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DEPENDENCE ON THE PARAMETERS OF THE SET OF TRAJECTORIES OF THE CONTROL SYSTEM DESCRIBED BY A NONLINEAR VOLTERRA INTEGRAL EQUATION

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Abstract. In this paper the control system with limited control resources is studied, where the behavior of the system is described by a nonlinear Volterra integral equation. The admissible control functions are chosen from the closed ball centered at the origin with radius μ in L_p ($p > 1$). It is proved that the set of trajectories generated by all admissible control functions is Lipschitz continuous with respect to μ for each fixed p, and is continuous with respect to p for each fixed μ. An upper estimate for the diameter of the set of trajectories is given.

Keywords: nonlinear Volterra integral equation; control system; integral constraint

MSC 2010: 45D05, 93C23

1. Introduction

Nowadays, the theory of control systems is one of the well developed fields of applied mathematics. Depending on the constraints which are satisfied by the admissible control functions, control systems can be classified as control systems with geometric constraint on the controls and control systems with integral constraint on the controls. The control systems with integral constraint on controls are generally needed in modelling the systems having limited energy resources which are exhausted by consumption, such as fuel or finance (see, e.g. [2], [3], [4], [5], [7], [9]). For example, the motion of a flying apparatus with variable mass is described in the form of a control system, where the control functions have integral constraint (see e.g. [2], [9]).

It is known that nonlinear integral equations arise in many problems of theory and applications (see e.g. [1], [6], [7], [8], [10], [11], [12], [13], [14], [15]), and many problems of nonlinear mechanics lead to nonlinear integral equations (see, e.g. [8], [12], [15]). In this paper the control system with integral constraint on the controls whose behavior is described by a nonlinear Volterra integral equation is considered. It is assumed that the integral equation is nonlinear with respect to the state and the control vectors. The closed ball of the space L_p ($p > 1$) with radius μ and centered at the origin is chosen as the set of admissible control functions. The set of trajectories of the system generated by all admissible control functions is studied. It is proved that for fixed μ the set of trajectories is continuous with respect to p, and for fixed p it is Lipschitz continuous with respect to μ, where p is the parameter of the space L_p from which the admissible control functions are chosen, μ is the parameter which characterizes the recourse of the control effort. This fact allows one to assert that the set of trajectories has a minor perturbation if in the modeling process the measurements of the parameters p and μ tolerate small errors.

Dependence of the set of trajectories and attainable sets on p and μ is studied in [4], where the behavior of the control system is described by an ordinary differential equation. The background of the continuity of the set of trajectories with respect to p is a theorem proved in [5] which asserts that the closed balls of the space L_p ($p > 1$) with radius μ and centered at the origin are continuous with respect to p in Hausdorff metric. In [3], an approximation method for the construction of attainable sets of a control system with integral constraint on the controls is given, where it is assumed that the dynamics of the system is described by a nonlinear ordinary differential equation. Precompactness of the set of trajectories is discussed in [7], where the behavior of the system is described by a nonlinear Volterra integral equation. The results obtained in this paper extend the ones presented in [4].

The paper is organized as follows. In Section 2 the basic conditions are formulated which are satisfied by the system (Conditions 2.A, 2.B and 2.C). In Section 3 it is proved that for each fixed p the set of trajectories is Lipschitz continuous with respect to μ (Theorem 1). In Section 4 it is shown that for each fixed μ the set of trajectories is continuous with respect to p (Theorem 3). In Section 5 an upper estimate for the diameter of the set of trajectories is given (Theorem 4).

2. Preliminaries

Consider a control system the behavior of which is described by a nonlinear Volterra integral equation

$$x(t) = a(t, x(t)) + \lambda \int_{t_0}^{t} K(t, s, x(s), u(s))\, \mathrm{d}s, \tag{2.1}$$

where $x(s) \in \mathbb{R}^n$ is the state vector, $u(s) \in \mathbb{R}^m$ is the control vector, $t \in [t_0, \theta]$, $\lambda \in \mathbb{R}^1$.

Let $p > 1$ and $\mu > 0$ be given numbers. A function $u(\cdot) \in L_p([t_0, \theta]; \mathbb{R}^m)$ such that

$$\|u(\cdot)\|_p \leqslant \mu \tag{2.2}$$

is said to be an admissible control function, where $\|u(\cdot)\|_p = \big(\int_{t_0}^{\theta} \|u(t)\|^p \,\mathrm{d}t\big)^{1/p}$. The set of all admissible control functions is denoted by $U_{p,\mu}$. Thus

$$U_{p,\mu} = \{u(\cdot) \in L_p([t_0, \theta]; \mathbb{R}^m) \colon \ \|u(\cdot)\|_p \leqslant \mu\}. \tag{2.3}$$

It is obvious that the set of admissible control functions is the closed ball with radius μ and centered at the origin in the space $L_p([t_0, \theta]; \mathbb{R}^m)$. Let us choose an arbitrary $u(\cdot) \in U_{p,\mu}$. Then Hölder's inequality yields

$$\int_{t_0}^{\theta} \|u(t)\| \,\mathrm{d}t \leqslant (\theta - t_0)^{(p-1)/p} \bigg(\int_{t_0}^{\theta} \|u(t)\|^p \,\mathrm{d}t\bigg)^{1/p} \leqslant (\theta - t_0)^{(p-1)/p} \mu. \tag{2.4}$$

It is assumed that the following conditions are satisfied:

2.A The functions $a(\cdot)\colon\ [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^n$ and $K(\cdot)\colon\ [t_0, \theta] \times [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous.

2.B There exist $L_0 \in [0, 1)$, $L_1 \geqslant 0$, $H_1 \geqslant 0$, $L_2 \geqslant 0$, $H_2 \geqslant 0$, $L_3 \geqslant 0$, and $H_3 \geqslant 0$ such that

$$\begin{aligned} \|a(t, x_1) - a(t, x_2)\| &\leqslant L_0 \|x_1 - x_2\|, \\ \|K(t_1, s, x_1, u_1) - K(t_2, s, x_2, u_2)\| &\leqslant [L_1 + H_1(\|u_1\| + \|u_2\|)] |t_1 - t_2| \\ &\quad + [L_2 + H_2(\|u_1\| + \|u_2\|)] \|x_1 - x_2\| \\ &\quad + [L_3 + H_3(\|x_1\| + \|x_2\|)] \|u_1 - u_2\| \end{aligned}$$

for every $(t_1, s, x_1, u_1) \in [t_0, \theta] \times [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m$, $(t_2, s, x_2, u_2) \in [t_0, \theta] \times [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

2.C There exist $p_0 > 1$ and $\mu_0 > 0$ such that the inequality

$$0 \leqslant \lambda(L_2(\theta - t_0) + 2H_2(\theta - t_0)^{(p_0-1)/p_0} \mu_0) < 1 - L_0$$

holds.

If $K(t, s, x, u) = \varphi(t, s, x) + B(t, s, x)u$, where the functions $(t, s, x) \to \varphi(t, s, x)$ and $(t, s, x) \to B(t, s, x)$ are continuous with respect to (t, s, x) and Lipschitz continuous with respect to (t, x), then the function $K(\cdot)\colon\ [t_0, \theta] \times [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the conditions 2.A and 2.B.

R e m a r k 1. Let us denote

$$L(\lambda; p, \mu) = L_0 + \lambda[L_2(\theta - t_0) + 2H_2(\theta - t_0)^{(p-1)/p}\mu]. \tag{2.5}$$

According to the condition 2.C we obtain that $0 \leqslant L(\lambda; p_0, \mu_0) < 1$. Then there exists $\alpha > 0$ such that

$$0 \leqslant L(\lambda; p, \mu) < 1$$

for every $p \in [p_0 - \alpha, p_0 + \alpha]$ and $\mu \in [\mu_0 - \alpha, \mu_0 + \alpha]$, where $p_0 - \alpha > 1$ and $\mu_0 - \alpha > 0$.

We set

$$L_*(\lambda) = \max\{L(\lambda; p, \mu)\colon\ p \in [p_0 - \alpha, p_0 + \alpha],\ \mu \in [\mu_0 - \alpha, \mu_0 + \alpha]\}. \tag{2.6}$$

From now on, it will be assumed that $p \in [p_0 - \alpha, p_0 + \alpha]$ and $\mu \in [\mu_0 - \alpha, \mu_0 + \alpha]$.

Now, let us define the trajectory of the system (2.1) generated by an admissible control function. Let p and μ be fixed and $u_*(\cdot) \in U_{p,\mu}$. A continuous function $x_*(\cdot)\colon\ [t_0, \theta] \to \mathbb{R}^n$ satisfying the integral equation

$$x_*(t) = a(t, x_*(t)) + \lambda \int_{t_0}^{t} K(t, s, x_*(s), u_*(s))\,\mathrm{d}s$$

for every $t \in [t_0, \theta]$ is said to be the trajectory of the system (2.1) generated by the admissible control function $u_*(\cdot) \in U_{p,\mu}$. The trajectory of the system (2.1) generated by the control function $u(\cdot) \in U_{p,\mu}$ is denoted by $x(\cdot; u(\cdot))$.

Proposition 1 ([7])**.** *Let the conditions* 2.A, 2.B, *and* 2.C *be satisfied. Then every admissible control function* $u(\cdot) \in U_{p,\mu}$ *generates a unique trajectory* $x(\cdot; u(\cdot))$ *of the system* (2.1)*, where* $p \in [p_0 - \alpha, p_0 + \alpha]$*,* $\mu \in [\mu_0 - \alpha, \mu_0 + \alpha]$*, and* $\alpha > 0$ *is defined in Remark* 1.

We set

$$\mathbf{X}_{p,\mu} = \{x(\cdot; u(\cdot))\colon\ u(\cdot) \in U_{p,\mu}\},$$

where $\mathbf{X}_{p,\mu}$ is called the set of trajectories of the system (2.1) with integral constraint (2.2). It is obvious that $\mathbf{X}_{p,\mu} \subset C([t_0, \theta]; \mathbb{R}^n)$ where $C([t_0, \theta]; \mathbb{R}^n)$ is the space of continuous functions $x(\cdot)\colon\ [t_0, \theta] \to \mathbb{R}^n$ with the norm

$$\|x(\cdot)\|_C = \max\{\|x(t)\|\colon\ t \in [t_0, \theta]\}.$$

For $t \in [t_0, \theta]$ we denote

$$\mathbf{X}_{p,\mu}(t) = \{x(t) \in \mathbb{R}^n\colon\ x(\cdot) \in \mathbf{X}_{p,\mu}\}.$$

The set $\mathbf{X}_{p,\mu}(t)$ consists of points to which the trajectories of the system arrive at the instant t.

Let (Y, d_Y) be a metric space. The Hausdorff distance between the sets $F \subset Y$ and $E \subset Y$ is denoted by $h(F, E)$ and defined by

$$h(F, E) = \max\Big\{ \sup_{x \in F} d_Y(x, E), \sup_{y \in E} d_Y(y, F) \Big\},$$

where $d_Y(x, E) = \inf\{d_Y(x, y)\colon\ y \in E\}$.

Definition 1. Let (W, d_W) and (Y, d_Y) be metric spaces, $w \to F(w)$ a set valued map, where $w \in W$, $F(w) \subset Y$, and $w_* \in W$.

If $h(F(w), F(w_*)) \to 0$ as $w \to w_*$, then the set valued map $F(\cdot)$ is called continuous at w_*.

If there exists $R \geqslant 0$ such that $h(F(w_1), F(w_2)) \leqslant R \cdot d_W(w_1, w_2)$ for every $w_1 \in W$ and $w_2 \in W$, then the set valued map $F(\cdot)$ is called Lipschitz continuous with Lipschitz constant R.

Now, let us give propositions which characterize boundedness and precompactness of the set of trajectories $\mathbf{X}_{p,\mu}$, and continuity of the set valued map $t \mapsto \mathbf{X}_{p,\mu}(t)$, $t \in [t_0, \theta]$.

Proposition 2 ([7])**.** *Let the conditions* 2.A, 2.B, *and* 2.C *be satisfied. Then the set of trajectories* $\mathbf{X}_{p,\mu}$ *is a precompact subset of the space* $C([t_0, \theta]; \mathbb{R}^n)$ *and* $h(\mathbf{X}_{p,\mu}(t), \mathbf{X}_{p,\mu}(t_*)) \to 0$ *as* $t \to t_*$, *where* $t_* \in [t_0, \theta]$, $p \in [p_0 - \alpha, p_0 + \alpha]$, $\mu \in [\mu_0 - \alpha, \mu_0 + \alpha]$, *and* $\alpha > 0$ *is defined in Remark* 1.

Proposition 3 ([7])**.** *Let the conditions* 2.A, 2.B *and* 2.C *be satisfied. Then there exists* $r_* > 0$ *such that*

$$\|x(\cdot)\|_C \leqslant r_* \tag{2.7}$$

for every $x(\cdot) \in \mathbf{X}_{p,\mu}$, $p \in [p_0 - \alpha, p_0 + \alpha]$ *and* $\mu \in [\mu_0 - \alpha, \mu_0 + \alpha]$, *where* $\alpha > 0$ *is defined in Remark* 1.

3. Dependence of the set of trajectories on μ

In this section we prove that for each fixed $p \in (p_0-\alpha, p_0+\alpha)$ the map $\mu \mapsto \mathbf{X}_{p,\mu}$, $\mu \in (\mu_0-\alpha, \mu_0+\alpha)$, is Lipschitz continuous, where $\alpha > 0$ is defined in Remark 1. Denote

$$\gamma_* = \max\{(\theta - t_0)^{(p-1)/p}\colon\ p \in [p_0-\alpha, p_0+\alpha]\}, \tag{3.1}$$

$$R_* = \frac{\lambda}{1-L_0}(L_3 + 2r_* H_3)\gamma_* \cdot \exp\Big[\frac{L_*(\lambda) - L_0}{1-L_0}\Big], \tag{3.2}$$

where $L_*(\lambda)$ is defined by (2.6), r_* is given in (2.7).

Theorem 1. *Let $p \in (p_0-\alpha, p_0+\alpha)$, $\mu \in (\mu_0-\alpha, \mu_0+\alpha)$, $\mu_* \in (\mu_0-\alpha, \mu_0+\alpha)$, where $\alpha > 0$ is defined in Remark 1. Then*

$$h(\mathbf{X}_{p,\mu}, \mathbf{X}_{p,\mu_*}) \leqslant R_*|\mu - \mu_*|$$

and hence,

$$h(\mathbf{X}_{p,\mu}(t), \mathbf{X}_{p,\mu_*}(t)) \leqslant R_*|\mu - \mu_*|$$

for every $t \in [t_0, \theta]$, where $R_* \geqslant 0$ is defined by (3.2).

Proof. Let us choose an arbitrary $x_*(\cdot) \in \mathbf{X}_{p,\mu_*}$. Then there exists $u_*(\cdot) \in U_{p,\mu_*}$ such that

$$x_*(t) = a(t, x_*(t)) + \lambda \int_{t_0}^{t} K(t, s, x_*(s), u_*(s))\,\mathrm{d}s \tag{3.3}$$

for every $t \in [t_0, \theta]$. Since $u_*(\cdot) \in U_{p,\mu_*}$, we have

$$\Big(\int_{t_0}^{\theta} \|u_*(t)\|^p\,\mathrm{d}t\Big)^{1/p} \leqslant \mu_*. \tag{3.4}$$

Given μ, define a function $u(\cdot)\colon [t_0, \theta] \to \mathbb{R}^m$ setting

$$u(t) = \frac{\mu}{\mu_*} u_*(t), \quad t \in [t_0, \theta]. \tag{3.5}$$

Then (3.4) and (3.5) yield

$$\begin{aligned}\Big(\int_{t_0}^{\theta} \|u(t)\|^p\,\mathrm{d}t\Big)^{1/p} &= \Big(\int_{t_0}^{\theta} \frac{\mu^p}{\mu_*^p}\|u_*(t)\|^p\,\mathrm{d}t\Big)^{1/p} \\ &= \frac{\mu}{\mu_*}\Big(\int_{t_0}^{\theta} \|u_*(t)\|^p\,\mathrm{d}t\Big)^{1/p} \leqslant \frac{\mu}{\mu_*} \cdot \mu_* = \mu\end{aligned}$$

and consequently $u(\cdot) \in U_{p,\mu}$. Let $x(\cdot)\colon [t_0, \theta] \to \mathbb{R}^m$ be the trajectory of the system (2.1) generated by the control function $u(\cdot) \in U_{p,\mu}$. Then $x(\cdot) \in \mathbf{X}_{p,\mu}$ and

$$x(t) = a(t, x(t)) + \lambda \int_{t_0}^{t} K(t, s, x(s), u(s))\,\mathrm{d}s \tag{3.6}$$

for every $t \in [t_0, \theta]$. From condition 2.B, (3.3) and (3.6) we obtain

$$\begin{aligned} \|x(t) - x_*(t)\| &\leqslant \|a(t, x(t)) - a(t, x_*(t))\| \\ &\quad + \lambda \int_{t_0}^{t} \|K(t, s, x(s), u(s)) - K(t, s, x_*(s), u_*(s))\|\,\mathrm{d}s \\ &\leqslant L_0 \|x(t) - x_*(t)\| \\ &\quad + \lambda \int_{t_0}^{t} [L_2 + H_2(\|u(s)\| + \|u_*(s)\|)] \|x(s) - x_*(s)\|\,\mathrm{d}s \\ &\quad + \lambda \int_{t_0}^{t} [L_3 + H_3(\|x(s)\| + \|x_*(s)\|)] \|u(s) - u_*(s)\|\,\mathrm{d}s \end{aligned} \tag{3.7}$$

for every $t \in [t_0, \theta]$. From Proposition 3, (2.4), (3.1), and (3.5) we have

$$\begin{aligned} &\int_{t_0}^{t} [L_3 + H_3(\|x(s)\| + \|x_*(s)\|)] \|u(s) - u_*(s)\|\,\mathrm{d}s \\ &\qquad \leqslant \int_{t_0}^{t} (L_3 + 2r_* H_3) \Big\|\frac{\mu}{\mu_*} u_*(s) - u_*(s)\Big\|\,\mathrm{d}s \\ &\qquad = (L_3 + 2r_* H_3) \frac{|\mu - \mu_*|}{\mu_*} \int_{t_0}^{t} \|u_*(s)\|\,\mathrm{d}s \\ &\qquad \leqslant (L_3 + 2r_* H_3) \frac{|\mu - \mu_*|}{\mu_*} (t - t_0)^{(p-1)/p} \mu_* \\ &\qquad \leqslant (L_3 + 2r_* H_3) \gamma_* |\mu - \mu_*| \end{aligned} \tag{3.8}$$

for every $t \in [t_0, \theta]$. Since $L_0 \in [0, 1)$, from (3.7) and (3.8) we conclude that

$$\begin{aligned} \|x(t) - x_*(t)\| &\leqslant \frac{\lambda}{1 - L_0} \int_{t_0}^{t} [L_2 + H_2(\|u(s)\| + \|u_*(s)\|)] \|x(s) - x_*(s)\|\,\mathrm{d}s \\ &\quad + \frac{\lambda}{1 - L_0} (L_3 + 2r_* H_3) \gamma_* |\mu - \mu_*| \end{aligned} \tag{3.9}$$

for every $t \in [t_0, \theta]$. Finally, since $u_*(\cdot) \in U_{p,\mu_*}$, $u(\cdot) \in U_{p,\mu}$, Gronwall's inequality, (2.4), (3.2), and (3.9) imply that

$$
\begin{aligned}
(3.10)\quad \|x(t) - x_*(t)\| &\leqslant \frac{\lambda}{1 - L_0}(L_3 + 2r_* H_3)\gamma_* |\mu - \mu_*| \\
&\quad \times \exp\left[\frac{\lambda}{1 - L_0}\int_{t_0}^{t}[L_2 + H_2(\|u(s)\| + \|u_*(s)\|)]\,\mathrm{d}s\right] \\
&\leqslant \frac{\lambda}{1 - L_0}(L_3 + 2r_* H_3)\gamma_* |\mu - \mu_*| \\
&\quad \times \exp\left[\frac{\lambda}{1 - L_0}[L_2(\theta - t_0) + H_2(\mu + \mu_*)(\theta - t_0)^{(p-1)/p}]\right] \\
&\leqslant \frac{\lambda}{1 - L_0}(L_3 + 2r_* H_3)\gamma_* |\mu - \mu_*| \cdot \exp\left[\frac{L_*(\lambda) - L_0}{1 - L_0}\right] \\
&= R_* |\mu - \mu_*|
\end{aligned}
$$

for every $t \in [t_0, \theta]$. So (3.10) yields that for each $x_*(\cdot) \in \mathbf{X}_{p,\mu_*}$ and μ there exists $x(\cdot) \in \mathbf{X}_{p,\mu}$ such that

$$\|x_*(\cdot) - x(\cdot)\|_C \leqslant R_* |\mu - \mu_*|$$

and hence,

$$\sup_{x_*(\cdot) \in \mathbf{X}_{p,\mu_*}} d(x_*(\cdot), \mathbf{X}_{p,\mu}) \leqslant R_* |\mu - \mu_*|. \tag{3.11}$$

Similarly, it is possible to verify that the inequality

$$\sup_{x(\cdot) \in \mathbf{X}_{p,\mu}} d(x(\cdot), \mathbf{X}_{p,\mu_*}) \leqslant R_* |\mu - \mu_*| \tag{3.12}$$

holds. By virtue of (3.11) and (3.12) we complete the proof. □

From Theorem 1 we obtain the validity of the following corollary.

Corollary 1. *Let $p \in (p_0 - \alpha, p_0 + \alpha)$ be fixed. Then the set valued map $\mu \mapsto \mathbf{X}_{p,\mu}$, $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$, is Lipschitz continuous with Lipschitz constant R_*, where $R_* \geqslant 0$ is defined by (3.2).*

4. Dependence of the set of trajectories on p

In this section the dependence of the set of trajectories on p is studied. First, let us define a distance between the subsets of the spaces $L_{p_1}([t_0,\theta];\mathbb{R}^m)$ and $L_{p_2}([t_0,\theta];\mathbb{R}^m)$, where $p_1 \in [1,\infty)$, $p_2 \in [1,\infty)$. Let $U \subset L_{p_1}([t_0,\theta];\mathbb{R}^m)$ and $V \subset L_{p_2}([t_0,\theta];\mathbb{R}^m)$, where $1 \leqslant p_1 < \infty$, $1 \leqslant p_2 < \infty$. The Hausdorff distance between the sets U and V is denoted by $\hbar_1(U,V)$ and defined as

$$\hbar_1(U,V) = \max\Big\{ \sup_{x(\cdot)\in U} d_{L_1}(x(\cdot),V),\ \sup_{y(\cdot)\in V} d_{L_1}(y(\cdot),U)\Big\},$$

where

$$d_{L_1}(x(\cdot),V) = \inf_{y(\cdot)\in V} \|x(\cdot)-y(\cdot)\|_1, \quad \|x(\cdot)-y(\cdot)\|_1 = \int_{t_0}^{\theta} \|x(t)-y(t)\|\,\mathrm{d}t.$$

The closed ball of the space $L_p([t_0,\theta];\mathbb{R}^m)$ ($p\in[1,\infty)$) with radius μ and centered at the origin is denoted by $B_{L_p}(0,\mu)$, that is

$$B_{L_p}(0,\mu) = \{u(\cdot)\in L_p([t_0,\theta];\mathbb{R}^m)\colon\ \|u(\cdot)\|_p \leqslant \mu\}.$$

By virtue of (2.3) we have $B_{L_p}(0,\mu) = U_{p,\mu}$. Now let us give a theorem which characterizes continuity of the balls $B_{L_p}(0,\mu)$ with respect to p.

Theorem 2 ([5]). *Let $\mu>0$, $p_*>1$, and $\varepsilon>0$. Then there exists $\delta_* = \delta_*(\varepsilon,p_*,\mu)\in(0,p_*-1)$ such that for every $p\in(p_*-\delta_*,p_*+\delta_*)$ the inequality*

$$\hbar_1(B_{L_p}(0,\mu), B_{L_{p_*}}(0,\mu)) < \varepsilon$$

is satisfied.

Theorem 2 implies that for each fixed $\mu\in(\mu_0-\alpha,\mu_0+\alpha)$ the set valued map $p\to \mathbf{X}_{p,\mu}$, $p\in(p_0-\alpha,p_0+\alpha)$, is continuous, where $\alpha>0$ is defined in Remark 1. We denote

$$k_* = \frac{\lambda(L_3+2r_*H_3)}{1-L_0}\exp\Big[\frac{L_*(\lambda)-L_0}{1-L_0}\Big], \tag{4.1}$$

where $L_*(\lambda)$ is defined by (2.6), r_* is defined by (2.7).

Theorem 3. *Let $\alpha > 0$ be given in Remark 1, let $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$ and $p_* \in (p_0 - \alpha, p_0 + \alpha)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, p_*, \mu) > 0$ such that for every $p \in (p_* - \delta, p_* + \delta)$ the inequality*

$$h(\mathbf{X}_{p,\mu}, \mathbf{X}_{p_*,\mu}) \leqslant \varepsilon$$

holds and consequently

$$h(\mathbf{X}_{p,\mu}(t), \mathbf{X}_{p_*,\mu}(t)) \leqslant \varepsilon$$

for every $t \in [t_0, \theta]$.

Proof. We have from Theorem 2 that for given $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$, $p_* \in (p_0 - \alpha, p_0 + \alpha)$ and ε/k_* there exists $\delta = \delta(\varepsilon, p_*, \mu) \in (0, p_* - 1)$ such that

$$\hbar_1(B_{L_p}(0,\mu), B_{L_{p_*}}(0,\mu)) < \frac{\varepsilon}{k_*} \tag{4.2}$$

for every $p \in (p_* - \delta, p_* + \delta)$, where k_* is defined by (4.1). Without loss of generality let us assume that

$$\delta = \delta(\varepsilon, p_*, \mu) < \min\{p_* - p_0 + \alpha, p_0 - p_* + \alpha\}.$$

Since $|p_* - p_0| < \alpha$, we obtain

$$(p_* - \delta, p_* + \delta) \subset (p_0 - \alpha, p_0 + \alpha). \tag{4.3}$$

Now, let us choose arbitrary $p \in (p_* - \delta, p_* + \delta)$ and $x(\cdot) \in \mathbf{X}_{p,\mu}$. Then there exists $u(\cdot) \in U_{p,\mu} = B_{L_p}(0,\mu)$ such that

$$x(t) = a(t, x(t)) + \lambda \int_{t_0}^{t} K(t, s, x(s), u(s))\,\mathrm{d}s \tag{4.4}$$

for every $t \in [t_0, \theta]$. It follows from (4.2) that there exists $u_*(\cdot) \in U_{p_*,\mu} = B_{L_{p_*}}(0,\mu)$ such that

$$\|u(\cdot) - u_*(\cdot)\|_1 \leqslant \frac{\varepsilon}{k_*},$$

that is

$$\int_{t_0}^{\theta} \|u(s) - u_*(s)\|\,\mathrm{d}s \leqslant \frac{\varepsilon}{k_*}, \tag{4.5}$$

where $k_* > 0$ is defined by (4.1). Let $x_*(\cdot)\colon [t_0, \theta] \to \mathbb{R}^n$ be the trajectory of the system (2.1) generated by the admissible control function $u_*(\cdot) \in U_{p_*,\mu} = B_{L_{p_*}}(0,\mu)$. Then $x_*(\cdot) \in \mathbf{X}_{p_*,\mu}$ and

$$x_*(t) = a(t, x_*(t)) + \lambda \int_{t_0}^{t} K(t, s, x_*(s), u_*(s))\,\mathrm{d}s \tag{4.6}$$

for every $t \in [t_0, \theta]$. Now (4.4), (4.6), and condition 2.B imply that

$$
\begin{aligned}
(4.7)\quad \|x(t) - x_*(t)\| &\leqslant \|a(t, x(t)) - a(t, x_*(t))\| \\
&\quad + \lambda \int_{t_0}^{t} \|K(t, s, x(s), u(s)) - K(t, s, x_*(s), u_*(s))\| \,\mathrm{d}s \\
&\leqslant L_0 \|x(t) - x_*(t)\| + \lambda \int_{t_0}^{t} [L_2 + H_2(\|u(s)\| + \|u_*(s)\|)] \|x(s) - x_*(s)\| \,\mathrm{d}s \\
&\quad + \lambda \int_{t_0}^{t} [L_3 + H_3(\|x(s)\| + \|x_*(s)\|)] \|u(s) - u_*(s)\| \,\mathrm{d}s
\end{aligned}
$$

for every $t \in [t_0, \theta]$. From (4.5) and Proposition 3 it follows

$$
\begin{aligned}
(4.8)\qquad & \int_{t_0}^{t} [L_3 + H_3(\|x(s)\| + \|x_*(s)\|)] \|u(s) - u_*(s)\| \,\mathrm{d}s \\
&\quad \leqslant \int_{t_0}^{\theta} (L_3 + 2r_* H_3) \|u(s) - u_*(s)\| \,\mathrm{d}s \leqslant (L_3 + 2r_* H_3) \frac{\varepsilon}{k_*}
\end{aligned}
$$

for every $t \in [t_0, \theta]$. Since $L_0 \in [0, 1)$, we have from (4.7) and (4.8)

$$
\begin{aligned}
(4.9)\quad \|x(t) - x_*(t)\| &\leqslant \frac{\lambda (L_3 + 2r_* H_3)}{k_* (1 - L_0)} \varepsilon \\
&\quad + \frac{\lambda}{1 - L_0} \int_{t_0}^{t} [L_2 + H_2(\|u(s)\| + \|u_*(s)\|)] \|x(s) - x_*(s)\| \,\mathrm{d}s
\end{aligned}
$$

for every $t \in [t_0, \theta]$. Further, (2.4)–(2.6), (4.1), (4.3), (4.9), and Gronwall's inequality yield

(4.10)

$$
\begin{aligned}
&\|x(t) - x_*(t)\| \\
&\quad \leqslant \frac{\lambda (L_3 + 2r_* H_3)}{k_* (1 - L_0)} \varepsilon \exp\left(\frac{\lambda}{1 - L_0} \int_{t_0}^{t} [L_2 + H_2(\|u(s)\| + \|u_*(s)\|)] \,\mathrm{d}s \right) \\
&\quad \leqslant \frac{\lambda (L_3 + 2r_* H_3)}{k_* (1 - L_0)} \varepsilon \\
&\qquad \times \exp\left[\frac{\lambda}{1 - L_0} (L_2(\theta - t_0) + H_2((\theta - t_0)^{(p-1)/p} + (\theta - t_0)^{(p_*-1)/p_*}) \mu) \right] \\
&\quad \leqslant \frac{\lambda (L_3 + 2r_* H_3)}{k_* (1 - L_0)} \varepsilon \\
&\qquad \times \exp\left[\frac{\lambda}{1 - L_0} (L_2(\theta - t_0) + 2H_2 \max\{(\theta - t_0)^{(p-1)/p}, (\theta - t_0)^{(p_*-1)/p_*}\} \mu) \right] \\
&\quad \leqslant \frac{\lambda (L_3 + 2r_* H_3)}{k_* (1 - L_0)} \varepsilon \cdot \exp\left[\frac{L_*(\lambda) - L_0}{1 - L_0} \right] = \varepsilon
\end{aligned}
$$

for every $t \in [t_0, \theta]$. Thus, from (4.10) we get that for each $p \in (p_* - \delta, p_* + \delta)$ and $x(\cdot) \in \mathbf{X}_{p,\mu}$ there exists $x_*(\cdot) \in \mathbf{X}_{p_*,\mu}$ such that

$$\|x(\cdot) - x_*(\cdot)\|_C \leqslant \varepsilon$$

and hence,

$$\sup_{x(\cdot) \in \mathbf{X}_{p,\mu}} d(x(\cdot), \mathbf{X}_{p_*,\mu}) \leqslant \varepsilon. \tag{4.11}$$

Analogously, one can prove that for each $p \in (p_* - \delta, p_* + \delta)$ the inequality

$$\sup_{x_*(\cdot) \in \mathbf{X}_{p_*,\mu}} d(x_*(\cdot), \mathbf{X}_{p,\mu}) \leqslant \varepsilon \tag{4.12}$$

holds.

Inequalities (4.11) and (4.12) complete the proof. □

From Theorem 3 we find that the following corollary holds.

Corollary 2. *For each fixed $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$ the set valued map $p \to \mathbf{X}_{p,\mu}$ is continuous in the interval $(p_0 - \alpha, p_0 + \alpha)$.*

For each fixed $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$ the set valued map $p \mapsto \mathbf{X}_{p,\mu}(t)$ is continuous in the interval $(p_0 - \alpha, p_0 + \alpha)$. This continuity is uniform with respect to $t \in [t_0, \theta]$.

5. Upper estimate of the diameter of the set of trajectories

In this section we present an upper estimate for the diameter of the set of trajectories $X_{p,\mu}$. For a given metric space (Y, d_Y) and a set $E \subset Y$ the diameter of E is denoted by $\operatorname{diam} E$ and defined as

$$\operatorname{diam} E = \sup\{d_Y(x, y) \colon\ x \in E,\ y \in E\}.$$

For $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$, $p \in (p_0 - \alpha, p_0 + \alpha)$ and $t \in [t_0, \theta]$ we set

$$\begin{aligned} g(t; p, \mu) = {} & \frac{2\lambda}{1 - L_0}(L_3 + 2H_3 r_*)\mu(t - t_0)^{(p-1)/p} \\ & \times \exp\left[\frac{\lambda}{1 - L_0}(L_2(t - t_0) + 2H_2\mu(t - t_0)^{(p-1)/p})\right], \end{aligned} \tag{5.1}$$

where $\alpha > 0$ is given in Remark 1.

Theorem 4. *For each $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$, $p \in (p_0 - \alpha, p_0 + \alpha)$ the inequalities*

$$\operatorname{diam} \mathbf{X}_{p,\mu}(t) \leqslant g(t; p, \mu), \quad t \in [t_0, \theta], \tag{5.2}$$

$$\operatorname{diam} \mathbf{X}_{p,\mu} \leqslant g(\theta; p, \mu) \tag{5.3}$$

are satisfied.

Proof. Let us choose arbitrary $\mu \in (\mu_0 - \alpha, \mu_0 + \alpha)$, $p \in (p_0 - \alpha, p_0 + \alpha)$, $x_1(\cdot) \in \mathbf{X}_{p,\mu}$ and $x_2(\cdot) \in \mathbf{X}_{p,\mu}$. Then there exist $u_1(\cdot) \in U_{p,\mu}$ and $u_2(\cdot) \in U_{p,\mu}$ such that

$$x_1(t) = a(t, x_1(t)) + \lambda \int_{t_0}^{t} K(t, s, x_1(s), u_1(s))\, \mathrm{d}s, \tag{5.4}$$

$$x_2(t) = a(t, x_2(t)) + \lambda \int_{t_0}^{t} K(t, s, x_2(s), u_2(s))\, \mathrm{d}s \tag{5.5}$$

for every $t \in [t_0, \theta]$. From (5.4), (5.5), and condition 2.B it follows that

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\leqslant L_0 \|x_1(t) - x_2(t)\| \\ &\quad + \lambda \int_{t_0}^{t} [L_2 + H_2(\|u_1(s)\| + \|u_2(s)\|)] \|x_1(s) - x_2(s)\|\, \mathrm{d}s \\ &\quad + \lambda \int_{t_0}^{t} [L_3 + H_3(\|x_1(s)\| + \|x_2(s)\|)] \|u_1(s) - u_2(s)\|\, \mathrm{d}s \end{aligned} \tag{5.6}$$

for every $t \in [t_0, \theta]$. Since $u_1(\cdot) \in U_{p,\mu}$, $u_2(\cdot) \in U_{p,\mu}$, by (2.4) we have

$$\int_{t_0}^{t} \|u_1(s)\| + \|u_2(s)\|\, \mathrm{d}s \leqslant 2(t - t_0)^{(p-1)/p} \mu. \tag{5.7}$$

From $x_1(\cdot) \in \mathbf{X}_{p,\mu}$, $x_2(\cdot) \in \mathbf{X}_{p,\mu}$ and Proposition 3 we obtain

$$\|x_1(\cdot)\|_C \leqslant r_*, \quad \|x_2(\cdot)\|_C \leqslant r_*. \tag{5.8}$$

Then inequalities (5.7) and (5.8) yield

$$\begin{aligned} &\int_{t_0}^{t} [L_3 + H_3(\|x_1(s)\| + \|x_2(s)\|)] \|u_1(s) - u_2(s)\|\, \mathrm{d}s \\ &\qquad \leqslant \int_{t_0}^{t} (L_3 + 2H_3 r_*)(\|u_1(s)\| + \|u_2(s)\|)\, \mathrm{d}s \\ &\qquad \leqslant 2(L_3 + 2H_3 r_*) \mu (t - t_0)^{(p-1)/p} \end{aligned} \tag{5.9}$$

for every $t \in [t_0, \theta]$. Since $L_0 \in [0,1)$, inequalities (5.6) and (5.9) imply

$$
\begin{aligned}
(5.10)\quad \|x_1(t) - x_2(t)\| \leqslant{}& \frac{\lambda}{1-L_0} \int_{t_0}^{t} [L_2 + H_2(\|u_1(s)\| + \|u_2(s)\|)]\|x_1(s) - x_2(s)\| \,\mathrm{d}s \\
&+ \frac{2\lambda}{1-L_0}(L_3 + 2H_3 r_*)\mu(t-t_0)^{(p-1)/p}
\end{aligned}
$$

for every $t \in [t_0, \theta]$. Thus, from (5.1), (5.7), (5.10) and Gronwall's inequality we obtain

$$
\begin{aligned}
(5.11)\quad \|x_1(t) - x_2(t)\| &\leqslant \frac{2\lambda}{1-L_0}(L_3 + 2H_3 r_*)\mu(t-t_0)^{(p-1)/p} \\
&\quad \times \exp\left[\frac{\lambda}{1-L_0} \int_{t_0}^{t} (L_2 + H_2(\|u_1(s)\| + \|u_2(s)\|)) \,\mathrm{d}s\right] \\
&\leqslant \frac{2\lambda}{1-L_0}(L_3 + 2H_3 r_*)\mu(t-t_0)^{(p-1)/p} \\
&\quad \times \exp\left[\frac{\lambda}{1-L_0}(L_2(t-t_0) + 2H_2\mu(t-t_0)^{(p-1)/p})\right] \\
&= g(t; p, \mu)
\end{aligned}
$$

for every $t \in [t_0, \theta]$. Since $x_1(\cdot) \in \mathbf{X}_{p,\mu}$ and $x_2(\cdot) \in \mathbf{X}_{p,\mu}$ are arbitrarily chosen, (5.11) yields the validity of estimate (5.2).

Since the function $g(\cdot, p, \mu)\colon [t_0, \theta] \to [0, \infty)$ is monotone increasing, we have from (5.11) that

$$\|x(t) - y(t)\| \leqslant g(\theta; p, \mu)$$

for every $t \in [t_0, \theta]$ and consequently,

$$\|x(\cdot) - y(\cdot)\|_C = \max\{\|x(t) - y(t)\|\colon t \in [t_0, \theta]\} \leqslant g(\theta; p, \mu),$$

which implies the validity of inequality (5.3). $\square$

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