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Mark Greer, Lee Raney

Abstract. We investigate loops which can be written as the semidirect product of a loop and a group, and we provide a necessary and sufficient condition for such a loop to be Moufang. We also examine a class of loop extensions which arise as a result of a finite cyclic group acting as a group of semiautomorphisms on an inverse property loop. In particular, we consider closure properties of certain extensions similar to those as in [S. Gagola III, *Cyclic extensions of Moufang loops induced by semiautomorphisms*, J. Algebra Appl. **13** (2014), no. 4, 1350128], but from an external point of view.

Keywords: extensions; semidirect products; Moufang loops; inverse property loops

Classification: 20N05

1. Introduction

A *loop* $(Q, \cdot)$ consists of a set Q with a binary operation $\cdot : Q \times Q \to Q$ such that (i) for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$, and (ii) there exists $1 \in Q$ such that $1x = x1 = x$ for all $x \in Q$. We denote these unique solutions by $x = a\backslash b$ and $y = b/a$, respectively. Standard references in loop theory are [2], [12]. All loops considered here are finite loops.

If G, N, and H are groups, then G is an *extension* of H by N if $N \trianglelefteq G$ and $G/N \simeq H$. Extensions of groups are of great interest in group theory. Of particular interest are the ideas of *internal* and *external* semidirect products of groups. That is, for $G = NH$ with $N \trianglelefteq G$ and $N \cap H = 1$, the structure of G is uniquely determined by N, H, and the action of H on N by conjugation. In this case, G is the internal *semidirect product of H acting on N*. Alternatively, given groups N and H and a group homomorphism $\phi : H \to \mathrm{Aut}(N)$ from H into the group of automorphisms of N, one can construct the external semidirect product $G = N \rtimes_\phi H$ in a standard way: namely, $G = \{(n, h) \mid n \in N, h \in H\}$ with multiplication defined by $(n_1, h_1)(n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2)$. Note that for all $h \in H$, $\phi(h)$ corresponds to conjugation by h in N. There is a natural equivalence between internal and external semidirect products of groups, and both are usually referred to as *semidirect products*.

Now, let N be a loop and H a group. Let $\phi : H \to \mathrm{Sym}(N)$ be a group homomorphism from H into the symmetric group on N. Then, the external semidirect product $G = N \rtimes_\phi H$ is the quasigroup defined as $G = \{(x, g) \mid x \in N, g \in H\}$ with multiplication given by

$$(x, g)(y, h) = (x\phi(g)(y), gh). \tag{1}$$

Such semidirect products have been studied in [8]. Note that if $\phi : H \to \mathrm{Sym}(N)_1$, where $\mathrm{Sym}(N)_1$ is the stabilizer of 1 in $\mathrm{Sym}(N)$, then G is a loop. In general, properties of the loop N do not necessarily extend to all of G. For example, if N is a Moufang loop, G need not be Moufang, even in the case that H acts on N as a group of automorphisms.

In §2, we provide a necessary and sufficient condition on N and ϕ in order for $G = N \rtimes_\phi H$ to be a Moufang loop (Theorem 5). We also provide an example which shows that it is possible to satisfy the hypotheses of Theorem 5.

In [7], the author studies Moufang loops that can be written as the product of a normal Moufang subloop and a cyclic subgroup. There it is shown that given a Moufang loop $G = NH$, where N is a normal subloop and H is a cyclic subgroup of order coprime to 3, the multiplication in G is completely determined by a particular semiautomorphism of N. Conversely, the author notes that an arbitrary extension of a cyclic group of order coprime to 3 by a Moufang loop (with multiplication given as in the conclusion of Theorem 8) is not necessarily Moufang. In §3, our approach is to consider the *external* extension of a cyclic group H (whose order is not divisible by 3) by a loop N with multiplication defined as in [7]. In particular, we show that the class of IP loops is closed under such extensions (Theorem 10). We conclude with examples which illustrate that, in general, certain classes of loops (*i.e.* diassociative loops, power associative loops, flexible loops, etc.) are not closed under such extensions.

2. Semidirect products and Moufang loops

Throughout juxtaposition binds more tightly than an explicit $\cdot$ so that, for instance, $xy \cdot z$ means $(xy)z$. For $x \in Q$, where Q is a loop, define the *right* and *left translations* by x by, respectively, $R_x(y) = yx$ and $L_x(y) = xy$ for all $y \in Q$. The fact that these mappings are permutations of Q follows easily from the definition of a loop. It is easy to see that $L_x^{-1}(y) = x\backslash y$ and $R_x^{-1}(y) = y/x$. We define the *multiplication group of* Q, $\mathrm{Mlt}(Q) = \langle L_x, R_x \mid \forall x \in Q\rangle$ and the *inner mapping group of* Q, $\mathrm{Mlt}(Q)_1 = \mathrm{Inn}(Q) = \{\theta \in \mathrm{Mlt}(Q) \mid \theta(1) = 1\}$, the stabilizer of $1 \in Q$.

A bijection $f : Q \to Q$ is a *semiautomorphism* of Q if (i) $f(1) = 1$ and (ii) $f(x(yx)) = f(x)(f(y)f(x))$. The *semiautomorphism group of* Q, $\mathrm{SemiAut}(Q)$ is defined as the set of semiautomorphisms of Q under composition. The following proposition will be helpful throughout.

Proposition 1. *Let Q be a loop with two-sided inverses* (i.e. $1\backslash x = 1/x = x^{-1}$ *for all $x \in Q$) and $f \in \mathrm{SemiAut}(Q)$. Then for all $x \in Q$, $f(x^{-1}) = (f(x))^{-1}$.*

PROOF: Since f is a semiautomorphism, $f(x) = f(x\cdot(x^{-1}x)) = f(x)(f(x^{-1})f(x))$. Canceling gives $f(x)^{-1} = f(x^{-1})$. ☐

Moufang loops, which are easily the most studied class of loops, are defined by any one of the following four equivalent identities:

$$(xy)(zx) = x(yz \cdot x) \qquad (xy)(zx) = (x \cdot yz)x$$

$$(xy \cdot x)z = x(y \cdot xz) \qquad (zx \cdot y)x = z(x \cdot yx).$$

In particular, Moufang loops are *diassociative* (*i.e.* every subloop generated by two elements is a group). If Q is Moufang, then $\mathrm{Inn}(Q) \leq \mathrm{SemiAut}(Q)$ [1].

The following shows that if $G = N \rtimes_\phi H$ is Moufang, then H acts on N as a group of automorphisms.

Lemma 2. *Let N be a loop and H a group. Let $\phi : H \to \mathrm{Sym}(N)_1$ be a group homomorphism. If $G = N \rtimes_\phi H$ is Moufang, then $\phi(H) \subseteq \mathrm{Aut}(N)$.*

PROOF: Since G is Moufang, $(xy)(zx) = x((yz)x)$ for all $x, y, z \in G$. Therefore for any $h \in H$,

$$\begin{aligned} [(x, 1_H)(1, h)][(z, 1_H)(x, 1_H)] = (x, 1_H)[(1, h)(z, 1_H) \cdot (x, 1_H)] \qquad &\Leftrightarrow \\ (x \cdot \phi(h)(zx), h) = (x \cdot \phi(h)(z)\phi(h)(x), h) \end{aligned}$$

Cancellation gives $\phi(h)(zx) = \phi(h)(z)\phi(h)(x)$. $\square$

A loop Q is an *inverse property* loop (or IP loop) if x has a two-sided inverse for all $x \in Q$, and $(yx^{-1})x = y = x(x^{-1}y)$ holds for all $x, y \in Q$.

Lemma 3. *Let N be an IP loop. Then there exists $f \in \mathrm{Aut}(N)$ such that*

$$(*) \qquad (xy)(zf(x)) = x(yz \cdot f(x))$$

for all $x, y, z \in N$ if and only if N is Moufang.

PROOF: Firstly, if N is Moufang, then the trivial isomorphism satisfies $(*)$ for all $x, y, z \in N$. Conversely, let $f \in \mathrm{Aut}(N)$ such that $(*)$ holds for all $x, y, z \in N$. Note that setting $z = 1$ in $(*)$ gives

$$(2) \qquad x \cdot yf(x) = xy \cdot f(x).$$

Since f is an automorphism and the above holds for all $x \in N$, replacing x with $f^{-1}(x)$ yields

$$(3) \qquad f^{-1}(x) \cdot yx = f^{-1}(x)y \cdot x.$$

Now, compute

$$\begin{aligned} (x \cdot zf(x))f(y) &= [x \cdot (y \cdot y^{-1}z)f(x)]f(y) \overset{(*)}{=} [(xy)(y^{-1}z \cdot f(x))]f(y) \\ &= [(xy)(y^{-1}z \cdot f(x))](f(x)^{-1} \cdot f(xy)) \\ &\overset{(*)}{=} (xy)[(y^{-1}z \cdot f(x))f(x)^{-1} \cdot f(xy)] \end{aligned}$$

$$
\begin{aligned}
&= (xy) \cdot (y^{-1}z)f(xy) \overset{(*)}{=} (xy \cdot y^{-1}) \cdot zf(xy) \\
&= x \cdot zf(xy).
\end{aligned}
$$

As before, replacing x with $f^{-1}(x)$ yields

$$(f^{-1}(x) \cdot zx)f(y) = f^{-1}(x) \cdot (z \cdot xf(y)). \tag{4}$$

Hence, we have

$$
\begin{aligned}
(x \cdot f^{-1}(y)z)(yf(x)) &\overset{(*)}{=} x[(f^{-1}(y)z \cdot y)f(x)] \quad \overset{(3)}{=} x[(f^{-1}(y) \cdot zy)f(x)] \\
&\overset{(4)}{=} x[f^{-1}(y) \cdot (z \cdot yf(x))].
\end{aligned}
$$

Multiplying by $f^{-1}(y)$ on the left gives

$$f^{-1}(y)[x \cdot f^{-1}(y)(z \cdot yf(x))] = f^{-1}(y)[(x \cdot f^{-1}(y)z)(yf(x))]. \tag{5}$$

Now, compute

$$
\begin{aligned}
f^{-1}(y)x \cdot z(yf(x)) &= f^{-1}(y)x \cdot zf(f^{-1}(y)x) \\
&= f^{-1}(y)x \cdot (x^{-1} \cdot xz)f(f^{-1}(y)x) \\
&\overset{(*)}{=} (f^{-1}(y)x \cdot x^{-1})(xz \cdot f(f^{-1}(y)x)) \\
&= f^{-1}(y)[xz \cdot yf(x)].
\end{aligned}
$$

Replacing z with $f^{-1}(y)z$ gives

$$(f^{-1}(y)x) \cdot (f^{-1}(y)z) \cdot (yf(x)) = f^{-1}(y)[(x \cdot f^{-1}(y)z) \cdot (yf(x))]. \tag{6}$$

Combining (5) and (6), we have

$$
\begin{aligned}
f^{-1}(y)[x \cdot f^{-1}(y)(z \cdot yf(x))] &\overset{(5)}{=} f^{-1}(y)[(x \cdot f^{-1}(y)z) \cdot (yf(x))] \\
&\overset{(6)}{=} (f^{-1}(y)x)[f^{-1}(y)z \cdot yf(x)].
\end{aligned}
$$

Replacing y with $f(y)$ yields

$$y \cdot x(y \cdot zf(yx)) = yx \cdot (yz \cdot f(yx)). \tag{7}$$

Finally, we have

$$
\begin{aligned}
(xy \cdot x)z &= (xy \cdot x) \cdot (zf(xy)^{-1})f(xy) \overset{(*)}{=} xy \cdot [(x \cdot zf(xy)^{-1})f(xy)] \\
&\overset{(7)}{=} x \cdot y(x \cdot (zf(xy)^{-1} \cdot f(xy))) \\
&= x(y \cdot xz).
\end{aligned}
$$

Therefore, N is Moufang. $\square$

Theorem 4. *Let N be a loop. Then N is Moufang if and only if there exists some $f \in \mathrm{Aut}(N)$ which satisfies $(*)$ for all $x, y, z \in N$.*

Proof: As before, if N is Moufang, then the trivial automorphism satisfies $(*)$ for all $x, y, z \in N$. Conversely, suppose there exists an $f \in \mathrm{Aut}(N)$ that satisfies $(*)$ for all $x, y, z \in N$. By Lemma 3, it is enough to show that N is an IP loop. As before, we have

$$x \cdot yf(x) = xy \cdot f(x), \tag{2}$$

$$f^{-1}(x) \cdot yx = f^{-1}(x)y \cdot x. \tag{3}$$

Moreover, $f(x) = (x \cdot x\backslash 1)f(x) = x \cdot (x\backslash 1)f(x)$, implying $x\backslash f(x) = x\backslash 1 \cdot f(x)$. Applying f^{-1} yields

$$f^{-1}(x)\backslash x = f^{-1}(x)\backslash 1 \cdot x.$$

We compute

$$\begin{aligned}\underbrace{(f^{-1}(x)\backslash x)}(x\backslash 1) &= [(f^{-1}(x)\backslash 1) \cdot x](x\backslash 1) = [(f^{-1}(x)\backslash 1) \cdot x] \cdot f(f^{-1}(x)\backslash 1)\\ &\overset{(2)}{=} (f^{-1}(x)\backslash 1)(x \cdot f(f^{-1}(x)\backslash 1)) = (f^{-1}(x)\backslash 1) \cdot x(x\backslash 1)\\ &= f^{-1}(x)\backslash 1.\end{aligned}$$

Finally, we compute

$$\begin{aligned}x &= f^{-1}(x)(f^{-1}(x)\backslash 1) \cdot x \overset{(3)}{=} f^{-1}(x)[\underbrace{(f^{-1}(x)\backslash 1)} \cdot x]\\ &= f^{-1}(x)[(f^{-1}(x)\backslash x)(x\backslash 1) \cdot x] = f^{-1}(x)[(f^{-1}(x)\backslash x)(x\backslash 1) \cdot f(f^{-1}(x))]\\ &\overset{(*)}{=} [f^{-1}(x)(f^{-1}(x)\backslash x)] \cdot [(x\backslash 1)(f(f^{-1}(x)))] = x \cdot (x\backslash 1)x.\end{aligned}$$

This immediately implies $1/x = x\backslash 1 = x^{-1}$.

To show that N is IP, observe that

$$xy \cdot f(x) = x \cdot yf(x) = x[x^{-1}(x^{-1}\backslash y) \cdot f(x)] \overset{(*)}{=} xx^{-1} \cdot (x^{-1}\backslash y)f(x) = (x^{-1}\backslash y)f(x),$$

which implies $xy = x^{-1}\backslash y$. Hence, $x^{-1} \cdot xy = y$.

To show that $yx \cdot x^{-1} = y$, note that $y^{-1} \cdot yx = x$ implies $x/(yx) = y^{-1}$. Also note that $(*)$ is equivalent to $[x \cdot (yz \cdot f(x))]/(zf(x)) = xy$. Using these two identities combined with $x^{-1} \cdot xy = y$, we have

$$y^{-1} = f((xy)^{-1})/[yf((xy)^{-1})] = [(xy)^{-1} \cdot (xy \cdot f((xy)^{-1}))]/[yf((xy)^{-1})] = (xy)^{-1}x.$$

Hence, $xy \cdot y^{-1} = (xy)[(xy)^{-1}x] = x$. □

Theorem 4 yields an equivalent definition for a loop to be Moufang in terms of the behavior of one of its automorphisms. In the same spirit, the next theorem gives a necessary and sufficient condition for a semidirect product of a loop and group to be Moufang.

Theorem 5. *Let N be a loop and H a group. Let $\phi : H \to \mathrm{Sym}(N)_1$ be a group homomorphism. Then $G = N \rtimes_\phi H$ is a Moufang loop if and only if $\phi(H) \subseteq \mathrm{Aut}(N)$ and $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$.*

Proof: Suppose $\phi(H) \subseteq \mathrm{Aut}(N)$ and $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$, and let $(x, g), (y, h), (z, k) \in G$. Then, by hypothesis,

$$[x\phi(g)(y)] \cdot [\phi(gh)(z)\phi(ghk)(x)] \overset{(*)}{=} x[\phi(g)(y)\phi(gh)(z) \cdot \phi(ghk)(x)].$$

Hence,

$$([x\phi(g)(y)] \cdot [\phi(gh)(z)\phi(ghk)(x)], ghkg) = (x[\phi(g)(y)\phi(gh)(z) \cdot \phi(ghk)(x)], ghkg),$$

and thus

$$[(x, g)(y, h)][(z, k)(x, g)] = (x, g) \cdot [(y, h)(z, k) \cdot (x, g)],$$

which shows that G is Moufang.

Conversely, if G is Moufang, then by Lemma 2 we have $\phi(H) \subseteq \mathrm{Aut}(N)$. Therefore

$$[(x, 1_H)(y, 1_H)][(z, h)(x, 1_H)] = (x, 1_H)[(y, 1_H)(z, h) \cdot (x, 1_H)],$$

which is equivalent to

$$(xy \cdot z\phi(h)(x), h) = (x(yz \cdot \phi(h)(x)), h).$$

Hence, $\phi(h)$ satisfies $(*)$ for all $h \in H$, for all $x, y, z \in N$. □

Note that if $N =$ `MoufangLoop(12,1)`, then it can be verified that there are no nontrivial automorphisms $f \in \mathrm{Aut}(N)$ which satisfy $(*)$ for all $x, y, z \in N$. Hence, for H a group and $\phi : H \to \mathrm{Aut}(N)$ a nontrivial group homomorphism, then $G = N \rtimes_\phi H$ is not Moufang, which shows that the converse of Lemma 2 is false. The next example illustrates that Theorem 5 can hold nontrivially.

Example 6. Let $N =$ `MoufangLoop(16,1)` and H a group of order 2. Let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism with $\phi(H) = \langle$`(2,6)(3,7)(10,14)(11,15)`$\rangle$. It can be verified that $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$. Then $G = N \rtimes_\phi H$ is a Moufang loop of order 32 (`MoufangLoop(32,18)`).

We end this section by showing that the class of IP loops is closed under semidirect products with groups acting as automorphisms.

Proposition 7. *Let N be a loop and H a group. Let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism. Then $G = N \rtimes_\phi H$ is an IP loop if and only if N is an IP loop.*

Proof: Clearly, if G is IP, then N is IP. Conversely, if N is IP, x^{-1} and g^{-1} exist for all $x \in N$ and $g \in G$. It is easy to verify that $(x,g)^{-1} = (\phi(g^{-1})(x^{-1}), g^{-1})$. To show G is IP, we compute

$$\begin{aligned}
(x,g)^{-1}[(x,g)(y,h)] &= (\phi(g^{-1})(x^{-1}), g^{-1})(x \cdot \phi(g)(y), gh) \\
&= (\phi(g^{-1})(x^{-1}) \cdot \phi(g^{-1})[x \cdot \phi(g)(y)], g^{-1}gh) \\
&= ((\phi(g^{-1})(x))^{-1}[\phi(g^{-1})(x) \cdot y], h) \\
&= (y,h). \\
[(y,h)(x,g)](x,g)^{-1} &= (y \cdot \phi(h)(x), hg)(\phi(g^{-1})(x^{-1}), g^{-1}) \\
&= ([y \cdot \phi(h)(x)]\phi(hg)(\phi(g^{-1})(x^{-1})), hgg^{-1}) \\
&= ([y \cdot \phi(h)(x)](\phi(h)(x))^{-1}, h) \\
&= (y,h).
\end{aligned}$$

□

3. Extensions and inverse property loops

We now turn our attention to extensions of loops as considered in [7].

Theorem 8 ([7]). *Suppose $G = NH$ is a Moufang loop, where N is a normal subloop of G and $H = \langle u \rangle$ is a finite cyclic subgroup of G whose order is coprime to 3. Then for any $xu^m, yu^n \in G$ where $x, y \in N$, we have*

$$(xu^m)(yu^n) = f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)) u^{m+n}, \tag{8}$$

where

$$f : N \to N \qquad x \to uxu^{-1}$$

is a semiautomorphism of N. Moreover, G is a group if and only if N is a group and f is an automorphism of N.

Note that if $H = \langle u \rangle \simeq \mathbb{Z}_2$, then the multiplication in G given by (8) is

$$xu \cdot y = xf(y) \cdot u, \qquad x \cdot yu = f(f(x)f(y)) \cdot u, \qquad xu \cdot yu = f(f(x)y),$$

for all $x, y \in N$. Hence, multiplication in such a loop G is easily seen to be equivalent to the multiplication given in [4, Lemma 1, p. 21]. These types of extensions have been well-studied, see [3], [4], [5].

Example 9. A loop Q is said to be a *semiautomorphic, inverse property loop* (or just semiautomorphic IP loop) if

(1) Q is flexible; that is, $(xy)x = x(yx)$ for all $x, y \in Q$;
(2) Q is an IP loop; and
(3) every inner mapping is a semiautomorphism.

It is known that Moufang loops are semiautomorphic IP loops and that semiautomorphic IP loops are diassociative loops [9]. Let N be a semiautomorphic IP loop and $H = \langle u \rangle$ have order 2, and let $\phi : H \to \mathrm{SemiAut}(N)$ be the group homomorphism such that $\phi(u)(x) = x^{-1}$ for all $x \in N$. Then $G = (N, H, \phi)$ with multiplication extended by (8) is a semiautomorphic IP loop (see [5]), but not necessarily Moufang, since it is shown in [4] that G is Moufang if and only if N is a group.

Here, we consider the *external* viewpoint. That is, the external extension of a cyclic group $H = \langle u \rangle$ by a loop N with $\phi : H \to \mathrm{SemiAut}(N)$ a group homomorphism. We extend the multiplication from N to $G = (N, H, \phi)$ as

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)), u^{m+n}), \tag{9}$$

where $f(x) = \phi(u)(x)$. As the previous example shows, if N is Moufang, G need not be Moufang. However, as the next theorem illustrates, the class of IP loops is closed under such extensions.

Theorem 10. *Let N be a loop, $H = \langle u \rangle$ a cyclic group, and let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism. If $|\phi(u)|$ is coprime to 3, define an extension $G = (N, H, \phi)$ with multiplication given by*

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)), u^{m+n}),$$

where $f(x) = \phi(u)(x)$. Then G is an IP loop if and only if N is an IP loop.

Proof: Clearly, N is an IP loop if G is an IP loop. Now, suppose N is an IP loop. Note that since the order of f in $\mathrm{SemiAut}(N)$ is coprime to 3, f must have a unique cubed root, denoted by $f^{\frac{1}{3}}$. It is clear that G is a loop with identity $(1, u^0)$. For $(x, u^m) \in G$, consider the element $(f^{-m}(x^{-1}), u^{-m})$. Note that this is well-defined, since N is an IP loop and H is a group. Then,

$$\begin{aligned}
(x, u^m)(f^{-m}(x^{-1}), u^{-m}) &= (f^{\frac{2m-m}{3}}(f^{\frac{-2m+m}{3}}(x) f^{\frac{m+m}{3}}(f^{-m}(x^{-1}))), u^0) \\
&= (f^{\frac{m}{3}}(f^{\frac{-m}{3}}(x) f^{\frac{-m}{3}}(x^{-1})), u^0) \\
&= (f^{\frac{m}{3}}(f^{\frac{-m}{3}}(x)(f^{\frac{-m}{3}}(x))^{-1}), u^0) \\
&= (f^{\frac{m}{3}}(1), u^0) \\
&= (1, u^0) \\
&= (f^{\frac{-m}{3}}(1), u^0) \\
&= (f^{\frac{-m}{3}}((f^{\frac{-2m}{3}}(x))^{-1} f^{\frac{-2m}{3}}(x)), u^0) \\
&= (f^{\frac{-m}{3}}(f^{\frac{-2m}{3}}(x^{-1}) f^{\frac{-2m}{3}}(x)), u^0) \\
&= (f^{\frac{-2m+m}{3}}(f^{\frac{2m-m}{3}}(f^{-m}(x^{-1}) f^{\frac{-m-m}{3}}(x))), u^0) \\
&= (f^{-m}(x^{-1}), u^{-m})(x, u^m).
\end{aligned}$$

Hence, G has two sided inverses for all $(x, u^m) \in G$, namely

$$(x, u^m)^{-1} = (f^{-m}(x^{-1}), u^{-m}).$$

To show that G is an IP loop, we compute

$$\begin{aligned}
&(x,u^m)[(f^{-m}(x^{-1}),u^{-m})(y,u^n)]\\
&\quad=(x,u^m)(f^{\frac{-2m+n}{3}}(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y)),u^{-m+n})\\
&\quad=(f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[f^{\frac{2m-n}{3}}(f^{\frac{-2m+n}{3}}(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y)))]),u^n)\\
&\quad=(f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y))]),u^n)\\
&\quad=(f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[f^{\frac{-m-n}{3}}(x^{-1})f^{\frac{-m-n}{3}}(y)]),u^n)\\
&\quad=(f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[(f^{\frac{-m-n}{3}}(x))^{-1}f^{\frac{-m-n}{3}}(y)]),u^n)\\
&\quad=(f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(y)),u^n)\\
&\quad=(y,u^n).
\end{aligned}$$

$$\begin{aligned}
&[(y,u^n)(f^{-m}(x^{-1}),u^{-m})](x,u^m)\\
&\quad=(f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))],u^{n-m})(x,u^m)\\
&\quad=(f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))])f^{\frac{n-2m}{3}}(x)],u^n)\\
&\quad=(f^{\frac{2n-m}{3}}(([f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))])f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad=(f^{\frac{2n-m}{3}}([f^{\frac{-2n+m}{3}}(y)f^{\frac{n-2m}{3}}(x^{-1})]f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad=(f^{\frac{2n-m}{3}}([f^{\frac{-2n+m}{3}}(y)(f^{\frac{n-2m}{3}}(x))^{-1}]f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad=(f^{\frac{2n-m}{3}}(f^{\frac{-2n+m}{3}}(y),u^n))\\
&\quad=(y,u^n).
\end{aligned}$$

□

The next two examples illustrate that if certain stronger conditions are imposed on N, then an extension as in Theorem 10 need not inherit a similar structure.

Example 11. Let N be the quaternion group of order 8 (GAP `SmallGroup(8,4)`) and H a group of order 2. Let $\phi : H \to \mathrm{SemiAut}(N)$ a group homomorphism with $\phi(H) = \langle$`(2,5)(6,8)`$\rangle$. Then the extension $G = (N, H, \phi)$ as in Theorem 10 is a power-associative (the subloop generated by x is associative for all $x \in G$), IP loop. It can be verified that G is neither flexible nor diassociative, although it is clear that N has both properties.

Steiner loops, which arise from Steiner triple systems in combinatorics, are loops satisfying the identities $xy = yx$, $x(yx) = y$. In particular, Steiner loops are semiautomorphic IP loops.

Example 12. Let N be a Steiner loop of order 8 (`SteinerLoop(8,1)`) and $H = \langle u\rangle$ a group of order 4. Let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism with $\phi(H) = \langle$`(2,4,8,3)`$\rangle$. Then the extension $G = (N, H, \phi)$ as in Theorem 10 is an

IP loop of order 32 that is neither power-associative nor flexible, although it is clear that N has both properties.

Corollary 13. *Let N be a Moufang loop, $H = \langle u \rangle$ a cyclic group, and let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism. If $|\phi(u)|$ is coprime to 3, define an extension $G = (N, H, \phi)$ with multiplication given by*

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)), u^{m+n}),$$

where $f(x) = \phi(u)(x)$. Then G is an IP loop.

Proof: Since Moufang loops are IP loops, the desired result follows from Theorem 10. $\square$

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