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A NEW ERROR ESTIMATE FOR A FULLY FINITE ELEMENT DISCRETIZATION SCHEME FOR PARABOLIC EQUATIONS USING CRANK-NICOLSON METHOD

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Abstract. Finite element methods with piecewise polynomial spaces in space for solving the nonstationary heat equation, as a model for parabolic equations are considered. The discretization in time is performed using the Crank-Nicolson method.

A new *a priori estimate* is proved. Thanks to this new *a priori estimate*, a new error estimate in the discrete norm of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$ is proved. An $\mathcal{L}^\infty(\mathcal{H}^1)$-error estimate is also shown.

These error estimates are useful since they allow us to get second order time accurate approximations for not only the exact solution of the heat equation but also for its first derivatives (both spatial and temporal).

Even the proof presented in this note is in some sense standard but the stated $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$-error estimate seems not to be present in the existing literature of the Crank-Nicolson finite element schemes for parabolic equations.

Keywords: parabolic equation; finite element method; Crank-Nicolson method; new error estimate

MSC 2010: 65N30, 65N15, 65M15, 35K15, 35K05

1. Preliminaries and a brief description of the main results

Let us consider the following the nonstationary heat equation, as a model for parabolic equations:

$$u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ ($d = 1$, 2 or 3) with a polyhedral boundary $\partial\Omega$, $T > 0$, and f is a given function.

An initial condition is given by:

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{1.2}$$

and, for the sake of simplicity, we consider homogeneous Dirichlet boundary conditions, that is

$$u(x,t) = u(1,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{1.3}$$

Heat equation is typically used in different applications, such as fluid mechanics, heat and mass transfer, etc., and it is the prototypical parabolic partial differential equation which in turn arises, for instance, in many different models like Navier-Stokes and reaction-diffusion systems. It describes the distribution of heat (or variation in temperature) in a given region over time. Therefore parabolic equations are important from the mathematical viewpoint as well as in practice.

Let $\{\mathcal{T}_h\colon\ h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain Ω. The elements of $\mathcal{T}_h$ will be denoted by K. For each triangulation $\mathcal{T}_h$, the subscript h refers to the level of refinement of the triangulation, which is defined by $h = \max\limits_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of the element K.

Let $\mathcal{V}^h$ be the standard finite element space of continuous, piecewise polynomial functions of degree $k \geqslant 1$, i.e.,

$$\mathcal{V}^h = \{v \in \mathcal{C}(\overline{\Omega})\colon\ v|_K \in \mathcal{P}_k,\ \forall K \in \mathcal{T}_h\},$$

and we denote by

$$\mathcal{V}_0^h = \mathcal{V}^h \cap \mathcal{H}_0^1(\Omega). \tag{1.4}$$

The time discretization is performed using a constant time step $\tau = T/(M+1)$, where $M \in \mathbb{N} \setminus \{0\}$, and we shall denote $t_n = n\tau$, for $n \in [\![0, M+1]\!]$.

Throughout this paper, the notations C_i, where $i \in \mathbb{N} \setminus \{0\}$, stand for positive constants independent of the parameters of the discretization.

The discretization scheme we want to consider is implicit and it is based on the use of the Crank-Nicolson method as discretization in time and on the use of the finite element mesh described above. We shall denote by $v^{n-1/2}$ the following arithmetic mean value, when $(v^n)_{n=0}^{M+1}$ is a discrete function, between the two time levels $n-1$ and n:

$$v^{n-1/2} = \frac{v^n + v^{n-1}}{2}, \tag{1.5}$$

and $v^{n-1/2}$ denotes the arithmetic mean value (1.5) by replacing v^n with $v(t_n)$ and v^{n-1} by $v(t_{n-1})$, when $v \in \mathcal{C}([0,T])$, i.e.,

$$(1.6) \qquad v^{n-1/2} = \frac{v(t_n) + v(t_{n-1})}{2}, \quad \forall\, v \in \mathcal{C}([0,T]).$$

To define the finite element approximation for our problem (1.1)–(1.3), we need to use the following discrete first time derivative

$$(1.7) \qquad \partial^1 v^n = \frac{v^n - v^{n-1}}{\tau}, \quad \forall\, n \in [\![1, M+1]\!].$$

The discretization of the initial condition (1.2) is performed as: Find $u_h^0 \in \mathcal{V}_0^h$ (see (1.4)) such that

$$(1.8) \qquad \mathbf{a}(u_h^0, v) = -(\Delta u^0, v)_{\mathcal{L}^2(\Omega)} = \mathbf{a}(u^0, v),\ \forall\, v \in \mathcal{V}_0^h.$$

The discretization of the heat equation (1.1) is: for any $n \in [\![0, M]\!]$, find $u_h^n \in \mathcal{V}_0^h$ such that, for all $v \in \mathcal{V}_0^h$

$$(1.9) \qquad (\partial^1 u_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \mathbf{a}(u_h^{n+1/2}, v) = \left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(t)\,\mathrm{d}t, v\right)_{\mathcal{L}^2(\Omega)},$$

where $\mathbf{a}(\cdot,\cdot)$ denotes the bilinear form defined for all $(u,v) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ by

$$\mathbf{a}(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x.$$

To compute the solution of the finite element scheme (1.8)–(1.9) (see either [3], page 385, or [4], pages 172–173), we first compute the initial solution u_h^0 using (1.8). Equation (1.8) can be written in to the following matrix form

$$(1.10) \qquad AU^0 = \eta^0,$$

where A is a symmetric and positive definite matrix, η^0 is known, and $U^n = (U_1^n, \ldots, U_N^n)^T$ with $u_h^n = \sum_{i=1}^{N} U_i^n \varphi_i$ where φ_i are the basis functions of $\mathcal{V}^h$.

Equation (1.9) leads to the following linear systems, for each time step $n \in [\![1, M+1]\!]$

$$(1.11) \qquad \Big(M + \frac{\tau}{2}A\Big)U^n = \eta^n,$$

where $M + \frac{\tau}{2}A$ is a symmetric, positive definite matrix and η^n is known from the previous steps.

The following discrete second time derivative will help us during the convergence analysis of the finite element scheme (1.8)–(1.9):

$$\partial^2 v^{n+1} = \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2} = \frac{1}{\tau}(\partial^1 v^{n+1} - \partial^1 v^n), \quad \forall\, n \in [\![1, M]\!]. \tag{1.12}$$

The following rules will be useful for our analysis, for any *smooth* function ψ defined on $[0, T]$

$$\partial^1 \psi(t_{n+1}) = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \psi_t(t)\,\mathrm{d}t \quad \text{and} \quad \partial^2 \psi(t_{n+1}) = \frac{1}{\tau^2}\int_{t_n}^{t_{n+1}}\int_{t-\tau}^{t} \psi_{tt}(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{1.13}$$

The existence and uniqueness of a solution to the finite element scheme (1.8)–(1.9) stems from the fact the matrices A and $M + \frac{1}{2}\tau A$ are positive definite (see also [3], pages 385–386, and [4], pages 171–173).

The following error estimates are known:

1. $\mathcal{L}^\infty(\mathcal{L}^2)$-*error estimate.* Under the regularity assumption $u^0 \in \mathcal{H}^{k+1}(\Omega)$, $u_t \in \mathcal{L}^1(0, T; \mathcal{H}^{k+1}(\Omega))$, and $u_{ttt} \in \mathcal{L}^1(0, T;\ \mathcal{L}^2(\Omega))$, the following $\mathcal{L}^\infty(\mathcal{L}^2)$-error estimate holds (see [2], where the piecewise linear finite element space is used, [3], Corollary 11.3.1, page 394, and [4], Theorem 7.5.2, pages 177–178), for all $n \in [\![0, M+1]\!]$:

$$\begin{aligned}
&\|u_h^n - u(t_n)\|_{\mathcal{L}^2(\Omega)} \\
&\quad \leqslant C_1 h^{k+1}\left(\|u^0\|_{\mathcal{H}^{k+1}(\Omega)} + \int_0^T \|u_t(t)\|_{\mathcal{H}^{k+1}(\Omega)}\,\mathrm{d}t\right) + \frac{\tau^2}{8}\int_0^T \|u_{ttt}(t)\|_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t \\
&\quad \leqslant C_2(h^{k+1} + \tau^2)\left(\|u^0\|_{\mathcal{H}^{k+1}(\Omega)} + \int_0^T \|u_t(t)\|_{\mathcal{H}^{k+1}(\Omega)}\,\mathrm{d}t + \int_0^T \|u_{ttt}(t)\|_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t\right).
\end{aligned}$$

2. $\mathcal{L}^2(\mathcal{H}^1)$-*error estimate.* Under the regularity assumption $u \in \mathcal{C}(0, T; \mathcal{H}^{k+1}(\Omega))$ and $u_{ttt} \in \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega))$, the following $\mathcal{L}^2(\mathcal{H}^1)$-error estimate holds (see [1]):

$$\left(\sum_{n=0}^{M} \tau \|e_h^{n+1/2}\|^2_{\mathcal{H}^1(\Omega)}\right)^{1/2} \leqslant C_3(h^k + \tau^2)(\|u\|_{\mathcal{C}(0,T;\ \mathcal{H}^{k+1}(\Omega))} + \|u_{ttt}\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}),$$

where $e_h^n = u(t_n) - u_h^n$ and the notation $v^{n+1/2}$ is defined in (1.5)–(1.6).

3. $\mathcal{L}^\infty(\mathcal{H}^1)$-*error estimate when the piecewise linear finite element space* ($k = 1$) *is used.* Under the assumption that $f \equiv 0$, $u^0 \in H_0^4(\Omega)$, $k = 1$ (piecewise linear finite element space), and a suitable choice for quadrature replacing the first term on the left of (1.9), it is proved in [2] that the error is of order $h + \tau^2$ in the discrete norm of $\mathcal{L}^\infty(\mathcal{H}^1)$.

4. $\mathcal{L}^\infty(\mathcal{H}^1)$-*error estimate when biquadratic finite volume element methods are used on rectangular spatial domain* Ω. Thanks to the estimate in [5], Theorem 5.1,

pages 1064–1065, the error is of order $h^2 + \tau^2$ in $\mathcal{L}^\infty(\mathcal{H}^1)$-norm when biquadratic *finite volume element* methods based on some *optimal stress points* are used in the particular case of a *rectangular spatial domain* Ω.

However, we are not aware of the existence of any error estimate in the discrete norm of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$. We think also that there is no explicit statement for an error in the discrete norm of $\mathcal{L}^\infty(\mathcal{H}^1)$ for arbitrary k (recall that the $\mathcal{L}^\infty(\mathcal{H}^1)$-error estimate provided in [2] is stated when $k = 1$, i.e., using piecewise linear trial functions).

We aim in this contribution to provide a new $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$-error estimate. An $\mathcal{L}^\infty(\mathcal{H}^1)$-error estimate for arbitrary k will be also derived. We will prove, under the assumption that the exact solution is smooth, that the error is of order $h^{k+1} + \tau^2$ in the discrete norm of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$ and is of order $h^k + \tau^2$ in the discrete norm of $\mathcal{L}^\infty(\mathcal{H}^1)$. The proof of the results we want to present is based on a new *a priori estimate* stated in Lemma 2.1.

It is clear that deriving error estimate of order $h^{k+s} + \tau^2$, with s is either 0 or 1, in the norms of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$ and $\mathcal{L}^\infty(\mathcal{H}^1)$ yields approximations of order $h^{k+s} + \tau^2$ in the discrete norm $\mathcal{L}^\infty(\mathcal{L}^2)$ for the first derivatives (both temporal and spatial) of the exact solution. Such results are important from mathematical point of view. In addition to this, the approximation of the first derivatives of the exact solution is useful in practice since the *time derivative* u_t represents the rate of changes of temperature at point over time and *temperature gradient* is a physical quantity that describes in which direction and at what rate the temperature changes the most rapidly around a particular location. The temperature gradient is a dimensional quantity expressed in units of degrees (on a particular temperature scale) per unit length. Temperature gradients in the atmosphere are important in the atmospheric sciences (meteorology, climatology and related fields).

If $\omega(x,t)$ is a function of the space variable x and time t, then it is sometimes suitable to separate these variables and consider ω as a function $\omega(t) = \omega(\cdot,t)$ which for each t under consideration attains a value $\omega(t)$ that is a function of x and belongs to a suitable space of functions depending on x, see [4], page 155.

We assume that $f \in \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ and $u_0 \in \mathcal{L}^2(\Omega)$. Then, see for instance [4], pages 156–158, for more details, there exists a unique weak solution for (1.1)–(1.3) in the following sense: there exists a function $u \in \mathcal{L}^2(0,T;\mathcal{H}^1_0(\Omega)) \cap \mathcal{C}(0,T;\mathcal{L}^2(\Omega))$ such that:

(1) In the sense of distributions on $]0,T[$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v\rangle + \mathbf{a}(u(t), v) = \langle f(t), v\rangle, \quad \forall v \in \mathcal{H}^1_0(\Omega). \tag{1.14}$$

(2) and

$$u(0) = u_0. \tag{1.15}$$

The following *coercivity* will be useful, for all $v \in \mathcal{H}_0^1(\Omega)$

$$\mathbf{a}(v,v) = \int_\Omega |\nabla v|^2(x)\,\mathrm{d}x = |v|_{1,\Omega}^2. \tag{1.16}$$

The Poincaré inequality (see [3], Theorem 1.3.3, page 11) states, for some positive constant C_p

$$\|v\|_{\mathcal{L}^2(\Omega)} \leqslant C_p |v|_{1,\Omega}, \quad \forall\, v \in \mathcal{H}_0^1(\Omega). \tag{1.17}$$

The convergence of the finite element scheme is analyzed using the spaces $\mathcal{C}^m([0,T];\mathcal{H}^l(\Omega))$, where m and l are integers, of m-times continuously differentiable mappings of the interval $[0,T]$ with values in the Sobolev space $\mathcal{H}^l(\Omega)$, see [4], page 156. The space $\mathcal{C}^m([0,T];\mathcal{H}^l(\Omega))$ is equipped with the norm

$$\|u\|_{\mathcal{C}^m([0,T]\,;\,\mathcal{H}^l(\Omega))} = \max_{j\in[\![0,m]\!]}\Big\{\sup_{t\in[0,T]}\Big\|\frac{\mathrm{d}^j u}{\mathrm{d}t^j}(t)\Big\|_{\mathcal{H}^l(\Omega)}\Big\}.$$

2. Statement of the main results

We first begin by a regularity assumption for the problem (2.1) below. For any $r \in \mathcal{L}^2(\Omega)$, let $\varphi(r) \in \mathcal{H}_0^1(\Omega)$ be the exact solution of the following problem (the existence and uniqueness are ensured by the Lax-Milgram lemma)

$$\mathbf{a}(\varphi(r), v) = \langle r, v\rangle,\ \forall v \in \mathcal{H}_0^1(\Omega). \tag{2.1}$$

Assumption 2.1 (Regularity assumption, see [3], Remark 6.2.1, page 173)**.** For any $r \in \mathcal{L}^2(\Omega)$, we assume that the solution $\varphi(r)$ of (2.1) belongs to $\mathcal{H}^2(\Omega)$ and there exists a constant $C_{\mathrm{reg}} > 0$ such that $\|\varphi(r)\|_{\mathcal{H}^2(\Omega)} \leqslant C_{\mathrm{reg}}\|r\|_{\mathcal{L}^2(\Omega)}$, for all $r \in \mathcal{L}^2(\Omega)$.

Among the main results of the present contribution is the following theorem:

Theorem 2.2 (New $\mathcal{W}^{1,\infty}(0,T;\mathcal{L}^2(\Omega))$-error estimate)**.** *Under Assumption* 2.1, *let* $u \in \mathcal{L}^2(0,T;H_0^1(\Omega))$ *be the weak solution of* (1.1)–(1.3) *in the sense of* (1.14)–(1.15). *Let* $\{\mathcal{T}_h; h > 0\}$ *be a family of shape regular and quasi-uniform triangulations of the domain* Ω *and* $h = \max_{K\in\mathcal{T}_h} h_K$, *where* h_K *denotes the diameter of the element* K. *Let* $\mathcal{V}_0^h$ *be the standard finite element space defined by* (1.4) *where* $k \in \mathbb{N}\setminus\{0\}$. *We assume that the time discretization is performed using a constant time step* $\tau = T/(M+1)$, *where* $M \in \mathbb{N}\setminus\{0\}$, *and we define* $t_n = n\tau$, *for* $n \in [\![0, M+1]\!]$.

Then, there exists a unique solution $(u_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ *for* (1.8)–(1.9). *Assume that the exact solution* u *is of class* $\mathcal{C}^3([0,T];\mathcal{H}^{k+1}(\Omega))$. *Then, the following error estimates hold:*

▷ *Discrete* $\mathcal{W}^{1,\infty}(0,T;\mathcal{L}^2(\Omega))$*-estimate: for all* $n \in [\![1, M+1]\!]$

$$\|\partial^1(u_h^n - u(t_n))\|_{\mathcal{L}^2(\Omega)} \leqslant C_3(h^{k+1} + \tau^2)\|u\|_{\mathcal{C}^3([0,T]\,;\,\mathcal{H}^{k+1}(\Omega))}, \tag{2.2}$$

▷ *Discrete* $\mathcal{L}^\infty(0,T;\mathcal{H}_0^1(\Omega))$*-estimate: for all* $n \in [\![0, M]\!]$

$$|u_h^{n+1/2} - u^{n+1/2}|_{1,\Omega} \leqslant C_4(h^k + \tau^2)\|u\|_{\mathcal{C}^3([0,T]\,;\,\mathcal{H}^{k+1}(\Omega))}, \tag{2.3}$$

where ∂^1 *denotes the discrete temporal derivative* (1.7), *and the notation* $v^{n+1/2}$ *is defined in* (1.5)–(1.6).

To prove Theorem 2.2, we need to use the following new *a priori estimate.*

Lemma 2.1 (A new a priori estimate)**.** *Under Assumption* 2.1, *let* $\{\mathcal{T}_h; h > 0\}$ *be a family of shape regular and quasi-uniform triangulations of the domain* Ω *and* $h = \max_{K\in\mathcal{T}_h} h_K$, *where* h_K *denotes the diameter of the element* K. *Let* $\mathcal{V}_0^h$ *be the standard finite element space defined by* (1.4). *We assume that the time discretization is performed using a constant time step* $\tau = T/(M+1)$, *where* $M \in \mathbb{N}\setminus\{0\}$, *and we define* $t_n = n\tau$, *for* $n \in [\![0, M+1]\!]$.

Assume that there exits $(\eta_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ *such that* $\eta_h^0 = 0$ *and for all* $n \in [\![0, M]\!]$

$$(\partial^1\eta_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \mathbf{a}(\eta_h^{n+1/2}, v) = (\gamma^n, v)_{\mathcal{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{V}_0^h, \tag{2.4}$$

where $\gamma^n \in \mathcal{L}^2(\Omega)$, *for all* $n \in [\![0, M]\!]$.

Then the following estimates hold:

$$\|\partial^1\eta_h^n\|_{\mathcal{L}^2(\Omega)} \leqslant C_5(\gamma + \overline{\gamma}), \quad \forall\, n \in [\![1, M+1]\!], \tag{2.5}$$

and

$$|\eta_h^{n+1/2}|_{1,\Omega} \leqslant C_6(\gamma + \overline{\gamma}), \quad \forall\, n \in [\![0, M]\!], \tag{2.6}$$

where the notation $v^{n+1/2}$ *is defined in* (1.6), *the seminorm (it is a norm on* $\mathcal{H}_0^1(\Omega)$*)* $|\cdot|_{1,\Omega}$ *is given in* (1.16), *and*

$$\gamma = \max_{n=0}^{M}\|\gamma^n\|_{\mathcal{L}^2(\Omega)} \quad \text{and} \quad \overline{\gamma} = \max_{n=1}^{M}\|\partial^1\gamma^n\|_{\mathcal{L}^2(\Omega)}. \tag{2.7}$$

P r o o f. We will prove Lemma 2.1 item by item.

P r o o f of (2.5): acting the discrete operator ∂^1 on (2.4) we get, for all $n \in [\![1, M]\!]$

$$(\partial^2 \eta_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \mathbf{a}(\partial^1 \eta_h^{n+1/2}, v) = (\partial^1 \gamma^n, \ v)_{\mathcal{L}^2(\Omega)}. \tag{2.8}$$

Taking $v = \partial^1 \eta_h^{n+1} + \partial^1 \eta_h^n$ in (2.8), using (1.12) together with (1.16), and multiplying the result by τ we get

$$\begin{aligned}\|\partial^1 \eta_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)} - \|\partial^1 \eta_h^n\|^2_{\mathcal{L}^2(\Omega)} + \frac{\tau}{2}\left|\partial^1(\eta_h^{n+1} + \eta_h^n)\right|^2_{1,\Omega} \\ = \tau(\partial^1 \gamma^n, \partial^1(\eta_h^{n+1} + \eta_h^n))_{\mathcal{L}^2(\Omega)}.\end{aligned}$$

Thanks to the use of the Cauchy-Schwarz inequality together with (1.17), the previous inequality implies that

$$\begin{aligned}\|\partial^1 \eta_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)} - \|\partial^1 \eta_h^n\|^2_{\mathcal{L}^2(\Omega)} + \frac{\tau}{2}|\partial^1(\eta_h^{n+1} + \eta_h^n)|^2_{1,\Omega} \\ \leqslant \tau C_p \|\partial^1 \gamma^n\|_{\mathcal{L}^2(\Omega)} |\partial^1(\eta_h^{n+1} + \eta_h^n)|_{1,\Omega}.\end{aligned} \tag{2.9}$$

Using the inequality $ab \leqslant 4a^2 + b^2/4$, the previous inequality yields that

$$\|\partial^1 \eta_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)} - \|\partial^1 \eta_h^n\|^2_{\mathcal{L}^2(\Omega)} \leqslant 4(C_p)^2 \tau (\overline{\gamma})^2.$$

Summing the previous inequality over $n \in [\![1, j]\!]$, where $j \in [\![1, M]\!]$, and using the fact that $M\tau < T$ we get

$$\|\partial^1 \eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)} \leqslant \|\partial^1 \eta_h^1\|^2_{\mathcal{L}^2(\Omega)} + 4T(C_p)^2 (\overline{\gamma})^2. \tag{2.10}$$

Thanks to the previous inequality, any estimate on $\|\partial^1 \eta_h^1\|^2_{\mathcal{L}^2(\Omega)}$ will yield an estimate on $\|\partial^1 \eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}$. Let us move to estimate $\|\partial^1 \eta_h^1\|^2_{\mathcal{L}^2(\Omega)}$. Take $n = 0$ in (2.4) to get (note that $\eta_h^0 = 0$)

$$(\partial^1 \eta_h^1, v)_{\mathcal{L}^2(\Omega)} + \frac{1}{2}\mathbf{a}(\eta_h^1, v) = (\gamma^0, v)_{\mathcal{L}^2(\Omega)}.$$

Taking $v = \partial^1 \eta_h^1$ in the previous equality, using that fact that $\partial^1 \eta_h^1 = \eta_h^1 / k$ (since $\eta_h^0 = 0$), and using (1.16) leads to

$$\|\partial^1 \eta_h^1\|^2_{\mathcal{L}^2(\Omega)} \leqslant (\gamma^0, \partial^1 \eta_h^1)_{\mathcal{L}^2(\Omega)}. \tag{2.11}$$

This with the Cauchy-Schwarz inequality yields that

$$\|\partial^1 \eta_h^1\|_{\mathcal{L}^2(\Omega)} \leqslant \gamma. \tag{2.12}$$

This with (2.10) implies that, for all $j \in [\![0, M]\!]$

$$\|\partial^1 \eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)} \leqslant (\gamma)^2 + 4T(C_p)^2(\overline{\gamma})^2.$$

The last inequality with the rule $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$, for all positive a and b, implies (2.5) with $C_5 = \max(1, 2\sqrt{T}C_p)$.

Proof of (2.6): Taking $v = \eta_h^{n+1} + \eta_h^n$ in (2.4) and using the Cauchy-Schwarz inequality together with (1.17), we get, for all $n \in [\![0, M]\!]$

$$|\eta_h^{n+1/2}|_{1,\Omega} \leqslant C_p(\|\partial^1 \eta_h^{n+1}\|_{\mathcal{L}^2(\Omega)} + \gamma).$$

This with (2.5) which has been proved in the previous item yields (2.6). This completes the proof of Lemma 2.1. □

Proof of Theorem 2.2

1. Existence and uniqueness of the discrete solution. The existence and uniqueness for the scheme (1.8)–(1.9) stems from the fact that the matrices involved in the linear systems of this scheme are either the matrix A or the matrix $M + \frac{1}{2}\tau A$ (see (1.10) and (1.11)) which are positive definite.

2. Proof of estimates (2.2)–(2.3)**.** The proof of the estimates (2.2)–(2.3) of Theorem 2.2 is based essentially on the comparison with the following finite element scheme: for each $n \in [\![0, M+1]\!]$, we compute $\bar{u}_h^n \in \mathcal{V}_0^h$ (see (1.4)) such that

$$\mathbf{a}(\bar{u}_h^n, v) = -(\Delta u(t_n),\ v)_{\mathcal{L}^2(\Omega)} = \mathbf{a}(u(t_n), v), \quad \forall\, v \in \mathcal{V}_0^h. \tag{2.13}$$

The scheme (2.13) has a unique solution thanks to the coercivity (1.16).

The following convergence result is known (see for instance [3], Theorem 6.2.1, pages 171–172)

$$|\bar{u}_h^n - u(t_n)|_{1,\Omega} \leqslant C_7 h^k \|u\|_{\mathcal{C}([0,T];\, \mathcal{H}^{k+1}(\Omega))}. \tag{2.14}$$

Acting the discrete operator ∂^j (see (1.7) and (1.12)), $j \in \{1, 2\}$, on the both sides of (2.13) yields

$$\mathbf{a}(\partial^j \bar{u}_h^n, v) = -(\Delta \partial^j u(t_n),\ v)_{\mathcal{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{V}_0^h. \tag{2.15}$$

This with the known convergence result [3], Proposition 6.2.2, page 173, and (1.13) implies

$$\|\partial^j \bar{u}_h^n - \partial^j u(t_n)\|_{\mathcal{L}^2(\Omega)} \leqslant C_7 h^{k+1} \|u\|_{\mathcal{C}^j([0,T];\, \mathcal{H}^{k+1}(\Omega))}.$$

This with the fact that $\|u\|_{\mathcal{C}^j([0,T];\mathcal{H}^{k+1}(\Omega))} \leqslant \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}$ (since $j \leqslant 2$) implies that

$$\|\partial^j \bar{u}_h^n - \partial^j u(t_n)\|_{\mathcal{L}^2(\Omega)} \leqslant C_7 h^{k+1} \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.16}$$

The convergence results (2.14) and (2.16) will be used later.

We consider the auxiliary error given by

$$\bar{e}_h^n = u_h^n - \bar{u}_h^n. \tag{2.17}$$

Taking $n = 0$ in (2.13), using (1.2), and comparing the result with (1.8) we get the following useful property

$$\bar{e}_h^0 = 0. \tag{2.18}$$

Writing (2.13) in the level $n+1$ and adding the result to (2.13) we get

$$\mathbf{a}(\bar{u}_h^{n+1/2}, v) = -(\Delta u^{n+1/2}, v)_{\mathcal{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{V}_0^h. \tag{2.19}$$

Subtracting (2.19) from (1.9), adding $-\partial^1 \bar{u}_h^n$ to the result, and substituting f by $u_t - \Delta u$ (the subject of (1.1)), we get

$$(\partial^1 \bar{e}_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \mathbf{a}(\bar{e}_h^{n+1/2}, v) = (\mathbb{K}^{n,1} - \mathbb{K}^{n,2}, v)_{\mathcal{L}^2(\Omega)}, \tag{2.20}$$

where

$$\mathbb{K}^{n,1} = \partial^1 (u(t_{n+1}) - \bar{u}_h^{n+1}) \text{ and } \mathbb{K}^{n,2} = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \Delta u(t)\, \mathrm{d}t - \frac{\Delta u(t_{n+1}) + \Delta u(t_n)}{2}.$$

Note that $\bar{e}_h^n$ satisfies (2.18) and (2.20) and therefore $\bar{e}_h^n$ satisfies the hypotheses of Lemma 2.1, so one can apply Lemma 2.1 to obtain

$$\|\partial^1 \bar{e}_h^n\|_{\mathcal{L}^2(\Omega)} \leqslant C_5(\gamma + \overline{\gamma}), \quad \forall n \in [\![1, M+1]\!], \tag{2.21}$$

and

$$|\bar{e}_h^{n+1/2}|_{1,\Omega} \leqslant C_6(\gamma + \overline{\gamma}), \quad \forall n \in [\![0, M]\!]. \tag{2.22}$$

where

$$\gamma = \max_{n=0}^{M} \|\mathbb{K}^{n,1} - \mathbb{K}^{n,2}\|_{\mathcal{L}^2(\Omega)} \quad \text{and} \quad \overline{\gamma} = \max_{n=1}^{M} \|\partial^1(\mathbb{K}^{n,1} - \mathbb{K}^{n,2})\|_{\mathcal{L}^2(\Omega)}. \tag{2.23}$$

The error estimates (2.16) imply that, for $j \in \{0,1\}$

$$\|\partial^j \mathbb{K}^{n,1}\|_{\mathcal{L}^2(\Omega)} \leqslant C_7 h^{k+1} \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.24}$$

We use the following known representation (it is easy to check using integration by parts):

$$\mathbb{K}^{n,2} = \frac{1}{\tau}\int_0^\tau \Big(\frac{(t-\tau/2)^2}{2} - \frac{\tau^2}{8}\Big)\Delta u_{tt}(t+t_n)\,\mathrm{d}t. \tag{2.25}$$

We can easily check that $\frac{1}{2}(t-\tau/2)^2 - \frac{1}{8}\tau^2$ is non-positive for $t \in [0,\tau]$ and by some elementary calculations, we get

$$\int_0^\tau \Big(\frac{(t-\tau/2)^2}{2} - \frac{\tau^2}{8}\Big)\,\mathrm{d}t = -\frac{\tau^3}{12}.$$

This with the first representation of (1.13) and the triangle inequality implies that, for $j \in \{0,1\}$ (recall that Ω is a subset of $\mathbb{R}^d$)

$$\|\partial^j \mathbb{K}^{n,2}\|_{\mathcal{L}^2(\Omega)} \leqslant d\frac{\tau^2}{12}\|u\|_{\mathcal{C}^3([0,T];\mathcal{H}^2(\Omega))}. \tag{2.26}$$

Gathering now (2.21)–(2.24) and (2.26) we get

$$\|\partial^1 \bar{e}_h^n\|_{\mathcal{L}^2(\Omega)} \leqslant C_8(h^{k+1}+\tau^2)\|u\|_{\mathcal{C}^3([0,T];\mathcal{H}^{k+1}(\Omega))} \tag{2.27}$$

and

$$|\bar{e}_h^{n+1/2}|_{1,\Omega} \leqslant C_9(h^{k+1}+\tau^2)\|u\|_{\mathcal{C}^3([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.28}$$

Noting that $u_h^n - u(t_n) = u_h^n - \bar{u}_h^n + \bar{u}_h^n - u(t_n) = \bar{e}_h^n + \bar{u}_h^n - u(t_n)$, one can deduce from (2.14), (2.16), and (2.27)–(2.28) together with the triangle inequality the desired estimates (2.2)–(2.3). □

3. Conclusion

We considered a Crank-Nicolson finite element scheme for the nonstationary heat equation. The existing literature, see for instance [1], [3], [4] and the references therein, concerning the convergence of the error state that the convergence order is $h^{k+1} + \tau^2$ or $h^k + \tau^2$ in the discrete norms $\mathcal{L}^\infty(\mathcal{L}^2)$ or $\mathcal{L}^2(\mathcal{H}^1)$, respectively, and [2] states that for $k = 1$ (piecewise linear finite elements) the order is $h + \tau^2$ in $\mathcal{L}^\infty(\mathcal{H}^1)$. We proved in the present contribution that the error is of order $h^{k+1} + \tau^2$ in the discrete norm of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$. It is also shown that the error is of order $h^k + \tau^2$ in the discrete norm of $\mathcal{L}^\infty(\mathcal{H}^1)$. These simple results seem not to be present in the existing literature. The stated results can be extended to parabolic equations with time independent variable coefficients.

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