$$\|u\|^2_{H^s(\Omega)} \leqslant C_s\Big(\|\nabla u\|^2_{L^2(\Omega)} - \frac{(N-2)^2}{4}\Big\|\frac{u}{|x|}\Big\|^2_{L^2(\Omega)}\Big) \qquad \forall\, u \in H^1_0(\Omega),\ \forall\, s \in [0,1). \tag{1.3}$$

Note that $C_s \to \infty$ ($s \to 1-0$). In view of (1.3) we define the norm

$$\|u\|_{X^1(\Omega)} := \Big[\int_\Omega \Big(|u|^2 + |\nabla u|^2 - \frac{(N-2)^2}{4|x|^2}|u|^2\Big)\,\mathrm{d}x\Big]^{1/2}$$

and the space $X^1(\Omega)$, which is the closure of $H^1_0(\Omega)$ in the norm $\|\cdot\|_{X^1(\Omega)}$. Also we denote $X^{-1}(\Omega) := X^1(\Omega)^*$. Now we can see from (1.3) that

$$H^1_0(\Omega) \subset X^1(\Omega) \subset H^s(\Omega), \quad \forall\, s \in [0,1);$$

note that all the injections are continuous. Hence we may construct a weak solution to $(\mathrm{NLS})_a$ even when $a = -(N-2)^2/4$. The main result in this paper is the following theorem which asserts the global existence of weak solutions to $(\mathrm{NLS})_a$.

Theorem 1.1. *Let $N \geqslant 3$, $a = -(N-2)^2/4$. Assume either $\lambda \geqslant 0$ and $1 \leqslant p < (N+2)/(N-2)$ or $\lambda < 0$ and $1 \leqslant p < 1 + 4/N$. Then for any $u_0 \in X^1(\Omega)$ there exists a global weak solution u to $(\mathrm{NLS})_a$ such that u belongs to $C_{\mathrm{w}}(\mathbb{R}; X^1(\Omega)) \cap L^\infty(\mathbb{R}; X^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}; X^{-1}(\Omega))$ and satisfies the equation in $(\mathrm{NLS})_a$ in $L^\infty(\mathbb{R}; X^{-1}(\Omega))$ as well as the initial condition. Moreover, the following weak conservation laws hold:*

$$\|u(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}, \quad E(u(t)) \leqslant E(u_0), \quad \forall\, t \in \mathbb{R},$$

where the energy E is defined as

$$E(\varphi) := \frac{1}{2}\int_\Omega \Big(|\nabla\varphi|^2 + \frac{a|\varphi|^2}{|x|^2}\Big)\,\mathrm{d}x + \frac{\lambda}{p+1}\|\varphi\|^{p+1}_{L^{p+1}(\Omega)}, \quad \varphi \in X^1(\Omega).$$

Remark 1.1. The uniqueness for $(\mathrm{NLS})_a$ is unknown as in the case where $a = 0$.

This paper is divided into three sections. In Section 2 we give preliminaries for the proof of Theorem 1.1, that is, we introduce the energy methods developed in [9] for nonlinear Schrödinger equations. Finally, we prove Theorem 1.1 in Section 3.

2. Abstract nonlinear Schrödinger equations

Let S be a nonnegative selfadjoint operator in a complex Hilbert space X. Put $X_S := D(S^{1/2})$ with norm $\|u\|_{X_S} := \|(1+S)^{1/2}u\|_X$. Then we have the usual triplet:

$$X_S \subset X = X^* \subset X_S^*.$$

Under this setting S can be extended to a nonnegative selfadjoint operator in X_S^* with domain X_S. Now we consider

$$\text{(ACP)} \qquad \begin{cases} \mathrm{i}\dfrac{\mathrm{d}u}{\mathrm{d}t} = Su + g(u), \\ u(0) = u_0, \end{cases}$$

where $g\colon X_S \to X_S^*$ is a nonlinear operator satisfying
(G1) Existence of energy functional: there exists $G \in C^1(X_S;\mathbb{R})$ such that $G' = g$, that is, given $u \in X_S$, for every $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$|G(u+v) - G(u) - \mathrm{Re}\langle g(u), v\rangle_{X_S^*,X_S}| \leqslant \varepsilon\|v\|_{X_S} \quad \forall\, v \in X_S \text{ with } \|v\|_{X_S} < \delta.$$

(G2) Local Lipschitz continuity: for all $M > 0$ there exists $C(M) > 0$ such that

$$\|g(u) - g(v)\|_{X_S^*} \leqslant C(M)\|u - v\|_{X_S} \quad \forall\, u, v \in X_S \text{ with } \|u\|_{X_S},\ \|v\|_{X_S} \leqslant M.$$

(G3) Hölder-like continuity of energy functional: given $M > 0$, for all $\delta > 0$ there exists a constant $C_\delta(M) > 0$ such that

$$|G(u) - G(v)| \leqslant \delta + C_\delta(M)\|u - v\|_X \quad \forall\, u, v \in X_S \text{ with } \|u\|_{X_S},\ \|v\|_{X_S} \leqslant M.$$

(G4) Gauge type condition for the conservation of charge:

$$\mathrm{Im}\langle g(u), u\rangle_{X_S^*,X_S} = 0, \quad \forall\, u \in X_S.$$

(G5) Closedness type condition: let $\{u_n\}_n$ be any bounded sequence in X_S such that

$$\begin{cases} u_n \to u \quad (n \to \infty) & \text{weakly in } X_S, \\ g(u_n) \to f \quad (n \to \infty) & \text{weakly in } X_S^*. \end{cases}$$

Then $f = g(u)$.
(G6) Lower boundedness of the energy: there exist $\varepsilon \in (0,1]$, $C_0(\cdot) \geqslant 0$ such that

$$G(u) \geqslant -\frac{1-\varepsilon}{2}\|u\|_{X_S}^2 - C_0(\|u\|_X), \quad \forall\, u \in X_S.$$

Here a function u is said to be a *local weak solution* on I to (ACP) if u belongs to $L^\infty(I;X_S) \cap W^{1,\infty}(I;X_S^*)$ and satisfies (ACP) in $L^\infty(I;X_S^*)$. If I coincides with $\mathbb{R}$, then the local weak solution is called a *global weak solution.*

Theorem 2.1 (cf. [9], Theorem 2.4). *Assume that* $g\colon X_S \to X_S^*$ *satisfies* (G1)–(G6). *Then for every* $u_0 \in X_S$ *there exists a global weak solution* u *to* (ACP). *Moreover,* u *belongs to* $C_{\rm w}(\mathbb{R}; X_S) \cap L^\infty(\mathbb{R}; X_S) \cap W^{1,\infty}(\mathbb{R}; X_S^*)$ *and satisfies the weak conservation laws*

$$\|u(t)\|_X = \|u_0\|_X, \quad E(u(t)) \leqslant E(u_0), \quad \forall\, t \in \mathbb{R},$$

where $E(\cdot)$ *is the energy given by* $E(\varphi) := (1/2)\|S^{1/2}\varphi\|_X^2 + G(\varphi)$, $\varphi \in X_S$.

In [9], Theorem 2.4, we assume the uniqueness of local weak solutions to show their regularity. As in [4], Theorem 3.4.1, we do not need to assume the uniqueness when we consider the global extension of local weak solutions.

Remark 2.1. Assume (G1)–(G5), not (G6). Then for every $u_0 \in X_S$ there exists a local weak solution u to (ACP).

3. Proof of Theorem 1.1

First we recall the fractional Sobolev spaces $H^s(\Omega)$.

Lemma 3.1. *Let* $N \geqslant 3$, $0 < s < 1$ *and let* Ω *be a bounded and smooth domain.*

(i) (Sobolev embeddings). *Assume* $2 \leqslant q \leqslant 2N/(N-2s)$. *Then the injection* $H^s(\Omega) \subset L^q(\Omega)$ *is continuous:*

$$\|\varphi\|_{L^q(\Omega)} \leqslant C_{s,q}\|\varphi\|_{H^s(\Omega)}, \quad \varphi \in H^s(\Omega). \tag{3.1}$$

(ii) (Rellich's compactness theorem). *Assume* $2 \leqslant q < 2N/(N-2s)$. *Then the embedding* $H^s(\Omega) \subset L^q(\Omega)$ *is compact.*

Proof. (i) is well known; see [11], Remark 1 in Section 4.6.1. To verify (ii) first note that $H^s(\Omega)$ is the same as the real interpolation space $(L^2(\Omega), H^1(\Omega))_{s,2}$; see [11], Theorem 1 in Section 4.3.1. Since $H^1(\Omega) \subset L^2(\Omega)$ is compact, then $(L^2(\Omega), H^1(\Omega))_{s,2} \subset L^2(\Omega)$ is also compact; see e.g., [11], Theorem 1 in Section 1.16.4. The Hölder inequality and (3.1) imply the interpolation inequality:

$$\|\varphi\|_{L^q(\Omega)} \leqslant \widetilde{C}_{s,q}\|\varphi\|_{L^2(\Omega)}^{1-\theta}\|\varphi\|_{H^s(\Omega)}^{\theta}, \quad \varphi \in H^s(\Omega),\ \theta = \frac{N}{s}\Big(\frac{1}{2} - \frac{1}{q}\Big) \in [0,1). \tag{3.2}$$

This yields part (ii). □

To show Theorem 1.1 it suffices to verify (G1)–(G6) with

$$g(u) := \lambda |u|^{p-1} u, \quad u \in X^1(\Omega), \tag{3.3}$$

$$G(u) := \frac{\lambda}{p+1} \|u\|_{L^{p+1}}^{p+1}, \quad u \in X^1(\Omega). \tag{3.4}$$

Now we put

$$s_0 := \frac{N(p-1)}{2(p+1)} \in [0,1), \quad \text{that is,} \quad \frac{1}{p+1} = \frac{1}{2} - \frac{s_0}{N}; \tag{3.5}$$

note that $H^{s_0}(\Omega) \subset L^{p+1}(\Omega)$ by Lemma 3.1 (i). Fix $s \in (s_0, 1)$. Then (1.3) and (3.1) imply that for every $u_j \in X^1(\Omega)$ ($j = 1,2,3$)

$$\begin{aligned} \Bigl| \int_\Omega |u_1|^{p-1} u_2 u_3 \, \mathrm{d}x \Bigr| &\leqslant C_{s,p+1}^{p+1} \|u_1\|_{H^s(\Omega)}^{p-1} \|u_2\|_{H^s(\Omega)} \|u_3\|_{H^s(\Omega)} \\ &\leqslant C_{s,p+1}^{p+1} C_s^{(p+1)/2} \|u_1\|_{X^1(\Omega)}^{p-1} \|u_2\|_{X^1(\Omega)} \|u_3\|_{X^1(\Omega)}. \end{aligned} \tag{3.6}$$

Verification of (G1). A simple calculation ensures that for all z_1, $z_2 \in \mathbb{C}$,

$$\Bigl| \frac{|z_1 + z_2|^{p+1} - |z_1|^{p+1}}{p+1} - \mathrm{Re}(|z_1|^{p-1} z_1 \overline{z_2}) \Bigr| \leqslant K_p (|z_1|^{p-1} + |z_2|^{p-1}) |z_2|^2.$$

Putting $z_1 := u(x)$, $z_2 := v(x)$ and integrating over Ω, we see from (3.6) that

$$\begin{aligned} &|G(u+v) - G(u) - \mathrm{Re}\langle g(u), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}| \\ &\qquad \leqslant |\lambda| C(s,p) \|v\|_{X^1(\Omega)}^2 (\|u\|_{X^1(\Omega)}^{p-1} + \|v\|_{X^1(\Omega)}^{p-1}), \quad \forall\, u, v \in X^1(\Omega). \end{aligned}$$

Hence we conclude (G1).

Verification of (G2). By using the embeddings $X^1(\Omega) \subset H^s(\Omega) \subset L^{p+1}(\Omega)$ and $L^{1+1/p}(\Omega) \subset X^{-1}(\Omega)$ we have for $u, v \in X^1(\Omega)$ with $\|u\|_{X^1(\Omega)} \leqslant M$, $\|v\|_{X^1(\Omega)} \leqslant M$,

$$\begin{aligned} c^{-1} \|g(u) - g(v)\|_{X^{-1}(\Omega)} &\leqslant \|g(u) - g(v)\|_{L^{1+1/p}(\Omega)} \\ &\leqslant C(\|u\|_{L^{p+1}(\Omega)}^{p-1} + \|v\|_{L^{p+1}(\Omega)}^{p-1}) \|u - v\|_{L^{p+1}(\Omega)} \\ &\leqslant C' M^{p-1} \|u - v\|_{X^1(\Omega)}. \end{aligned}$$

Thus we obtain (G2).

Verification of (G3). A simple calculation yields that for every $u, v \in X^1(\Omega)$

$$|G(u) - G(v)| \leqslant |\lambda| (\|u\|_{L^{p+1}(\Omega)}^p + \|v\|_{L^{p+1}(\Omega)}^p) \|u - v\|_{L^{p+1}(\Omega)}. \tag{3.7}$$

It follows from (3.2) that

$$\|\varphi\|_{L^{p+1}(\Omega)} \leqslant \widetilde{C}_{s,p+1}\|\varphi\|_{H^s(\Omega)}^{s_0/s}\|\varphi\|_{L^2(\Omega)}^{1-s_0/s}, \quad \forall\,\varphi \in X^1(\Omega). \tag{3.8}$$

Applying (3.8) to (3.7) and using the Young inequality, we conclude (G3).

Verification of (G4). The proof is easy and hence omitted.

Verification of (G5). Now let $\{u_n\}_n \subset X^1(\Omega)$ satisfy

$$\begin{cases} u_n \to u \quad (n\to\infty) & \text{weakly in } X^1(\Omega), \\ g(u_n) \to f \quad (n\to\infty) & \text{weakly in } X^{-1}(\Omega). \end{cases}$$

It follows from $X^1(\Omega) \subset H^s(\Omega)$ and Lemma 3.1 (ii) that

$$u_n \to u \quad (n\to\infty) \text{ strongly in } L^q(\Omega)\ (2 \leqslant q < 2N/(N-2s)).$$

Note that $p+1 = 2N/(N-2s_0) < 2N/(N-2s)$ [see (3.5)]. Hence

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad (n\to\infty) \text{ strongly in } L^{1+1/p}(\Omega).$$

Thus $L^{1+1/p}(\Omega) \subset X^{-1}(\Omega)$ implies that

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad (n\to\infty) \text{ strongly in } X^{-1}(\Omega).$$

Therefore $f = g(u)$.

Verification of (G6). If $\lambda \geqslant 0$, then $G(u) \geqslant 0$ for all $u \in X^1(\Omega)$. Thus we only consider $\lambda < 0$ and $p < 1 + 4/N$. Applying (3.8) we have

$$\begin{aligned} G(u) &\geqslant -\frac{\lambda_-}{p+1}\|u\|_{L^{p+1}(\Omega)}^{p+1} \\ &\geqslant -C(s,p,\lambda)(\|u\|_{L^2(\Omega)}^{1-(s_0/s)}\|u\|_{H^s(\Omega)}^{s_0/s})^{p+1}, \quad \forall\,u \in X^1(\Omega). \end{aligned}$$

Since we can take $s \in (s_0, 1)$ such that

$$\frac{(p+1)s_0}{s} = \frac{N(p-1)}{2s} < 2,$$

the Young inequality yields that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$G(u) \geqslant -C_\delta(\|u\|_{L^2(\Omega)}) - \delta\|u\|_{X^1(\Omega)}^2, \quad \forall\,u \in X^1(\Omega).$$

Putting $\delta := (1-\varepsilon)/2$ for some $\varepsilon \in (0,1)$, we conclude that (G6) is satisfied.

Since (G1)–(G6) are fulfilled, Theorem 1.1 is a consequence of Theorem 2.1. □

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