Amin Esfahani

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INSTABILITY OF THE STATIONARY SOLUTIONS OF GENERALIZED DISSIPATIVE BOUSSINESQ EQUATION

Amin Esfahani, Damghan

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Abstract. In this work we study the generalized Boussinesq equation with a dissipation term. We show that, under suitable conditions, a global solution for the initial value problem exists. In addition, we derive sufficient conditions for the blow-up of the solution to the problem. Furthermore, the instability of the stationary solutions of this equation is established.

Keywords: damped Boussinesq equation; stationary solution; instability

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1. Introduction

One of the equations describing the propagation of long waves on the surface of shallow water is the Boussinesq one which was first derived in 1872 by Boussinesq [6] (see also [5]). In his work he derived a nonlinear dissipative wave system which is now known as the Boussinesq equations. By using multiple scaling analysis, the Boussinesq equation, see for instance Boussinesq [4] and Craig [7], can be derived as the evolution equation

$$u_{tt} = u_{xx} - u_{xxxx} - (f(u))_{xx}, \tag{1.1}$$

where $u = u(x,t)$ is the vertical velocity component on the free surface of an irrotational fluid and $f(u) = u^2$. It also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice waves. Equation (1.1) in a cylindrical domain describes small nonlinear transverse

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oscillations of an elastic beam and is known in the literature as the "good" Boussinesq equation. A nice survey on the history of the derivation of model (1.1) can be found in Miles [16]. The Boussinesq equation (1.1) on the action of an internal strong damping, which means containing a structural damping u_{xxt}, models nonlinear beam oscillations in the presence of viscosity; thus (1.1) becomes

$$u_{tt} = u_{xx} + u_{xxt} - u_{xxxx} - (|u|u)_{xx}. \tag{1.2}$$

Considered in this paper is the generalized dissipative Boussinesq equation

$$u_{tt} = u_{xx} + u_{xxt} - u_{xxxx} - (|u|^{p-2}u)_{xx}, \tag{1.3}$$

where $u = u(x,t)$ is a complex-valued function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and $p > 2$ (see [3], [6], [20], [21], [23], [22]).

In [3], [6] an abstract Cauchy problem for the generalization of (1.3) has been studied. In [19], [24] equation (1.3) has been considered from the point of view of the theory of global attractors and inertial manifolds.

Under various assumptions of initial/boundary data in [20], [21], [23], [22], Varlamov constructed the classical solution of the problem and obtained the long-time asymptotics in explicit form. Using the eigenfunction expansion method, he also studied the long-time asymptotics of a damped Boussinesq equation which is similar to (1.3).

In the present paper, we aim at giving sharp criteria for the global existence of solution for the Cauchy problem associated to (1.3). In order to do this, we employ the variational methods and the existence of the invariant (stable and unstable) sets of solutions [10], [15], [13], [12], [14], [11].

The main difficulty is to prove the blowing up properties of the associated Cauchy problem. We borrow the method presented in [17] to overcome this difficulty. Also, we show the instability of the ground state for (1.3). This property is the same as in the case of the classical generalized Boussinesq equation, which can help us to comprehend the effect of the damping term to the Boussinesq system.

Throughout this paper we denote by $\widehat{\varphi}$ the Fourier transform of φ, defined as

$$\widehat{\varphi}(\zeta) = \int_{\mathbb{R}} \varphi(\omega) \mathrm{e}^{-\mathrm{i}\omega\zeta} \, \mathrm{d}\omega.$$

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \{\varphi \in \mathscr{S}'(\mathbb{R}) \colon \ \|\varphi\|_{H^s(\mathbb{R})} < \infty\},$$

where

$$\|\varphi\|_{H^s(\mathbb{R})} = \|(1+\zeta^2)^{s/2}\widehat{\varphi}(\zeta)\|_{L^2(\mathbb{R})},$$

and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions.

2. Main result

Note that the Cauchy problem (1.3) with initial data

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = (v_0)_x(x) \tag{2.1}$$

is equivalent to the system

$$\begin{cases} u_t = v_x, \\ v_t = (u + v_x - u_{xx} - |u|^{p-2}u)_x, \end{cases} \tag{2.2}$$

with initial data

$$u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x). \tag{2.3}$$

A local well-posedness result of the initial value problem (1.3) and (2.1), or equivalently problem (2.2)–(2.3), reads as follows.

Theorem 2.1. *Let $p > 2$. Then for initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ there exist $T > 0$ and a unique solution $(u, v) \in C([0,T);\ H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ of (2.2) such that $(u(0), v(0)) = (u_0, v_0)$. In addition, if we assume $\xi^{-1}\widehat{u_0} \in L^2(\mathbb{R})$, then $\xi^{-1}\hat{u} \in C^1([0,T);\ L^2(\mathbb{R}))$. Moreover, $T = \infty$, or $T < \infty$ and*

$$\lim_{t \to T^-} \|(u(t), v(t))\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} = \infty.$$

Furthermore, if we define the energy

$$E(t) = \frac{1}{2}\big(\|u(t)\|^2_{H^1(\mathbb{R})} + \|v(t)\|^2_{L^2(\mathbb{R})}\big) - \frac{1}{p}\|u(t)\|^p_{L^p(\mathbb{R})}, \quad t \in [0,T),$$

then

$$E(t) + \int_0^t \|u_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s = E(0), \quad t \in [0,T).$$

The local well-posedness of the initial value problem (1.3) and (2.1), and also the proof of Theorem 2.1, can be obtained from the results of Varlamov in [20], [21],

[23], [22], or by using the ideas of Liu in [9]. Indeed, it is straightforward to see that the initial value problem (1.3) and (2.1) is equivalent to the integral form

$$u(t) = U_1(t)u_0 + U_2(t)(v_0)_x + \int_0^t U_2(t-\tau)F(u(\tau))\,\mathrm{d}\tau,$$

where $F(u) = (|u|^{p-2}u)_{xx}$ and $U_1(t)$ and $U_2(t)$ are the C_0-semigroups

$$U_1(t)u_0 = \left(\left[\cos\left(\sqrt{4\xi^2+3\xi^4}t/2\right) + \frac{\xi^2\sin\left(\sqrt{4\xi^2+3\xi^4}t/2\right)}{\sqrt{4\xi^2+3\xi^4}}\right]\exp(-\xi^2t/2)\widehat{u_0}(\xi)\right)^{\vee}$$

and

$$U_2(t)(v_0)_x = \left(\frac{2\sin\left(\sqrt{4\xi^2+3\xi^4}t/2\right)}{\sqrt{4\xi^2+3\xi^4}}\exp(-\xi^2t/2)\widehat{(v_0)_x}(\xi)\right)^{\vee}.$$

Similarly, an integral form of the initial problem (2.2)–(2.3) is

$$\vec{u}(t) = U(t)\vec{u}_0 + \int_0^t U(t-\tau)\vec{f}(\vec{u}(\tau))\,\mathrm{d}\tau,$$

where $\vec{u} = \binom{u}{v}$, $\vec{u}_0 = \binom{u_0}{v_0}$, $\vec{f}(\vec{u}) = \binom{0}{F(u)}$ and $U(t)$ is the C_0-semigroup generated by the operator $\begin{pmatrix} 0 & \partial_x \\ \partial_x & -\partial_x^3\ \partial_x \end{pmatrix}$ in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. Now the proof follows directly from the above integral forms, the semigroup theory [18], the Sobolev embedding and the fixed-point argument.

It is obvious that if $\varphi(x)$ satisfies the semilinear elliptic equation

$$-\varphi'' + \varphi = |\varphi|^{p-2}\varphi, \quad \varphi\in H^1(\mathbb{R})\setminus\{0\}, \tag{2.4}$$

then $u(x,t) = \varphi(x)$, $t\geqslant 0$, $x\in\mathbb{R}$ satisfies (1.3), which is the stationary solution of (1.3).

When $p>2$, existence of a unique ground state solution, i.e. a solution of (2.4) with the minimal action, is well known. It follows, for example, from the results of Berestycki and Lions [1], [2].

Furthermore, for $\varphi\in H^1(\mathbb{R})$, we define the functionals

$$S(\varphi) = \frac{1}{2}\|\varphi\|^2_{H^1(\mathbb{R})} - \frac{1}{p}\|\varphi\|^p_{L^p(\mathbb{R})}, \tag{2.5}$$

$$R(\varphi) = \|\varphi\|^2_{H^1(\mathbb{R})} - \|\varphi\|^p_{L^p(\mathbb{R})}, \tag{2.6}$$

and define the manifold

$$\mathscr{M} = \{\psi\in H^1(\mathbb{R})\setminus\{0\};\ R(\psi)=0\}. \tag{2.7}$$

We consider the constrained variational problem

$$d_{\mathscr{M}} = \inf\{S(\varphi);\ \varphi \in \mathscr{M}\}. \tag{2.8}$$

From [9], we have

Lemma 2.1. *There exists $\varphi \in \mathscr{M}$ such that $S(\varphi) = d_{\mathscr{M}}$, and φ is a ground state solution of* (2.4). *Moreover, if φ is a ground state of* (2.4), *then*

$$S(\varphi) = \min\{S(\varphi);\ \varphi \in \mathscr{M}\}.$$

Proposition 2.1. *The functional S is bounded below on $\mathscr{M}$ and $d_{\mathscr{M}} > 0$.*

Proof. The proof follows easily from the definition of S and R. Indeed, for $\varphi \in \mathscr{M}$ we have

$$S(\varphi) = \frac{p-2}{p}\|\varphi\|^2_{H^1(\mathbb{R})}.$$

It follows that $S(\varphi) > 0$ on $\mathscr{M}$. So S is bounded below on $\mathscr{M}$. From (2.7) we have $d_{\mathscr{M}} > 0$. □

The main result of this paper is the following theorem

Theorem 2.2. *Let φ be a ground state solution of* (2.4). *Then for any $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R})$ such that $\|u_0 - \varphi\|_{H^1(\mathbb{R})} < \varepsilon$. Moreover, the solution $u(x,t)$ of* (2.2) *corresponding to the initial data $u(x,0) = u_0$ and $v(x,0) = 0$ is defined for $0 < T < \infty$, such that $(u,v) \in C([0,T);\ H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ and*

$$\lim_{t\to T} \|(u(t), v(t))\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} = \infty. \tag{2.2}$$

3. Global existence and blow-up

In this section, we define some invariant sets under the flow of the Cauchy problem associated to (1.3). Under suitable conditions, we show that a global solution for the initial value problem exists. Furthermore, we derive sufficient conditions for the blow-up of the solution to the problem.

Let us define the following sets:

$$\mathscr{K}_1 = \{\varphi \in H^1(\mathbb{R});\ R(\varphi) < 0,\ S(\varphi) < d_{\mathscr{M}}\} \tag{3.1}$$

and

$$\mathscr{K}_2 = \{\varphi \in H^1(\mathbb{R});\ R(\varphi) > 0,\ S(\varphi) < d_{\mathscr{M}}\} \cup \{0\}. \tag{3.2}$$

Lemma 3.1. *Let $E(0) < d_{\mathscr{M}}$, then $\mathscr{K}_1$ and $\mathscr{K}_2$ are invariant under the flow generated by the Cauchy problem associated with* (1.3).

P r o o f. Suppose that $u_0 \in \mathscr{K}_1$ and $u(t)$ is the solution of problem (2.2) with $u(0) = u_0$ and $v(0) = v_0$. From the definition of E and S we have

$$S(u(t)) \leqslant E(t) = E(0) - \int_0^t \|u_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s < d_{\mathscr{M}}, \quad t \in [0,T). \tag{3.3}$$

To see that $u(t) \in \mathscr{K}_1$, we need to prove that

$$R(u(t)) < 0, \quad t \in [0,T). \tag{3.4}$$

If (3.4) were not true, by continuity there would exist a $\tau > 0$ such that $R(u(\tau)) = 0$, because $R(u_0) < 0$. It follows that $u(\tau) \in \mathscr{M}$. This is impossible for $S(u(\tau)) < d_{\mathscr{M}} = S(\varphi)$ and $S(\varphi) = \min\limits_{\varphi \in \mathscr{M}} S(\varphi)$. Thus (3.4) is true. So $\mathscr{K}_1$ is invariant under the flow generated by the Cauchy problem associated with (1.3).

Similarly, we can show that $\mathscr{K}_2$ is also invariant under the flow generated by the Cauchy problem associated with (1.3). This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $\varphi_\lambda(x) = \lambda\varphi(x)$ for $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ and $\lambda > 0$. Then there exists a unique $\mu > 0$, depending on φ, such that $R(\varphi_\mu) = 0$. Moreover, $R(\varphi_\lambda) > 0$ for $\lambda \in (0,\mu)$, $R(\varphi_\lambda) < 0$ for $\lambda \in (\mu,\infty)$, and $S(\varphi_\mu) \geqslant S(\varphi_\lambda)$ for all $\lambda > 0$.*

P r o o f. By using (2.5) and (2.6), we arrive at

$$S(\varphi_\lambda) = \frac{\lambda^2}{2}\|\varphi\|^2_{H^1(\mathbb{R})} - \frac{\lambda^p}{p}\|\varphi\|^p_{L^p(\mathbb{R})}$$

and

$$R(\varphi_\lambda) = \lambda^2\|\varphi\|^2_{H^1(\mathbb{R})} - \lambda^p\|\varphi\|^p_{L^p(\mathbb{R})}.$$

From the definition of $\mathscr{M}$, there exists a unique $\mu > 0$ such that $R(\varphi_\mu) = 0$ and $R(\varphi_\lambda) > 0$ for $\lambda \in (0,\mu)$, $R(\varphi_\lambda) < 0$ for $\lambda \in (\mu,\infty)$.

On the other hand, $(\mathrm{d}/\mathrm{d}\lambda)S(\varphi_\lambda) = \lambda^{-1}R(\varphi_\lambda)$ and $R(\varphi_\mu) = 0$ reveal that $S(\varphi_\mu) \geqslant S(\varphi_\lambda)$ for all $\lambda > 0$. □

Lemma 3.3. *Let the initial data fulfil* $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$*, and let* $(u(t), v(t))$ *be a local solution of* (2.2) *corresponding to the initial data* (u_0, v_0) *on* $[0, T)$*. If* $u_0 \in \mathscr{K}_1$ *and* $E(0) < d_{\mathscr{M}}$*, then*

$$d_{\mathscr{M}} < \frac{p-2}{2p}\|u(t)\|^2_{H^1(\mathbb{R})}$$

for any $t \in [0, T)$.

Proof. First we note that since $u_0 \in \mathscr{K}_1$, by Lemma 3.1 we have $R(u(t)) < 0$ and $S(u(t)) < d_{\mathscr{M}}$ for $t \in [0, T)$. By continuity of R, we have that there exists $\eta_* \in (0, 1)$ such that $R(\eta_* u(t)) = 0$. Therefore, it follows for $t \in [0, T)$ that

$$\begin{aligned} d_{\mathscr{M}} &\leqslant S(\eta_* u(t)) = \frac{\eta_*^2}{2}\|u(t)\|^2_{H^1(\mathbb{R})} - \frac{\eta_*^p}{p}\|u(t)\|^p_{L^p(\mathbb{R})} \\ &< \frac{p-2}{2p}\eta_*^2\|u(t)\|^2_{H^1(\mathbb{R})} < \frac{p-2}{2p}\|u(t)\|^2_{H^1(\mathbb{R})}. \end{aligned}$$

This completes the proof. □

The following theorem gives sufficient conditions for the existence of a global solution. It also establishes the local solution of (1.3) will blow up if the initial data belongs to $\mathscr{K}_1$.

Theorem 3.1. *Let* $p > 2$, $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *and* $E(0) < S(\varphi)$.

(i) *If there exists* $t_0 \in [0, T)$ *such that* $u(t_0) \in \mathscr{K}_1$*, then the solution* $(u(t), v(t))$ *of* (2.2) *with initial data* (u_0, v_0) *blows up in a finite time.*

(ii) *If there exists* $t_0 \in [0, T)$ *such that* $u(t_0) \in \mathscr{K}_2$*, then the solution of* (2.2) *with initial data* (u_0, v_0) *exists globally on* $[0, \infty)$*. Moreover, for* $t \in [0, \infty)$, $u(t)$ *satisfies*

$$\|v(t)\|^2_{L^2(\mathbb{R})} + \frac{p-2}{p}\|u(t)\|^2_{H^1(\mathbb{R})} < \frac{p-2}{p}\|\varphi\|^2_{H^1(\mathbb{R})}. \tag{3.5}$$

Proof. First we note that $S(u_0) \leqslant E(0) < S(\varphi)$, by $E(0) < S(\varphi) = d_{\mathscr{M}}$. Now we prove (i).

If $u_0 \in \mathscr{K}_1$, then Lemma 3.1 implies that $R(u(t)) < 0$ for $t \in [0, T)$. Since $u(t)$ is a solution of (1.3) on $[0, T)$, denoting

$$I(t) = \|\xi^{-1}\hat{u}(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + (T-t)\|u_0\|^2_{L^2(\mathbb{R})},$$

we obtain that

$$I'(t) = 2\mathrm{Re}\,\langle \xi^{-1}\hat{u},\ \hat{v}\rangle_{L^2(\mathbb{R})} + 2\int_0^t\!\!\int_{\mathbb{R}} uu_s\,\mathrm{d}x\,\mathrm{d}s$$

and

$$I''(t) = 2\|v(t)\|^2_{L^2(\mathbb{R})} - 2R(u(t)) = (p+2)\|v(t)\|^2_{L^2(\mathbb{R})} + (p-2)\|u(t)\|^2_{H^1(\mathbb{R})} - 2pE(t).$$

Since $R(u(t)) < 0$, we have $I''(t) > 0$ for all $t \in [0,T)$. On the other hand, a straightforward calculation reveals that

$$\begin{aligned}(3.6)\qquad I(t)I''(t) - \frac{p+2}{4}(I'(t))^2 &= I(t)I''(t)\\ &+ (p+2)\Big[\mathscr{N}(t) - (I(t) - (T-t)\|u_0\|^2_{L^2(\mathbb{R})})\\ &\times\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)\Big],\end{aligned}$$

where

$$\begin{aligned}\mathscr{N}(t) = &\Big(\|\xi^{-1}\hat{u}\|^2_{L^2(\mathbb{R})} + \int_0^t \|u(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)\\ &- \Big(\mathrm{Re}\,\langle \xi^{-1}\hat{u},\ \hat{v}\rangle_{L^2(\mathbb{R})} + \int_0^t\int_{\mathbb{R}} uu_s\,\mathrm{d}x\,\mathrm{d}s\Big)^2.\end{aligned}$$

The Cauchy-Schwarz inequality implies that

$$\int_{\mathbb{R}} \xi^{-1}\hat{u}\xi^{-1}\hat{v}\,\mathrm{d}x \leqslant \|v(t)\|_{L^2(\mathbb{R})}\|\xi^{-1}\hat{u}\|_{L^2(\mathbb{R})},$$

$$\int_0^t\!\!\int_{\mathbb{R}} uu_s\,\mathrm{d}x\,\mathrm{d}s \leqslant \Big(\int_0^t \|u(s)\|^2_{L^2(\mathbb{R})}\Big)^{1/2}\Big(\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\Big)^{1/2},$$

and

$$\begin{aligned}&\int_{\mathbb{R}} \xi^{-1}\hat{u}\hat{v}\,\mathrm{d}x \int_0^t\!\!\int_{\mathbb{R}} uu_s\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\leqslant \|\xi^{-1}\hat{u}\|_{L^2(\mathbb{R})}\Big(\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)^{1/2}\|v(t)\|_{L^2(\mathbb{R})}\Big(\int_0^t \|u(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)^{1/2}.\end{aligned}$$

From the above three inequalities, we obtain that $\mathscr{N}(t) \geqslant 0$ for all $t \in [0,T)$. Hence, (3.6) gives

$$\begin{aligned}(3.7)\quad I(t)I''(t) - \frac{p+2}{4}(I'(t))^2 &\geqslant I(t)I''(t) - (p+2)(I(t) - (T-t)\|u_0\|^2_{L^2(\mathbb{R})})\\ &\quad\times\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)\\ &\geqslant I(t)\mathscr{Q}(t),\end{aligned}$$

where

$$\begin{aligned}\mathscr{Q}(t) &= I''(t) - (p+2)\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\Big)\\ &= -2pE(t) + (p+2)\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + (p-2)\|u(t)\|^2_{H^1(\mathbb{R})}\\ &= -2pE(0) + (p-2)\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + (p-2)\|u(t)\|^2_{H^1(\mathbb{R})}.\end{aligned}$$

Now since $E(0) < d_{\mathscr{M}}$, Lemma 3.3 leads to the formula

$$\begin{aligned}\mathscr{Q}(t) &> 2p(d_{\mathscr{M}} - E(0)) + (p-2)\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\ &\geqslant (p-2)\int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s > 0\end{aligned} \tag{3.8}$$

for $t \in [0, T)$. Combining (3.7) and (3.8), one has

$$I(t)I''(t) - \frac{p+2}{4}(I'(t))^2 \geqslant 0. \tag{3.9}$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{-(p-2)/4}(t) = -\frac{p-2}{4} I^{-(p+6)/4}\Big(I(t)I''(t) - \frac{p+2}{4}(I'(t))^2\Big),$$

from (3.9) one obtains that $(\mathrm{d}^2/\mathrm{d}t^2)I^{-(p-2)/4}(t) \leqslant 0$. Therefore $I^{-(p-2)/4}(t)$ is concave for sufficiently large t, and there exists a finite time T^* such that

$$\lim_{t\to T^*} I^{-(p-2)/4}(t) = 0, \quad \lim_{t\to T^*} I(t) = \infty.$$

Thus one has $T < \infty$ and $\lim_{t\to T} \|(u(t), v(t))\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} = \infty$. The proof of (i) will be complete once we have shown that $I'(t) > 0$ for some t_1. We will argue by contradiction and suppose $I'(t) \leqslant 0$ for all $t \geqslant 0$. Since $I(t) > 0$ and $I(t)$ is convex, $I(t)$ must tend to a finite nonnegative limit ϱ as $t \to \infty$. By Lemma 3.1, we assert that $\varrho > 0$. Therefore one has, as $t \to \infty$, $I(t) \to \varrho$, $I'(t) \to 0$ and $I''(t) \to 0$. From (3.6) one obtains that

$$\lim_{t\to\infty}\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_s(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + (T-t)\|v_0\|^2_{L^2(\mathbb{R})}\Big) = 0. \tag{3.10}$$

This means that $R(u(t)) \to 0$ as $t \to \infty$. Now for any fixed $t > 0$, because $R(u(t)) < 0$, there exists $0 < l < 1$ such that $R(lu) = 0$. Furthermore, one can check that

$$\begin{aligned}S(u) - S(lu) &= \frac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R})} - \frac{1}{p}\|u(t)\|^p_{L^p(\mathbb{R})} - \frac{l^2}{2}\|u(t)\|^2_{H^1(\mathbb{R})}\\ &\quad + \frac{l^p}{p}\|u(t)\|^p_{L^p(\mathbb{R})} \geqslant \frac{1}{2}R(u).\end{aligned} \tag{3.11}$$

By (3.10) and (3.11) we may conclude that

$$S(u) \geqslant S(lu) \geqslant S(\varphi) \tag{3.12}$$

as $t \to \infty$. From Lemma 3.1 we conclude that (3.12) is false, i.e. $I'(t) \leqslant 0$ is not true. So there exists some $t_1 > 0$ such that $I'(t_1) > 0$. Thus we completed the proof of (i).

To prove (ii), we note that since $u_0 \in \mathscr{K}_2$ and $E(0) < d_{\mathscr{M}}$, by Lemma 3.1 we obtain $R(u(t)) > 0$ for all $t \in [0, T)$. Hence, we get

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathbb{R})} + \frac{p-2}{2p}\|u(t)\|^2_{H^1(\mathbb{R})} \leqslant E(0) \tag{3.13}$$

for $t \in [0, T)$. Thus we establish the boundedness of $v(t)$ in $L^2(\mathbb{R})$ and the boundedness of $u(t)$ in $H^1(\mathbb{R})$ for $t \in [0, T)$. Hence we must have $T = \infty$. Then the solution $u(t)$ of (1.3) exists globally on $t \in [0, \infty)$. Furthermore, it follows from (3.13) that $E(0) < S(\varphi)$ and (3.5) hold. This completes the proof of Theorem 3.1. □

Lemma 3.4. *Let $p > 2$ and $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Moreover, suppose that the following inequality holds:*

$$\|(u_0, v_0)\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant \frac{1}{p}\|u_0\|^p_{L^p(\mathbb{R})}. \tag{3.14}$$

Then the solution $(u(t), v(t))$ of (2.2) with initial data (u_0, v_0) blows up in a finite time.

Proof. From (3.14) we have $E(0) < 0$. Since $p > 2$, it follows that

$$E(0) < 0 < \frac{p-2}{2p}\|\varphi\|^2_{H^1(\mathbb{R})}, \tag{3.15}$$

and

$$\|(u_0, v_0)\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant \frac{2}{p}\|u_0\|^p_{L^p(\mathbb{R})}. \tag{3.16}$$

Consequently, we obtain that

$$\|u_0\|^2_{H^1(\mathbb{R})} - \|u_0\|^p_{L^p(\mathbb{R})} \leqslant \frac{2-p}{p}\|u_0\|^p_{L^p(\mathbb{R})} < 0. \tag{3.17}$$

Finally, (3.15) and (3.17) complete the proof. □

Theorem 3.2. *Let $p > 2$ and let (u_0, v_0) be in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and satisfy*

$$\|(u_0, v_0)\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \frac{p-2}{p}\|\varphi\|^2_{H^1(\mathbb{R})}. \tag{3.18}$$

Then the solution $(u(t), v(t))$ of (2.2) with initial data $(u(0), v(0)) = (u_0, v_0)$ globally exists. Moreover, for $t \in [0, \infty)$ one has

$$\|(u(t), v(t))\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \frac{p-2}{p}\|\varphi\|^2_{H^1(\mathbb{R})}. \tag{3.19}$$

P r o o f. From (3.18) one has

$$E(0) < \frac{p-2}{2p}\|\varphi\|^2_{H^1(\mathbb{R})}.$$

Now we show that u_0 satisfies $R(u_0) > 0$. We get this fact by contradiction; so suppose that

$$\|u_0\|^2_{H^1(\mathbb{R})} \leqslant \|u_0\|^p_{L^p(\mathbb{R})}. \tag{3.20}$$

Thus there exists $l \in (0, 1]$ such that

$$l^2\|u_0\|^2_{H^1(\mathbb{R})} = l^p\|u_0\|^p_{L^p(\mathbb{R})},$$

from which we have $R(lu_0) = 0$ and $lu_0 \in \mathscr{M}$. On the other hand, for $l \in (0, 1]$, lu_0 still satisfies (3.18). It follows that

$$S(lu_0) \leqslant \frac{(p-2)}{2p}l^2\|\varphi\|^2_{H^1(\mathbb{R})} = S(\varphi),$$

which contradicts Lemma 2.1. Therefore, $R(u_0) > 0$ is true. From the proof of (ii) of Theorem 3.1, we get the conclusion of Theorem 3.2. □

The following lemma is essential in the proof of the instability result. For the sake of completeness, we give the proof (see [8, Lemma 4.4]).

Lemma 3.5. *The set* $\mathscr{A} = \{w \in H^1(\mathbb{R});\ \xi^{-1}\widehat{w}(\xi) \in L^2(\mathbb{R})\}$ *is dense in* $H^1(\mathbb{R})$.

Proof. For any $u \in H^1(\mathbb{R})$ and $\varepsilon > 0$, define, for $\delta > 0$, the function w_δ such that

$$\widehat{w_\delta}(\xi) = \begin{cases} \hat{u}(\xi), & |\xi| > \delta, \\ 0, & |\xi| \leqslant \delta. \end{cases}$$

Then we have

$$\|\xi^{-1}\widehat{w_\delta}\|_{L^2(\mathbb{R})}^2 = \int_{|\xi|>\delta} \xi^{-2}|\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \delta^{-2}\|u\|_{L^2(\mathbb{R})}^2 < \infty,$$

and

$$\|w_\delta\|_{H^1(\mathbb{R})} = \|(1+\xi^2)^{1/2}\widehat{w_\delta}\|_{L^2(\mathbb{R})} \leqslant \|(1+\xi^2)^{1/2}\hat{u}\|_{L^2(\mathbb{R})} = \|u\|_{H^1(\mathbb{R})} < \infty.$$

This implies that $w_\delta \in \mathscr{A}$. On the other hand, we have

$$\begin{aligned} \|w_\delta - u\|_{H^1(\mathbb{R})}^2 &= \|(1+\xi^2)^{1/2}(\widehat{w_\delta} - \hat{u})\|_{L^2(\mathbb{R})}^2 \\ &= \int_{|\xi|\leqslant\delta} (1+\xi^2)|\hat{u}(\xi)|^2 \,\mathrm{d}\xi \leqslant \|u\|_{H^1(\mathbb{R})}^2 < \infty. \end{aligned} \tag{3.21}$$

Therefore, we choose δ to be sufficiently small so that

$$\int_{|\xi|\leqslant\delta} (1+\xi^2)|\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \varepsilon,$$

and the proof of Lemma 3.5 is complete. □

A proof of Theorem 2.2 is now in sight.

Proof of Theorem 2.2. For any $\varepsilon > 0$, let $\varepsilon_0 \in (0, \min\{\varepsilon/2, \|\varphi\|_{H^1(\mathbb{R})}\})$ and $\varepsilon_1 < \varepsilon/4\|\varphi\|_{H^1(\mathbb{R})}$. By Lemma 3.5, there exists $w_0 \in \mathscr{A}$ such that $\|w_0 - \varphi\|_{H^1(\mathbb{R})} < \varepsilon_0$.

Let $u_0(x) = \lambda w_0$ with $\lambda = (1+\varepsilon_1) > 1$, so that $u_0 \in \mathscr{A}$. Then by using the proof of Lemma 3.5 and (3.21), we deduce that

$$\|u_0 - \varphi\|_{H^1(\mathbb{R})} \leqslant (\lambda - 1)\|w_0\|_{H^1(\mathbb{R})} + \varepsilon_0 < 2\|\varphi\|_{H^1(\mathbb{R})}\varepsilon_1 + \varepsilon_0 < \varepsilon. \tag{3.22}$$

Lemma 3.2 yields that

$$E(0) = S(u_0), \quad R(u_0) < R(\varphi) = 0, \quad \text{and} \quad S(u_0) < S(\varphi). \tag{3.23}$$

It follows from (3.23) that

$$E(0) < S(\varphi) = \frac{p-2}{2p}\|\varphi\|_{H^1(\mathbb{R})}^2. \tag{3.24}$$

Therefore, from (3.23) and (3.24) and (i) of Theorem 3.1, we get the result of Theorem 2.2. □

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Author's address: *Amin Esfahani*, School of Mathematics and Computer Science, Damghan University, Damghan, P. O. Box 36715-364, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box: 19395-5746, Tehran, Iran, e-mails: amin@impa.br, esfahani@du.ac.ir.