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DIRICHLET-NEUMANN ALTERNATING ALGORITHM FOR AN EXTERIOR ANISOTROPIC QUASILINEAR ELLIPTIC PROBLEM

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Abstract. In this paper, by the Kirchhoff transformation, a Dirichlet-Neumann (D-N) alternating algorithm which is a non-overlapping domain decomposition method based on natural boundary reduction is discussed for solving exterior anisotropic quasilinear problems with circular artificial boundary. By the principle of the natural boundary reduction, we obtain natural integral equation for the anisotropic quasilinear problems on circular artificial boundaries and construct the algorithm and analyze its convergence. Moreover, the convergence rate is obtained in detail for a typical domain. Finally, some numerical examples are presented to illustrate the feasibility of the method.

Keywords: quasilinear elliptic equation; domain decomposition method; natural integral equation

MSC 2010: 65N30, 35J65

1. Introduction

Based on natural boundary reduction [4], [13], the overlapping and non-overlapping domain decomposition methods can be viewed as effective ways to solve problems in unbounded domains. These techniques have been used to solve many linear problems [11], [12], [13], [14] and they have also been generalized to linear or nonlinear wave problems [2], [1], [3]. In this paper, we consider a non-overlapping domain decomposition method for an exterior anisotropic quasilinear elliptic problem with circular artificial boundary. By the Kirchhoff transformation, we shall discuss some

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exterior anisotropic quasilinear elliptic problems [5], [7], [6], [9], [10] using the non-overlapping domain decomposition method.

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega = \Gamma_0$ and let $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$. We consider the numerical solution of the exterior quasilinear problem

$$\text{(1.1)}\qquad \begin{cases} -\Big(\dfrac{\partial}{\partial x}\Big(\alpha a(\boldsymbol{x},u)\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(\beta a(\boldsymbol{x},u)\dfrac{\partial u}{\partial y}\Big)\Big) = f, & \text{in } \Omega^c,\\ u = 0, & \text{on } \Gamma_0,\\ u(\boldsymbol{x}) = \mathcal{O}(1), & \text{as } |\boldsymbol{x}| \to \infty, \end{cases}$$

with $\beta > \alpha > 0$ or $\alpha = \beta = 1$, $\boldsymbol{x} = (x, y)$, $a(\cdot,\cdot)$ and f are given functions which will be ranked as below. Following [5], [6], suppose that the given function $a(\cdot,\cdot)$ satisfies

$$\text{(1.2)}\qquad 0 < C_0 \leqslant a(\boldsymbol{x},u) \leqslant C_1, \quad \forall u \in \mathbb{R}, \text{ and for almost all } \boldsymbol{x} \in \Omega^c,$$

with two constants $C_0, C_1 \in \mathbb{R}$, and

$$\text{(1.3)}\qquad |a(\boldsymbol{x},u) - a(\boldsymbol{x},v)| \leqslant C_L|u - v|, \quad \forall u, v \in \mathbb{R}, \text{ and for almost all } \boldsymbol{x} \in \Omega^c,$$

with a constant $C_L > 0$. In the following, we suppose that the function $f \in L^2(\Omega^c)$ has compact support, i.e., there exists a constant $\Gamma_0 > 0$, such that

$$\text{(1.4)}\qquad \operatorname{supp} f \subset \Omega_{R_0} = \{\boldsymbol{x};\ \boldsymbol{x} \in \mathbb{R}^2, |\boldsymbol{x}| \leqslant \Gamma_0\}.$$

We also assume that

$$\text{(1.5)}\qquad a(\boldsymbol{x},u) \triangleq a_0(u), \quad \text{when } |\boldsymbol{x}| \geqslant \Gamma_0.$$

Now, we introduce a circular arc Γ_1 in Ω^c with radius R centered at the origin, enclosing Γ_0 such that $R > \Gamma_0 > 0$ and $\operatorname{dist}(\Gamma_1, \Gamma_0) = \delta_0 > 0$. Then, Ω^c is divided into two non-overlapping subdomains Ω_1 and Ω_2 (see Fig. 1.1), where Ω_1 denotes the bounded domain between Γ_0 and Γ_1 and Ω_2 refers to the unbounded domain outside Γ_1. The original problem (1.1) can be decomposed into two subproblems in domains Ω_1 and Ω_2, respectively, with $\Omega_1 \cap \Omega_2 = \emptyset$. We have the following D-N alternating algorithm:

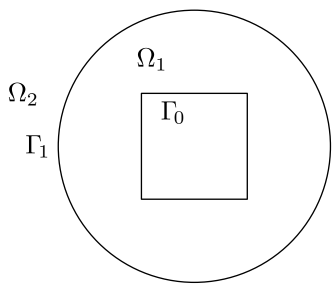

Figure 1.1. The illustration of the domains Ω_1 and Ω_2.

Step 1: Choose an initial value $\lambda^0 \in H^{1/2}(\Gamma_1)$, and put $k = 0$.

Step 2: Solve a Dirichlet boundary value problem in the exterior domain Ω_2

$$\begin{cases} -\Big(\dfrac{\partial}{\partial x}\Big(\alpha a(\boldsymbol{x}, u_2^k)\dfrac{\partial u_2^k}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(\beta a(\boldsymbol{x}, u_2^k)\dfrac{\partial u_2^k}{\partial y}\Big)\Big) = 0, & \text{in } \Omega_2, \\ u_2^k = \lambda^k, & \text{on } \Gamma_1, \\ u_2^k(\boldsymbol{x}) = \mathcal{O}(1), & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{1.6}$$

Step 3: Solve a mixed boundary value problem in the interior domain Ω_1

$$\begin{cases} -\Big(\dfrac{\partial}{\partial x}\Big(\alpha a(\boldsymbol{x}, u_1^k)\dfrac{\partial u_1^k}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(\beta a(\boldsymbol{x}, u_1^k)\dfrac{\partial u_1^k}{\partial y}\Big)\Big) = f, & \text{in } \Omega_1, \\ \dfrac{\partial u_1^k}{\partial \boldsymbol{n}_1} = -\dfrac{\partial u_2^k}{\partial \boldsymbol{n}_2}, & \text{on } \Gamma_1, \\ u_1^k = 0, & \text{on } \Gamma_0, \end{cases} \tag{1.7}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the unit exterior normal vectors on Γ_1.

Step 4: Update the boundary value $0 < \theta_k < 1$,

$$\lambda^{k+1} = \theta_k u_1^k + (1 - \theta_k)\lambda^k, \quad \text{on } \Gamma_1.$$

Step 5: Put $k = k + 1$, and go to Step 2.

The relaxation factor θ_k is a suitably chosen real number. Notice that, in Step 3 we solve the problem (1.7) by the standard finite element method and only need the normal derivative of the solution to the problem (1.6) in Step 2. So we need not to solve (1.6) directly, based on the Kirchhoff transformation, the natural integral equation for the quasilinear problem can be obtained by the natural boundary element method [11], [13]. In particular, when $a(\boldsymbol{x}, u) = c$ which is independent of $\boldsymbol{x}$ and u, [12], [13], [14] have discussed the corresponding problems by this technique. Now, we introduce the so-called Kirichhoff transformation [8]

$$w(\boldsymbol{x}) = \int_0^{u(\boldsymbol{x})} a_0(\xi)\,\mathrm{d}\xi, \quad \text{for } \boldsymbol{x} \in \Omega_2. \tag{1.8}$$

Then we have

$$\nabla w = a_0(u)\nabla u \tag{1.9}$$

and

$$\Big(\alpha\frac{\partial w}{\partial x}, \beta\frac{\partial w}{\partial y}\Big) = \Big(\alpha a_0(u)\frac{\partial u}{\partial x}, \beta a_0(u)\frac{\partial u}{\partial y}\Big). \tag{1.10}$$

By (1.6), one obtains that w satisfies the following problem

$$(1.11)\qquad \begin{cases} -\Big(\alpha\dfrac{\partial^2 w^k}{\partial x^2}+\beta\dfrac{\partial^2 w^k}{\partial y^2}\Big)=0, & \text{in } \Omega_2,\\ w^k=\displaystyle\int_0^{\lambda^k} a_0(\xi)\,\mathrm{d}\xi, & \text{on } \Gamma_1. \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we obtain the natural integral equation for the circular unbounded domain cases. In Section 3, we discuss the convergence of the D-N algorithm and analyze its convergence rate. At last, in Section 4, we present some numerical examples to present the efficiency and feasibility of our method.

2. Exact quasilinear artificial boundary condition

In this section, by virtue of the Poisson integral formula and natural integral equation for the linear problem, we shall obtain the corresponding results for the quasilinear problem in Ω_2.

2.1. Natural integral equation for $\alpha=\beta=1$. Assume that $w(\boldsymbol{x})$ is the solution of the problem (1.11), and the value $w|_{\Gamma_1}$ is given, namely

$$w|_{\Gamma_1}=w(R,\theta).$$

Then, based on the natural boundary reduction, there are the Poisson integral formulas in the Fourier expansion [5], [9], [13]:

$$(2.1)\qquad w(r,\theta)=\frac{c_0}{2}+\sum_{j=1}^{\infty}\Big(\frac{R}{r}\Big)^j(c_j\cos j\theta+d_j\sin j\theta),$$

with

$$(2.2)\qquad c_j=\frac{1}{\pi}\int_0^{2\pi} w(R,\theta)\cos j\theta\,\mathrm{d}\theta,\quad j=0,1,\dots,$$

$$(2.3)\qquad d_j=\frac{1}{\pi}\int_0^{2\pi} w(R,\theta)\sin j\theta\,\mathrm{d}\theta,\quad j=1,2,\dots$$

So, we have

$$(2.4)\qquad \frac{\partial w}{\partial r}(r,\theta)\Big|_{r=R}=-\frac{1}{R\pi}\sum_{j=1}^{\infty} j\int_0^{2\pi} w(R,\theta')\cos j(\theta'-\theta)\,\mathrm{d}\theta'.$$

From (1.10), we obtain

$$\frac{\partial w}{\partial \boldsymbol{n}} = a_0(u)\frac{\partial u}{\partial \boldsymbol{n}}. \tag{2.5}$$

Combining (1.9), (2.4), and (2.5), we get the exact artificial boundary condition for u on Γ_1,

$$\begin{aligned}\Big(a_0(u)\frac{\partial u(r,\theta)}{\partial \boldsymbol{n}}\Big)\Big|_{r=R} &= -\frac{1}{R\pi}\int_0^{2\pi}\sum_{j=1}^{\infty}\Big(\int_0^{u(R,\theta')} a_0(y)\,\mathrm{d}y\Big) j\cos j(\theta'-\theta)\,\mathrm{d}\theta' \\ &\triangleq \mathscr{K}_1(u(R,\theta)).\end{aligned} \tag{2.6}$$

2.2. Natural integral equation for $\beta > \alpha > 0$. Now we assume that $\beta > \alpha > 0$. We let $w(\boldsymbol{x})$ be a solution of problem (1.11) and let the value $w|_{\Gamma_1}$ be given, namely

$$w|_{\Gamma_1} = w(R,\theta).$$

Let $x = \sqrt{\alpha}\xi$ and $y = \sqrt{\beta}\eta$. Then the boundary Γ_1 is changed into the elliptic boundary $\widetilde{\Gamma}_1 = \{(\xi,\eta);\ \alpha\xi^2 + \beta\eta^2 = R^2\}$. Assume $\xi = (R/\sqrt{\alpha})\cos\varphi$, $\eta = (R/\sqrt{\beta})\sin\varphi$, then the unit exterior normal vector on $\widetilde{\Gamma}_1$ is

$$\boldsymbol{\nu} = -\frac{R}{\sqrt{\alpha x^2 + \beta y^2}}(\sqrt{\alpha}\cos\varphi, \sqrt{\beta}\sin\varphi).$$

By the above transformation, the problem (1.11) changes into

$$\begin{cases} -\Big(\dfrac{\partial^2 w}{\partial \xi^2} + \dfrac{\partial^2 w}{\partial \eta^2}\Big) = 0, & \text{in } \widetilde{\Omega}_2, \\ w = w_0, & \text{on } \widetilde{\Gamma}_1. \end{cases} \tag{2.7}$$

Now, we introduce elliptic coordinates (μ,φ):

$$\xi = f_0\cosh\mu\cos\varphi, \quad \eta = f_0\sinh\mu\sin\varphi,$$

with $f_0 = \sqrt{(\beta-\alpha)/(\alpha\beta)}R$, $\mu_0 = \ln\big((\sqrt{b}+\sqrt{a})/\sqrt{b-a}\big)$, $\widetilde{\Gamma}_1 = \{(\mu,\varphi);\ \mu = \mu_0, \varphi \in [0,2\pi]\}$ and $\widetilde{\Omega}_2 = \{(\mu,\varphi);\ \mu > \mu_0, \varphi \in [0,2\pi]\}$.

Letting

$$J(\mu,\varphi) = \begin{vmatrix} \frac{\partial\xi}{\partial\mu} & \frac{\partial\xi}{\partial\varphi} \\ \frac{\partial\eta}{\partial\mu} & \frac{\partial\eta}{\partial\varphi} \end{vmatrix},$$

then $J(\mu,\varphi) = f_0^2(\sinh^2\mu\cos^2\varphi + \cosh\mu\sin^2\varphi)$ and $J(\mu_0,\varphi) = (R^2/\alpha\beta)(\beta\sin^2\varphi + \alpha\cos^2\varphi) \triangleq J_0$. Based on the natural boundary reduction, there are the Poisson integral formulas

$$w(\mu,\varphi) = \frac{\mathrm{e}^{2\mu} - \mathrm{e}^{2\mu_0}}{2\pi}\int_0^{2\pi}\frac{w_0(\mu_0,\varphi')}{\mathrm{e}^{2\mu} + \mathrm{e}^{2\mu_0} - 2\mathrm{e}^{\mu+\mu_0}\cos(\varphi-\varphi')}\,\mathrm{d}\varphi', \quad \mu > \mu_0, \tag{2.8}$$

and the natural integral equation

$$\frac{\partial w}{\partial \boldsymbol{\nu}} = \frac{1}{\sqrt{J_0}}\Big[-\frac{1}{4\pi\sin^2\frac{\varphi}{2}} * w_0(\mu_0,\varphi)\Big] = \frac{1}{\pi\sqrt{J_0}}\sum_{j=1}^{\infty} j\int_0^{2\pi}\cos j(\varphi-\varphi')w_0(\mu_0,\varphi')\,\mathrm{d}\varphi. \tag{2.9}$$

Hence, for the original problem (1.11), we have the natural integral equation

$$\alpha n_x\frac{\partial w}{\partial x} + \beta n_y\frac{\partial w}{\partial y} = -\frac{\sqrt{\alpha\beta}}{4\pi R\sin^2\theta/2} * w_0(R,\theta) = -\frac{\sqrt{\alpha\beta}}{R\pi}\int_0^{2\pi}\sum_{j=1}^{\infty} w_0(R,\theta')j\cos j(\theta'-\theta)\,\mathrm{d}\theta', \tag{2.10}$$

where $(n_x, n_y) = (x/R, y/R)$ is the unit exterior normal vector on Γ_1. From (1.11), we obtain

$$\alpha n_x\frac{\partial w}{\partial x} + \beta n_y\frac{\partial w}{\partial y} = \alpha n_x a_0(u)\frac{\partial u}{\partial x} + \beta n_y a_0(u)\frac{\partial u}{\partial y}. \tag{2.11}$$

Combining (1.9), (2.10) and (2.11), we obtain the exact artificial boundary condition for u on Γ_1,

$$\Big(\alpha n_x a_0(u)\frac{\partial u}{\partial x} + \beta n_y a_0(u)\frac{\partial u}{\partial y}\Big)\Big|_{r=R} = -\frac{\sqrt{\alpha\beta}}{R\pi}\int_0^{2\pi}\sum_{j=1}^{\infty}\Big(\int_0^{u(R,\theta')} a_0(y)\,\mathrm{d}y\Big)j\cos j(\theta'-\theta)\,\mathrm{d}\theta' \triangleq \mathscr{K}_1(u(R,\theta)). \tag{2.12}$$

3. Variational problem and convergence analysis of the algorithm

Now, we consider the equation (1.7). We shall use $W^{m,p}$ to denote the standard Sobolev spaces, $\|\cdot\|$ and $|\cdot|$ referring to the corresponding norms and semi-norms. Especially, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,\Omega} = |\cdot|_{m,2,\Omega}$. Let us introduce the space

$$V = \{v;\ v\in H^1(\Omega_1), v|_{\Gamma_0} = 0\}, \tag{3.1}$$

and the corresponding norms

$$\|v\|_{0,\Omega_1} = \sqrt{\int_{\Omega_1}|v|^2\,\mathrm{d}\boldsymbol{x}},\quad \|v\|_{1,\Omega_1} = \sqrt{\int_{\Omega_1}(|v|^2+|\nabla v|^2)\,\mathrm{d}\boldsymbol{x}}.$$

The boundary value problem (1.7) is equivalent to the following variational problem

$$\tag{3.2} \begin{cases} \text{find } u \in V, \text{ such that} \\ D(u;u,v) + \widehat{D}(u;u,v) = F(v), \quad \forall v \in V, \end{cases}$$

where

$$\tag{3.3} D(w;u,v) = \int_{\Omega_1} a(\boldsymbol{x},w)\Big(\alpha \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big)\,\mathrm{d}\boldsymbol{x},$$

$$\tag{3.4} \begin{aligned} \widehat{D}(w;u,v) = \sum_{j=1}^{\infty} \frac{\sqrt{\alpha\beta}}{j\pi} \int_0^{2\pi}\!\!\int_0^{2\pi} a_0(w(R,\theta')) \frac{\partial u(R,\theta')}{\partial \theta'} \\ \times \frac{\partial v(R,\theta)}{\partial \theta} \cos j(\theta'-\theta)\,\mathrm{d}\theta'\,\mathrm{d}\theta, \end{aligned}$$

and

$$\tag{3.5} F(v) = \int_{\Omega_1} f(\boldsymbol{x}) v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

3.1. D-N alternating algorithm and convergence analysis. Divide the arc Γ_1 into M parts and take a finite element subdivision in Ω_1 such that their nodes on Γ_1 are coincident. That is, we make a regular and quasi-uniform triangulation $\mathscr{T}_h$ on Ω_1, such that

$$\tag{3.6} \Omega_1 = \bigcup_{K\in\mathscr{T}_h} K,$$

where K is a (curved) triangle and h the maximal diameter of the triangles. Let

$$\tag{3.7} V_h = \{v_h;\ v_h \in V, v|_K \text{ is a linear polynomial}, \ \forall K \in \mathscr{T}_h\}.$$

Then the approximate problem of (3.2) can be written as

$$\tag{3.8} \begin{cases} \text{find } u_h \in V_h, \text{ such that} \\ D(u_h;u_h,v_h) + \widehat{D}(u_h;u_h,v_h) = F(v_h), \quad \forall v_h \in V_h, \end{cases}$$

where

$$\tag{3.9} D(w_h;u_h,v_h) = \int_{\Omega_1} a(\boldsymbol{x},w_h)\Big(\alpha \frac{\partial u_h}{\partial x}\frac{\partial v_h}{\partial x} + \beta \frac{\partial u_h}{\partial y}\frac{\partial v_h}{\partial y}\Big)\,\mathrm{d}\boldsymbol{x},$$

$$\tag{3.10} \begin{aligned} \widehat{D}(w_h;u_h,v_h) = \sum_{j=1}^{\infty} \frac{\sqrt{\alpha\beta}}{j\pi} \int_0^{2\pi}\!\!\int_0^{2\pi} a_0(w_h(R,\theta')) \frac{\partial u_h(R,\theta')}{\partial \theta'} \\ \times \frac{\partial v_h(R,\theta)}{\partial \theta} \cos j(\theta'-\theta)\,\mathrm{d}\theta'\,\mathrm{d}\theta. \end{aligned}$$

Some existence and uniqueness results for this type of problem are given in [7], [6] under some conditions on the coefficients a. So, by the constraint conditions (1.2)–(1.3), we have

Lemma 3.1. *The problems* (3.2) *and* (3.8) *have unique solvability.*

In practice, the sum in (3.10) is truncated to a finite number of terms N. By the hypothesis on $a(\cdot,\cdot)$, it is not difficult to show that $D(\cdot;\cdot,\cdot)$ is a positive definite bilinear form on $V \times V$ and $V_h \times V_h$. For $\widehat{D}(\cdot;\cdot,\cdot)$, similarly as in the proof in [5], [9], we have the following conclusion.

Lemma 3.2. *There exists a constant $C > 0$ which has different meaning in different places and is related to α and β, such that*

$$|\widehat{D}(w;u,v)| \leqslant C\|u\|_{1,\Omega_1}\|v\|_{1,\Omega_1}, \ \widehat{D}(u;u,u) \geqslant C_0|u|^2_{1,\Omega_1}, \quad \forall u,v,w \in V.$$

From the discrete problem (3.8), we can get a system of algebraic equations of the following form

$$\begin{pmatrix} A_{11}+K_h & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{pmatrix}, \tag{3.11}$$

where $\boldsymbol{U}$ is a vector whose components are function values at the nodes on Γ_1, and $\boldsymbol{V}$ is a vector whose components are function values at the interior nodes of Ω_1. The matrix $A \triangleq A(u) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is the stiffness matrix obtained from the finite element method in Ω_1 while $K_h \triangleq K_h(u|_{\Gamma_1})$ is gotten from the natural boundary element method on Γ_1.

The equation (3.11) can also be rewritten as follows

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} -K_h\boldsymbol{U} \\ \boldsymbol{b} \end{pmatrix}. \tag{3.12}$$

Then, we have the iterative algorithm

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{U}_k \\ \boldsymbol{V}_k \end{pmatrix} = \begin{pmatrix} -K_h\Lambda_k \\ \boldsymbol{b} \end{pmatrix}, \tag{3.13}$$

with

$$\Lambda_{k+1} = \theta_k\boldsymbol{U}_k + (1-\theta_k)\Lambda_k, \quad k = 0,1,\ldots \tag{3.14}$$

Since A is a positive definite matrix, we know that A_{22}^{-1} exists. Now, we let $S_h = S_h^{(1)} + K_h$ be the discrete analogue of the Steklov-Poincaré operator on Γ_1, with $S_h^{(1)} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, and $\boldsymbol{B} = -A_{12}A_{22}^{-1}\boldsymbol{b}$. Then, similarly to the proof of [13], [14], we conclude that the alternating algorithm (3.13)–(3.14) is equivalent to the preconditioned Richardson iteration:

$$S_h^{(1)}(\Lambda^{k+1} - \Lambda^k) = \theta_k(\boldsymbol{B} - S_h\Lambda^k). \tag{3.15}$$

And we also have the following convergence result:

Theorem 3.1. *If* $0 < \min\theta_k \leqslant \max\theta_k < 1$, *then the discrete non-overlapping alternating method* (3.13)–(3.14) *is convergent, and both the convergence rate and the condition number of the iterative matrix* $[S_h^{(1)}]^{-1}S_h$ *are independent of the finite element mesh size* h.

3.2. Convergence analysis of the method in continuous cases. From (1.8)–(1.9), the original problem (1.1) can be changed to

$$\begin{cases} -\Big(\alpha\dfrac{\partial^2 w}{\partial x^2} + \beta\dfrac{\partial^2 w}{\partial y^2}\Big) = f(\boldsymbol{x}), & \text{in } \Omega^c, \\ w = 0, & \text{on } \Gamma_0, \\ w(\boldsymbol{x}) = \mathcal{O}(1), & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{3.16}$$

Then, we let $x = \sqrt{\alpha}\xi$, $y = \sqrt{\beta}\eta$. The equation (3.16) becomes

$$\begin{cases} -\Delta w = -\Big(\dfrac{\partial^2 w}{\partial \xi^2} + \dfrac{\partial^2 w}{\partial \eta^2}\Big) = f(\boldsymbol{x}), & \text{in } \widetilde{\Omega}^c, \\ w = 0, & \text{on } \widetilde{\Gamma}_0, \\ w(\boldsymbol{x}) = \mathcal{O}(1), & \text{as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{3.17}$$

where $\widetilde{\Omega}^c$ and $\widetilde{\Gamma}_0$ are the corresponding images of Ω^c and Γ_0, respectively. Let g be extended to $\widetilde{\Omega}^c$, $w = u - g$, $f = \Delta g$, then the equation (3.17) is equivalent to

$$\begin{cases} -\Delta u = 0, & \text{in } \widetilde{\Omega}^c, \\ u = g, & \text{on } \widetilde{\Gamma}_0, \\ u(\boldsymbol{x}) = \mathcal{O}(1), & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{3.18}$$

Since it is difficult to estimate the convergence rate for a general unbounded domain Ω^c, we here let Ω^c be an exterior domain of a circle Γ_0, with radius $r = R_0$ and Γ_1 is taken as stated in Section 1. For the case $\beta > \alpha > 0$, Γ_0 and Γ_1 will be changed to ellipses $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$, respectively. We introduce the following conclusions for $\alpha = \beta = 1$ and $\beta > \alpha > 0$, respectively.

Lemma 3.3. *If u is the solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega_1, \\ u = u_0, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = u_n, & \text{on } \Gamma_1, \end{cases} \tag{3.19}$$

where Ω_1 is the annular domain between Γ_0 and Γ_1,

$$u_0 = \sum_{m=-\infty}^{\infty} a_m \mathrm{e}^{\mathrm{i}m\varphi} \in H^{1/2}(\Gamma_0), \quad u_n = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} b_m |m| \mathrm{e}^{\mathrm{i}m\varphi} + b_0 \in H^{-1/2}(\Gamma_1),$$

then, there exists a unique $u \in H^1(\Omega_1)$ and

$$u(r,\varphi) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{a_m R_0^{|m|}(r^{|m|} + R^{2|m|}r^{-|m|}) + b_m R^{|m|+1}(r^{|m|} - R_0^{2|m|}r^{-|m|})}{R_0^{2|m|} + R^{2|m|}} \mathrm{e}^{\mathrm{i}m\varphi} + a_0 + R b_0 \ln \frac{r}{R_0}.$$

P r o o f. The result can be obtained directly from (3.19) by separation of variables. □

Lemma 3.4. *If u is the solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega_1, \\ u = u_0, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = u_n, & \text{on } \Gamma_1, \end{cases} \tag{3.20}$$

where Ω_1 is the elliptical ring domain between Γ_0 and Γ_1,

$$u_0 = \sum_{m=-\infty}^{\infty} c_m \mathrm{e}^{\mathrm{i}m\varphi} \in H^{1/2}(\Gamma_0), \quad u_n = \frac{1}{\sqrt{J_0}} \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} d_m |m| \mathrm{e}^{\mathrm{i}m\varphi} + d_0 \right) \in H^{-1/2}(\Gamma_1),$$

then, there exists a unique $u \in H^1(\Omega_1)$ and

$$u(\mu,\varphi) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{c_m(\mathrm{e}^{|m|(\mu-\mu_1)} + \mathrm{e}^{|m|(\mu_1-\mu)}) + d_m(\mathrm{e}^{|m|(\mu-\mu_0)} - \mathrm{e}^{|m|(\mu_0-\mu)})}{\mathrm{e}^{|m|(\mu_1-\mu_0)} + \mathrm{e}^{|m|(\mu_0-\mu_1)}} \mathrm{e}^{\mathrm{i}m\varphi} + c_0 + d_0(\mu - \mu_0).$$

P r o o f. The result also can be obtained directly from (3.20) by the separation of variables. □

Theorem 3.2. *If* $0 < \theta_k < 1$, *then the non-overlapping domain decomposition method* (1.6)–(1.7) *is convergent.*

Proof. We only focus on the case $\beta > \alpha > 0$, the other can be discussed similarly.

We assume the exact solution of (1.1) is u and we let $\lambda = u|_{\Gamma_1}$, $u_k = u|_{\Omega_k}$, $k = 1, 2$. Then, following (1.6)–(1.7), we let $e_1^k = \lambda - u_1^k$ and $e_1^k|_{\Gamma_1} = \lambda - \lambda^k \triangleq e_2^k$. We let $e_2^k = \sum\limits_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\varphi} \in H^{1/2}(\Gamma_1)$. By the natural integral equation, we have

$$\frac{\partial e_1^k}{\partial \boldsymbol{n}} = -\mathscr{K}_1(e_2^k) = -\frac{1}{\sqrt{J_0}} \sum_{n=-\infty}^{\infty} |n| a_n \mathrm{e}^{\mathrm{i}n\varphi}.$$

So, e_1^k satisfies

$$\begin{cases} -\Delta e_1^k = 0, & \text{in } \Omega_1, \\ e_1^k = 0, & \text{on } \Gamma_0, \\ \dfrac{\partial e_1^k}{\partial \boldsymbol{n}} = \dfrac{1}{\sqrt{J_0}} \displaystyle\sum_{n=-\infty}^{\infty} |n| a_n \mathrm{e}^{\mathrm{i}n\varphi}, & \text{on } \Gamma_1. \end{cases} \tag{3.21}$$

By Lemma 3.2, one obtains

$$e_1^k = -\sum_{n=-\infty}^{\infty} a_n H_n(\mu) \mathrm{e}^{\mathrm{i}n\varphi}, \tag{3.22}$$

with $H_n(\mu) = (\mathrm{e}^{|n|(\mu-\mu_0)} - \mathrm{e}^{|n|(\mu_0-\mu)})/(\mathrm{e}^{|n|(\mu_1-\mu_0)} + \mathrm{e}^{|n|(\mu_0-\mu_1)})$. From (3.22), the restriction of e_1^k to Γ_1 can be expressed as

$$e_1^k|_{\Gamma_1} = -\sum_{n=-\infty}^{\infty} a_n H_n(\mu_1) \mathrm{e}^{\mathrm{i}n\varphi},$$

and

$$\mathscr{K}_1(e_1^k) = -\frac{1}{\sqrt{J_0}} \sum_{n=-\infty}^{\infty} |n| a_n H_n(\mu_1) \mathrm{e}^{\mathrm{i}n\varphi}.$$

Thus, we have

$$\begin{aligned} \frac{\partial e_1^{k+1}}{\partial \boldsymbol{n}} &= -\mathscr{K}_1(\lambda - \lambda^{k+1}) = \mathscr{K}_1(\theta_k u_1^k + (1-\theta_k)\lambda^k - \lambda) \\ &= -\theta_k \mathscr{K}_1(e_1^k) - (1-\theta_k)\mathscr{K}_1(e_2^k) \\ &= \frac{1}{\sqrt{J_0}} \sum_{n=-\infty}^{\infty} |n| a_n (\theta_k H_n(\mu_1) - 1 + \theta_k) \mathrm{e}^{\mathrm{i}n\varphi}. \end{aligned}$$

Let $E^n \triangleq \|\partial e_1^k/\partial \boldsymbol{n}\|^2_{-1/2,\Gamma_1}$. Then $E^n = 2\pi \sum\limits_{n=-\infty}^{\infty} \big(n^2/\sqrt{1+n^2}\big)|a_n|^2$ and

$$(3.23)\quad E^{n+1} = 2\pi \sum_{n=-\infty}^{\infty} \frac{n^2}{\sqrt{1+n^2}}|a_n|^2(\theta_k H_n(\mu_1) - 1 + \theta_k)^2$$
$$= (1-\theta_k)^2 E^n + 2\pi \sum_{n=-\infty}^{\infty} \frac{n^2}{\sqrt{1+n^2}}|a_n|^2 \theta_k H_n(\mu_1)[\theta_k(H_n(\mu_1)+2) - 2].$$

Assume $\delta_1 = \inf\limits_{n\in\mathbb{Z}\setminus\{0\}} 2/(2+H_n(\mu_1))$, then $1 > \delta_1 \geqslant 2/3$.

If $0 < \theta_k \leqslant \delta_1$, $k = 0, 1, 2, \ldots$, then

$$(3.24)\qquad E^{n+1} < (1-\theta_k)^2 E^n,$$

or equally

$$(3.25)\qquad E^{n+1} < \prod_{j=1}^{n}(1-\theta_j)^2 E^1 \leqslant r^n E^1, \quad \frac{1}{9} \leqslant r < 1.$$

By the trace theorem, we have

$$(3.26)\qquad \|e_1^k\|^2_{1,\Omega_1} \leqslant C E^n \to 0, \quad n \to \infty.$$

From (3.23), one also has

$$(3.27)\quad E^{n+1} = 2\pi \sum_{n=-\infty}^{\infty} \frac{n^2}{\sqrt{1+n^2}}|a_n|^2(\theta_k H_n(\mu_1) - 1 + \theta_k)^2$$
$$= (1-2\theta_k)^2 E^n + 2\pi \sum_{n=-\infty}^{\infty} \frac{n^2}{\sqrt{1+n^2}}|a_n|^2 \theta_k I_n(\mu_1)[\theta_k(I_n(\mu_1)-2) + 1],$$

with $I_n(\mu_1) = (1 - H_n(\mu_1))/2$. Assume $\delta_2 = \sup\limits_{n\in\mathbb{Z}\setminus\{0\}} 1/(2 - I_n(\mu_1))$, then $0 < \delta_2 \leqslant 2/3$.

For $\delta_2 \leqslant \theta_k < 1$, $k = 0, 1, 2, \ldots$, the convergence result can be obtained similarly to (3.24)–(3.26). Therefore, for $0 < \theta_k < 1$, the non-overlapping domain decomposition method is convergent. □

4. Numerical examples

In this section, we shall give some examples to illustrate our theoretical results. In the following, we choose the finite element space as given in (3.7). For simplicity, we let

$$\Delta r = \frac{1}{m}, \quad \Delta\theta = \frac{2\pi}{M}, \quad e(k) = \|u - u_{h,N}^k\|_{L^\infty(\Omega_i)}.$$

Moreover, let $e_h(k)$ denote the maximal error between the iteration $k-1$ and k, that is, $e_h(k) = \|u_{h,N}^k - u_{h,N}^{k-1}\|_{L^\infty(\Omega_i)}$, and let $q_h(k) = e_h(k-1)/e_h(k)$ simulate the convergence rate.

Example 4.1. We take $\Omega^c = \{(x,y);\ x,y \in \mathbb{R}, r = \sqrt{x^2+y^2} > 1\}$ and with boundary $\Gamma_0 = \{(1,\theta);\ \theta \in [0,2\pi]\}$, $\Gamma_R = \{(2,\theta);\ \theta \in [0,2\pi]\}$. We show our numerical results for problem (1.1) with $\alpha = \beta = 1$, where

$$a(\boldsymbol{x},u) = \begin{cases} 4 - r^2 + \dfrac{1}{1+u^2}, & 1 \leqslant r \leqslant 2, \\ \dfrac{1}{1+u^2}, & r > 2, \end{cases} \tag{4.1}$$

$$f(\boldsymbol{x}) = \begin{cases} -\Big(1 + \tan^2 \dfrac{y}{r^2}\Big)\Big(\dfrac{2y}{r^2} + \dfrac{2(4-r^2)}{r^4} \tan \dfrac{y}{r^2}\Big), & 1 \leqslant r \leqslant 2, \\ 0, & r > 2. \end{cases} \tag{4.2}$$

The exact solution of Example 4.1 is $u = \tan(y/r^2)$. The numerical results are given in Table 4.1.

Example 4.2. We assume the exterior domain Ω^c with boundary $\Gamma_0 = \{(1.5,\theta);\ \theta \in [0,2\pi]\}$, $\Gamma_R = \{(3,\theta);\ \theta \in [0,2\pi]\}$. Now we consider the problem

$$\begin{cases} -\Big(\dfrac{\partial}{\partial x}\Big(\varepsilon a(x,u)\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(a(x,u)\dfrac{\partial u}{\partial y}\Big)\Big) = f(\boldsymbol{x}), & \text{in } \Omega_i, \\ u = 0, & \text{on } \Gamma_0, \\ \varepsilon n_x a_0(u)\dfrac{\partial u}{\partial x} + n_y a_0(u)\dfrac{\partial u}{\partial y} = \mathscr{K}_1(u(R,\theta)), & \text{on } \Gamma_R, \end{cases} \tag{4.3}$$

where $a(x,u) = 1/(1+u^2)$ and $f = (2y(1-\varepsilon)(3x^2-y^2))/(x^2+y^2)^3$.

The exact solution of Example 4.2 is $u = \tan(y/r^2)$. The numerical results are given in Table 4.2 and Figure 4.1.

Example 4.3. Similar with Example 4.2, $a(x,u)$ is replaced by $a(x,u) = 1/\sqrt{1-u^2}$. And we take $f = 2x(1-\varepsilon)(x^2-3y^2)/(x^2+y^2)^3$.

The exact solution of Example 4.3 is $u = \sin(x/r^2)$. The numerical results are given in Tables 4.3, 4.4, and Figure 4.2.

(m, M)	error	Iteration number							
		0	1	2	3	4	5	6	9
	e	3.1267E-01	2.6695E-01	2.3989E-01	2.2255E-01	2.1096E-01	2.0294E-01	1.9719E-01	1.8723E-01
$(2, 8)$	e_h	–	4.5716E-02	2.7057E-02	1.7339E-02	1.1590E-02	8.0264E-03	5.7445E-03	2.5046E-03
	q_h	–	–	1.6896	1.5605	1.4961	1.4440	1.3972	1.2847
	e	2.1516E-01	1.5965E-01	1.2520E-01	1.0330E-01	8.8956E-02	7.9245E-02	7.2452E-02	6.1109E-02
$(4, 16)$	e_h	–	5.5517E-02	3.4447E-02	2.1896E-02	1.4348E-02	9.7116E-03	6.7924E-03	2.7899E-03
	q_h	–	–	1.6117	1.5732	1.5261	1.4774	1.4298	1.3078
	e	1.8696E-01	1.2285E-01	8.4593E-02	6.1009E-02	4.5959E-02	3.6011E-02	2.9234E-02	1.8980E-02
$(8, 32)$	e_h	–	6.4629E-02	3.8734E-02	2.3925E-02	1.5290E-02	1.0119E-02	6.9347E-03	2.7112E-03
	q_h	–	–	1.6685	1.6190	1.5647	1.5110	1.4592	1.3270

Table 4.1. The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.50$)

θ	k	0	1	2	3	4	5	6	9
	e	2.4518E-01	2.1469E-01	1.8909E-01	1.6769E-01	1.4980E-01	1.3480E-01	1.2218E-01	9.4903E-02
0.18	e_h	–	3.0488E-02	2.5607E-02	2.1391E-02	1.7894E-02	1.5005E-02	1.2618E-02	7.6329E-03
	q_h	–	–	1.1906	1.1971	1.1955	1.1925	1.1892	1.1791
	e	2.4518E-01	1.8245E-01	1.4010E-01	1.1189E-01	9.2904E-02	7.9930E-02	7.0920E-02	5.6628E-02
0.38	e_h	–	6.2734E-02	4.2348E-02	2.8207E-02	1.8988E-02	1.2974E-02	9.0100E-03	3.3507E-03
	q_h	–	–	1.4814	1.5013	1.4855	1.4636	1.4399	1.3658
	e	2.4518E-01	1.6310E-01	1.1623E-01	8.9593E-02	7.4012E-02	6.4577E-02	5.8645E-02	5.0282E-02
0.50	e_h	–	8.082 E-02	4.6865E-02	2.642E-02	1.581E-02	9.4350E-03	5.9317E-03	1.8546E-03
	q_h	–	–	1.7515	1.7591	1.7099	1.6514	1.5906	1.4192
	e	2.4518E-01	1.5020E-01	1.0260E-01	7.8494E-02	6.5665E-02	5.8427E-02	5.4088E-02	4.8168E-02
0.58	e_h	–	9.4981E-02	4.7604E-02	2.4103E-02	1.2829E-02	7.2378E-03	4.3387E-03	1.3066E-03
	q_h	–	–	1.9952	1.9750	1.8788	1.7725	1.6682	1.4188
	e	2.4518E-01	1.3891E-01	9.2142E-02	7.0876E-02	6.0407E-02	5.4777E-02	5.1478E-02	4.6971E-02
0.65	e_h	–	1.0627E-01	4.6773E-02	2.1266E-02	1.0469E-02	5.6292E-03	3.2995E-03	1.0052E-03
	q_h	–	–	2.2720	2.1995	2.0313	1.8598	1.7060	1.4048

Table 4.2. The relationship between θ and convergence rate ($N = 10$, $\varepsilon = 0.50$, $m = 4$ and $M = 16$)

(m, M)	error	Iteration number							
		0	1	2	3	4	5	6	9
(2,8)	e	2.8435E-01	1.6224E-01	1.2442E-01	1.0754E-01	9.9263E-02	9.4815E-02	9.2202E-02	8.8600E-02
	e_h	–	1.5390E-01	5.7326E-02	2.4772E-02	1.1662E-02	5.9765E-03	3.3381E-03	9.0551E-04
	q_h	–	–	2.6847	2.3142	2.1242	1.9513	1.7904	1.4518
(4,16)	e	2.4395E-01	9.6279E-02	6.0029E-02	4.5237E-02	3.8537E-02	3.5193E-02	3.3353E-02	3.1593E-02
	e_h	–	1.6892E-01	4.9286E-02	1.8472E-02	7.6755E-03	3.6131E-03	1.9503E-03	5.4979E-04
	q_h	–	–	3.4274	2.6681	2.4066	2.1243	1.8526	1.4282

Table 4.3. The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.65$ and $\varepsilon = 0.50$)

(m, M)	error	Iteration number							
		0	1	2	3	4	5	6	9
$(2, 8)$	e	2.4989E-01	1.3435E-01	9.4821E-02	7.9172E-02	7.1187E-02	6.6744E-02	6.4061E-02	6.0244E-02
	e_h	–	1.1554E-01	4.6950E-02	2.1771E-02	1.1029E-02	6.0871E-03	3.6459E-03	1.1582E-03
	q_h	–	–	2.4609	2.1565	1.9739	1.8120	1.6695	1.3896
$(4, 16)$	e	2.1005E-01	8.4833E-02	4.4906E-02	3.0606E-02	2.3875E-02	2.0372E-02	1.8374E-02	1.5735E-02
	e_h	–	1.2576E-01	4.4282E-02	1.8891E-02	8.9007E-03	4.6036E-03	2.6097E-03	7.4866E-04
	q_h	–	–	2.8400	2.3441	2.1224	1.9334	1.7640	1.4246

Table 4.4. The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.65$ and $\varepsilon = 0.75$)

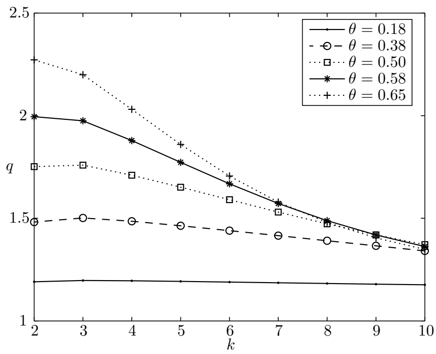

Figure 4.1. The relationship between θ and convergence rate ($N = 10$, $m = 4$, $M = 16$).

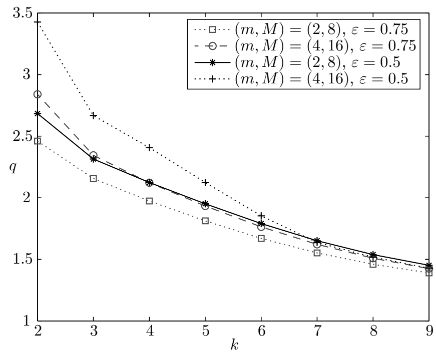

Figure 4.2. The relationship between meshes and convergence rate ($N = 10$, $\theta = 0.65$).

In Tables 4.1, 4.3, 4.4, and Figure 4.2, the relationship between the meshes and convergence rate is shown. We obtain that the convergence rate is independent of the finite element mesh size. In Table 4.2 and Figure 4.1, the convergence rates for different relaxation factors θ are compared. The results indicate that the choice of the relaxation factor is very important for the performance of the D-N alternating method. On the other hand, the convergence rate is not sensitive to the relaxation factor θ in the interval $(0.5, 0.67)$.

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