Mehdi Ghasemi; Mojtaba Fardi; Reza Khoshsiar Ghaziani

A new application of the homotopy analysis method in solving the fractional Volterra's population system

Applications of Mathematics, Vol. 59 (2014), No. 3, 319–330

Persistent URL: http://dml.cz/dmlcz/143775

A NEW APPLICATION OF THE HOMOTOPY ANALYSIS METHOD IN SOLVING THE FRACTIONAL VOLTERRA'S POPULATION SYSTEM

Mehdi Ghasemi, Shahrekord, Mojtaba Fardi[1], Najafabad,
Reza Khoshsiar Ghaziani, Shahrekord

(Received October 13, 2011)

Abstract. This paper considers a Volterra's population system of fractional order and describes a bi-parametric homotopy analysis method for solving this system. The homotopy method offers a possibility to increase the convergence region of the series solution. Two examples are presented to illustrate the convergence and accuracy of the method to the solution. Further, we define the averaged residual error to show that the obtained results have reasonable accuracy.

Keywords: Volterra's population system of fractional order; Caputo's fractional derivative; bi-parametric homotopy method; convergence region

MSC 2010: 26A33

1. Introduction

Recently, the fractional derivative has drawn much attention due to its wide application in engineering: for instance, the nonlinear oscillation of an earthquake can be modeled with fractional derivatives [5], and the fluid-dynamic traffic model with fractional derivatives [6] can eliminate the deficiency arising from the assumption of continuum traffic flow. A review of some applications of fractional derivatives in continuum and statistical mechanics is given in [3], [5], [6].

This paper outlines reliable numerical strategies for solving the fractional population growth model of a species within a closed system. The model which we con-

[1] Corresponding author
This research was partially supported by the Center of Excellence for Mathematics, University of Shahrekord, Iran.

sider, first introduced in [11], is defined by the nonlinear fractional Volterra integro-differential equation

$$D^{\alpha}p(t) = ap(t) - bp^2(t) - cp(t)\int_0^t p(x)\,\mathrm{d}x, \quad p(0) = p_0,\ 0 < \alpha \leqslant 1, \tag{1.1}$$

where $p = p(t)$ is the population of identical individuals at time t which exhibits crowding and sensitivity to the amount of toxins produced, see [12], α is a parameter describing the order of the time-fractional derivative, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run. If $c = 0$ the system reduces to the well-known logistic equation [12], [11]. The last term of equation (1.1) contains the integral that indicates the "total metabolism" or total amount of toxins produced since time zero. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor u. Since the system is closed, the presence of the toxic term always causes the population level to fall to zero in the long run, which will be seen later. The relative size of the sensitivity to toxins c determines the manner in which the population evolves before its extinction. The time-fractional derivative is considered in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative which can be varied to obtain various responses.

We employ the time and population scales by introducing the non-dimensional variables

$$t = \frac{ct}{b}, \quad u = \frac{bp}{a}$$

which produce the non-dimensional problem

$$\kappa D^{\alpha}u(t) = u(t) - u^2(t) - u(t)\int_0^t u(x)\,\mathrm{d}x, \quad u(0) = \beta,\ 0 < \alpha \leqslant 1, \tag{1.2}$$

where $\kappa = c/ab$ is a prescribed non-dimensional parameter. It is evident that the only equilibrium solution to equation (1.2) is the trivial solution $u(t) = 0$.

In the case of $\alpha = 1$, the fractional equation reduces to a classical logistic growth model. Several analytical and numerical methods have been proposed to solve the classical population growth model (1.1) when $\alpha = 1$, see [14], [1], [13]. In [11], the successive approximations were suggested to handle the population system (1.1), but the method was not implemented. In [12], singular perturbation methods were used to find a closed form approximations to the solutions of equation (1.1). In [12] it has been indicated that if κ is large, where the populations are strongly sensitive to toxins, the solution is proportional to $\operatorname{sech}^2(t)$. In this case the solution $u(t)$ has a smaller amplitude. For $k \ll 1$, where populations are weakly sensitive to toxins, it was shown by [12] that a rapid rise occurs along the logistic curve that will reach a peak and then follow a slow exponential decay. The results were obtained by considering a boundary layer near time $t = 0$ where the population grows rapidly inside the boundary layer. However, because of the toxin accumulation, the population decays steadily [4], [7] to zero outside the boundary layer.

This paper is organized as follows. In Section 2, we mention some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3, we extend the application of the homotopy analysis method (HAM) to construct our numerical solutions for fractional population growth model. Applications and numerical results are given in Section 4.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory [8], which are used in the following sections.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number p $(> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)}(t) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator J^α of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau)\,\mathrm{d}\tau \quad (\alpha, t > 0), \\ J^0 f(t) &= f(t), \end{aligned} \tag{2.1}$$

where Γ is the well-known Gamma function.

The properties of the operator J^α can be found in [8] and we mention only the following cases:

(1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
(2) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
(3) $J^\alpha t^\gamma = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)) t^{\alpha+\gamma}$,

where $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena by fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α which is proposed by Caputo [2].

Definition 3. The fractional derivative D^α of $f(t) \in C^n_{-1}$ in Caputo's sense is defined as

$$D^\alpha f(t) = 1/\Gamma(n-\alpha) \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau \quad (n-1 < \alpha \leqslant n,\ t > 0). \tag{2.2}$$

Now we give two basic properties of Caputo's fractional derivative:

(1) Let $f \in C^n_{-1}$. Then $D^\alpha f$, $0 \leqslant \alpha \leqslant n$, is well defined and $D^\alpha f \in C_{-1}$.
(2) Let $n-1 < \alpha \leqslant n$, and $f \in C^n_\mu$, $\mu \geqslant -1$.

Then

$$\begin{aligned} J^\alpha D^\alpha f(t) &= f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \\ D^\alpha J^\alpha f(t) &= f(t). \end{aligned} \tag{2.3}$$

In this study, we consider the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition is that to solve differential equations (both classical and fractional), we need to specify same additional conditions in order to produce a unique solution [3]. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $t = 0$, which are functions of t. These initial conditions are not physical. Furthermore, it is not clear how such quantities are to be measured from experiment. For more details on the geometric and physical interpretations for fractional derivatives of both the Riemann-Liouville and the Caputo types we refer to [10].

3. Description of the method

In this section, we extend the application of a generalized HAM [9], [10], [11], [12], [13], [14] to the Volterra population model of fractional order as defined in equation (1.2). We assume that the solution of equation (1.2) can be expressed by the set of base functions

$$\{t^{i\alpha+k};\ i \geqslant 0,\ k \geqslant 0\}. \tag{3.1}$$

By means of generalizing the HAM, we construct the zero order deformation equation

$$(3.2)\quad \begin{aligned}&(1-A(q))L[\varphi(t;q)-u_0(t)]\\ &\quad= -\hbar A(q)\Big[D^\alpha\varphi(t;q)-\varphi(t;q)+\varphi^2(t;q)+\varphi(t;q)\int_0^t\varphi(x;q)\,\mathrm{d}x\Big],\end{aligned}$$

subject to the initial condition

$$(3.3)\qquad \varphi(0;q)=\beta,$$

where $A(q)$ is a deformation function. We note that $A(0) = 0$ and $A(1) = 1$. Differentiating equation (3.2) n times with respect to the embedding parameter q, then setting $q = 0$ and dividing by $n!$, we obtain the so-called mth-order deformation equations

$$(3.4)\quad \begin{aligned}&L\Big[u_n(t)-\sum_{m=1}^{n-1}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}u_{n-m}\Big]\\ &\qquad= -\hbar\sum_{m=1}^{n}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}R_{n-m}(u_0,u_1,\dots,u_{n-m}),\end{aligned}$$

where

$$(3.5)\quad \begin{aligned}R_k(u_0,u_1,\dots,u_k)&=D^\alpha u_k(t)-u_k(t)\\ &\quad+\sum_{l=1}^{k}u_l(t)u_{k-l}(t)+\sum_{l=1}^{k}u_l(t)\int_0^t u_{k-l}(x)\,\mathrm{d}x,\end{aligned}$$

$$(3.6)\qquad u_m(t)=\frac{\partial^m\varphi(t;q)}{m!\partial q^m}\Big|_{q=0},$$

subject to the initial condition

$$(3.7)\qquad u_n(t)=0.$$

By defining $L = D^\alpha$ and applying the operator J^α to both sides of equation (3.4), we obtain

$$(3.8)\quad \begin{aligned}u_n(t)&=\sum_{m=1}^{n-1}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}u_{n-m}\\ &\quad-\hbar\sum_{m=1}^{n}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}J^\alpha R_{n-m}(u_0,u_1,\dots,u_{n-m}).\end{aligned}$$

There is an infinite number of deformation functions that satisfy the conditions $A(0) = 0$ and $A(1) = 1$, and we use here only the deformation function

$$A(q) = \sum_{k=1}^{\infty}(1-\varpi)\varpi^{k-1}q^k, \quad |\varpi| < 1, \tag{3.9}$$

where ϖ is an auxiliary parameter satisfying the inequality $|\varpi| < 1$.

For simplicity of numerical computation, let the expression

$$S(t) = \sum_{k=1}^{N} u_k(t) \tag{3.10}$$

be the N-term approximation to $u(t)$, which contains the auxiliary parameters $\hbar$ and ϖ.

3.1. Convergence theorem.

Theorem. *If the series*

$$\sum_{k=0}^{\infty} u_k(t) \tag{3.11}$$

is convergent, then it defines a solution to the fractional Volterra integral equation (1.2).

P r o o f. If the solution series

$$u_0(t) + \sum_{k=1}^{\infty} u_k(t) \tag{3.12}$$

is convergent, then

$$\lim_{k\to\infty} u_k(t) = 0. \tag{3.13}$$

Using equation (3.4) and equation (3.13), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} L\left[u_n(t) - \sum_{m=1}^{n-1} \frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0} u_{n-m}\right] \\ &\quad = L[(1-(1-\varpi)-(1-\varpi)\varpi-\ldots)(u_1(t)+u_2(t)+u_3(t)+\ldots)] = 0. \end{aligned} \tag{3.14}$$

Since $\hbar \neq 0$, we deduce

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0} R_{n-m}(u_0,u_1,\ldots,u_{n-m})=0. \tag{3.15}$$

From equation (3.5) we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}\Big[D^\alpha u_{n-m}(t)-u_{n-m}(t) \\
&\quad+\sum_{l=1}^{n-m}u_l(t)u_{n-m-l}(t)+\sum_{l=1}^{n-m}u_l(t)\int_0^t u_{n-m-l}(x)\,\mathrm{d}x\Big] \\
&=\sum_{j=0}^{\infty}\varpi^j\sum_{j=0}^{\infty}\Big[D^\alpha u_j(t)-u_j(t)+\sum_{l=1}^{j}u_l(t)u_{j-l}(t)+\sum_{l=1}^{j}u_l(t)\int_0^t u_{j-l}(x)\,\mathrm{d}x\Big] \\
&=\sum_{j=0}^{\infty}\varpi^j\Big[D^\alpha\sum_{j=0}^{\infty}u_j(t)-\sum_{j=0}^{\infty}u_j(t)+\sum_{j=0}^{\infty}\sum_{l=1}^{j}u_l(t)u_{j-l}(t) \\
&\quad+\sum_{j=0}^{\infty}\sum_{l=1}^{j}u_l(t)\int_0^t u_{j-l}(x)\,\mathrm{d}x\Big] \\
&=\sum_{j=0}^{\infty}\varpi^j\Big[D^\alpha u(t)-u(t)+u^2(t)+u(t)\int_0^t u(x)\,\mathrm{d}x\Big]=0.
\end{aligned} \tag{3.16}$$

Finally, by using equation (3.16) we obtain

$$\kappa D^\alpha u(t)-u(t)+u^2(t)+u(t)\int_0^t u(x)\,\mathrm{d}x=0. \tag{3.17}$$

This completes the proof. □

4. Experimental data and results

According to the convergence theorem we show the convergence of the solution series. We note that the series solutions contain the auxiliary parameters $\hbar$ and ϖ which influence the convergence region of the series solutions. We need only to concentrate on the convergence of the results obtained by properly choosing $\hbar$ and ϖ.

Example 1. We investigate the influence of $\hbar$ and ϖ on the convergence of $S(25)$, when $\alpha = 0.5$ and $\beta = 0.1$. Clearly, $S(25)$ is dependent on both $\hbar$ and ϖ. For a given ϖ, we can study the influence of $\hbar$ on the convergence region of $S(25)$ by means of the $\hbar$-curve as shown in Figure 1.

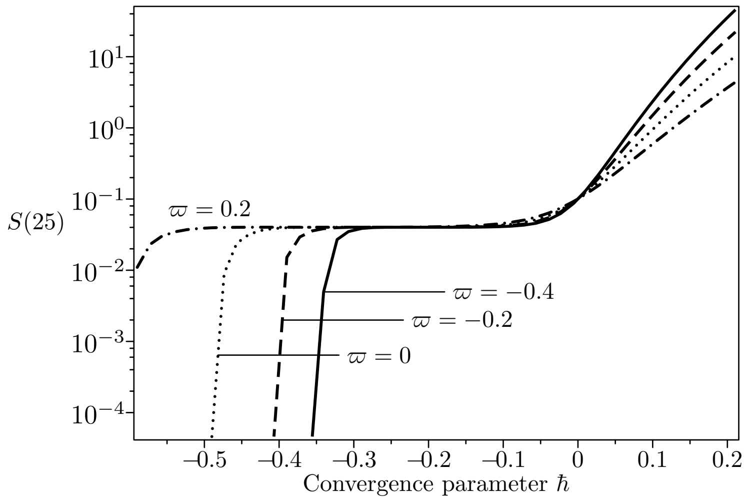

Figure 1. The $\hbar$-curve of $S(25)$ when $\alpha = 0.1$ and $\beta = 0.1$ (Example 1).

The residual errors (RE) by the 9th-order approximation, i.e.

$$D^{0.5}\sum_{k=0}^{M} u_k(t) - \sum_{k=0}^{M} u_k(t) + \left(\sum_{k=0}^{M} u_k(t)\right)^2 + \sum_{k=0}^{M} u_k(t)\int_0^t \sum_{k=0}^{M} u_k(x), \quad M = 9, \tag{4.1}$$

are listed in Table 1. The residual errors obtained by the present method are less than those obtained by the HAM method, as shown in Table 1.

t	$\varpi = -0.4$, $\hbar = -0.2$	$\varpi = -0.2$, $\hbar = -0.25$	$\varpi = 0$, $\hbar = -0.2$
0	$4.6E{-3}$	$3.6E{-3}$	$1.2E{-2}$
5	$3.3E{-3}$	$1.0E{-2}$	$5.1E{-2}$
10	$4.5E{-3}$	$1.0E{-2}$	$3.4E{-2}$
15	$1.2E{-2}$	$1.6E{-3}$	$5.9E{-3}$
20	$4.0E{-3}$	$5.2E{-4}$	$3.6E{-3}$
25	$1.9E{-4}$	$1.2E{-4}$	$8.3E{-5}$
t	$\varpi = 0$, $\hbar = -0.25$	$\varpi = 0$, $\hbar = -0.3$	$\varpi = 0.2$, $\hbar = -0.35$
0	$6.7E{-3}$	$3.6E{-3}$	$4.6E{-3}$
5	$2.4E{-2}$	$1.0E{-2}$	$3.3E{-3}$
10	$2.3E{-2}$	$1.0E{-2}$	$4.5E{-3}$
15	$1.8E{-2}$	$1.6E{-3}$	$1.2E{-2}$
20	$3.9E{-3}$	$5.2E{-4}$	$3.9E{-3}$
25	$2.1E{-4}$	$8.0E{-6}$	$1.0E{-4}$

Table 1. The residual error by 9th-order approximation in Example 1.

Example 2. We study the influence of $\hbar$ and ϖ on the convergence of $S(50)$, when $\alpha = 0.3$ and $\beta = 0.1$. For a given ϖ, we can investigate the influence of $\hbar$ on the convergence region of $S(50)$ by means of the $\hbar$-curve as shown in Figure 2. Clearly, the corresponding valid region of $\hbar$ increases for increasing ranges of ϖ, as depicted in Figure 2.

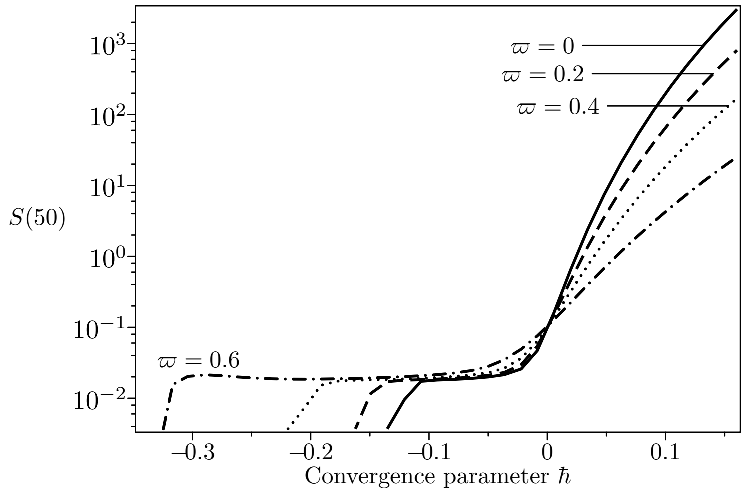

Figure 2. The $\hbar$-curve of $S(50)$ when $\alpha = 0.3$ and $\beta = 0.1$ (Example 2).

Residual errors (RE) by the 9th-order approximation, i.e.

$$D^{0.3}\sum_{k=0}^{M} u_k(t) - \sum_{k=0}^{M} u_k(t) + \left(\sum_{k=0}^{M} u_k(t)\right)^2 + \sum_{k=0}^{M} u_k(t)\int_0^t \sum_{k=0}^{M} u_k(x), \quad M = 9, \tag{4.2}$$

are listed in Table 2. The residual errors obtained by the present method are less than those obtained by the HAM method, as shown in Table 2.

5. Increasing the convergence rate of the approximation

In this section, we apply the homotopy-pade method to accelerate the convergence rate of the series solution.

The homotopy-pade approximant

$$R[M,M](t,\hbar,\varpi) \tag{5.1}$$

to

$$S_{2M}(t,\hbar,\varpi) = \sum_{j=0}^{2M} u_j(t) q^j, \tag{5.2}$$

t	$\varpi=0,\hbar=-0.07$	$\varpi=0,\hbar=-0.08$	$\varpi=0,\hbar=-0.09$
0	$4.6E{-2}$	$4.2E{-2}$	$3.8E{-2}$
5	$5.3E{-4}$	$5.6E{-3}$	$1.0E{-2}$
10	$2.9E{-2}$	$2.8E{-2}$	$2.5E{-2}$
15	$2.0E{-2}$	$1.5E{-2}$	$1.0E{-2}$
20	$1.0E{-2}$	$5.8E{-3}$	$2.8E{-3}$
25	$5.4E{-3}$	$3.1E{-3}$	$1.8E{-3}$
30	$4.6E{-3}$	$3.2E{-3}$	$2.4E{-3}$
35	$4.7E{-3}$	$3.7E{-3}$	$2.8E{-3}$
40	$4.8E{-3}$	$3.7E{-3}$	$2.8E{-3}$
45	$4.7E{-3}$	$3.7E{-3}$	$2.8E{-3}$
50	$4.7E{-3}$	$3.6E{-3}$	$2.8E{-3}$
t	$\varpi=0.2,\hbar=-0.13$	$\varpi=0.4,\hbar=-0.17$	$\varpi=0.6,\hbar=-0.29$
0	$3.3E{-2}$	$3.4E{-2}$	$2.9E{-2}$
5	$1.6E{-2}$	$1.5E{-2}$	$2.0E{-2}$
10	$2.2E{-2}$	$2.2E{-2}$	$1.8E{-2}$
15	$5.4E{-3}$	$6.1E{-3}$	$1.7E{-3}$
20	$3.2E{-3}$	$5.9E{-4}$	$7.8E{-4}$
25	$9.5E{-4}$	$1.0E{-3}$	$6.2E{-4}$
30	$1.7E{-3}$	$1.8E{-3}$	$1.2E{-3}$
35	$2.0E{-3}$	$2.1E{-3}$	$1.9E{-3}$
40	$2.0E{-3}$	$2.1E{-3}$	$2.2E{-3}$
45	$2.0E{-3}$	$2.1E{-3}$	$9.2E{-5}$
50	$1.7E{-3}$	$2.0E{-3}$	$1.7E{-3}$

Table 2. The residual errors by 9th-order approximation in Example 2.

is defined as the the rational function

$$R[M,M](t,\hbar,\varpi)=\frac{\sum_{j=0}^{M}P_j(t,\hbar,\varpi)q^j}{1+\sum_{j=1}^{M}Q_j(t,\hbar,\varpi)q^j}, \tag{5.3}$$

satisfying

$$\sum_{j=0}^{2M}u_j(t)+\sum_{j=1}^{M}Q_j(t,\hbar,\varpi)q^j\sum_{j=0}^{2M}u_j(t)-\sum_{j=0}^{M}P_j(t,\hbar,\varpi)q^j=O(q^{2M+1}), \quad \text{as } q\to 0. \tag{5.4}$$

The coefficients P_j and Q_j can be determined in terms of

$$u_0(t),u_1(t),\ldots,u_{2M}(t). \tag{5.5}$$

By setting $q = 1$, we have the $[M, M]$ homotopy-Pade approximation

$$R[M, M](t, \hbar, \varpi) = \frac{\sum_{j=0}^{M} P_j(t, \hbar, \varpi)}{1 + \sum_{j=1}^{M} Q_j(t, \hbar, \varpi)}. \tag{5.6}$$

In particular, for $M = 1$ we have

$$R[1, 1](t, \hbar, \varpi) = \frac{P_0(t, \hbar, \varpi) + P_1(t, \hbar, \varpi)}{1 + Q_1(t, \hbar, \varpi)}, \tag{5.7}$$

$$P_0(t, \hbar, \varpi) = u_0(t, \hbar, \varpi), \tag{5.8}$$

$$P_1(t, \hbar, \varpi) = -\frac{u_0(t, \hbar, \varpi)u_2(t, \hbar, \varpi) - u_1^2(t, \hbar, \varpi)}{u_1(t, \hbar, \varpi)}, \tag{5.9}$$

$$Q_1(t, \hbar, \varpi) = -\frac{u_2(t, \hbar, \varpi)}{u_1(t, \hbar, \varpi)}. \tag{5.10}$$

We employ the homotopy-pade technique to gain more accurate approximations of $S(10)$ and $S(25)$ in Example 1, as shown in Table 3.

$S(25)$	$\hbar = -0.2, \varpi = -0.4$	$\hbar = -0.35, \varpi = 0.2$
R[2,2]	0.01803	0.01803
R[3,3]	0.01508	0.01508
R[4,4]	0.01626	0.01625
R[5,5]	0.01626	0.01626
R[6,6]	0.01626	0.01626
$S(10)$	$\hbar = -0.2,\ \varpi = -0.4$	$\hbar = -0.35,\ \varpi = 0.2$
R[2,2]	0.05647	0.05647
R[3,3]	0.05656	0.05656
R[4,4]	0.05517	0.05517
R[5,5]	0.05540	0.05540
R[6,6]	0.05530	0.05534

Table 3. The homotopy-pade approximations (Example 1).

6. Conclusions

In this paper, a biparametric homotopy technique is proposed for solving a Volterra's population system of fractional order, which turns out to be successful. The bi-parametric homotopy method provides two auxiliary parameters $\hbar$ and ϖ which can control the convergence region of the series solution. Numerical examples reveal that the proposed zero-order deformation equation increases the convergence region of the series solution and the efficiency of this technique.

References

[1] *K. Al-Khaled*: Numerical approximations for population growth models. Appl. Math. Comput. *160* (2005), 865–873.

[2] *M. Caputo*: Linear models of dissipation whose Q is almost frequency independent II. Geophys. J. R. Astron. Soc. *13* (1967), 529–539.

[3] *K. Diethelm, N. J. Ford, A. D. Freed, Y. Luchko*: Algorithms for the fractional calculus: a selection of numerical methods. Comput. Methods Appl. Mech. Eng. *194* (2005), 743–773.

[4] *J. He*: Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Methods Appl. Mech. Eng. *167* (1998), 57–68.

[5] *J. He*: Nonlinear oscillation with fractional derivative and its applications. International Conference on Vibrating Engineering, Dalian, China, 1998, 288–291.

[6] *J. He*: Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. *15* (1999), 86–90.

[7] *S. Liao*: Boundary element method for general nonlinear differential operators. Eng. Anal. Bound. Elem. *20* (1997), 91–99.

[8] *Y. Luchko, R. Gorenflo*: The initial value problem for some fractional differential equations with the Caputo derivatives. Fachbereich Mathematik und Informatik, Freie Universität Berlin (1998), Preprint A-98-08.

[9] *I. Podlubny*: Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering 198, Academic Press, San Diego, 1999.

[10] *I. Podlubny*: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. *5* (2002), 367–386.

[11] *F. M. Scudo*: Vito Volterra and theoretical ecology. Theor. Population Biology *2* (1971), 1–23.

[12] *R. D. Small*: Population growth in a closed system. Mathematical Modelling: Classroom Notes in Applied Mathematics (M. S. Klamkin, ed.). Society for Industrial and Applied Mathematics, Philadelphia, 1989.

[13] *K. G. TeBeest*: Numerical and analytical solutions of Volterra's population model. SIAM Rev. *39* (1997), 484–493.

[14] *A.-M. Wazwaz*: Analytical approximations and Padé approximants for Volterra's population model. Appl. Math. Comput. *100* (1999), 13–25.

Authors' addresses: *Mehdi Ghasemi*, Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran, e-mail: `meh_ghasemi@yahoo.com`; *Mojtaba Fardi*, Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Iran, e-mail: `fardi_mojtaba@yahoo.com`; *Reza Khoshsiar Ghaziani*, Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran.