David Stanovský

Selfdistributive grupoids, Part A2: Non-idempotent left distributive quasigroups

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 52 (2011), No. 2, 7--28

Persistent URL: `http://dml.cz/dmlcz/143675`

Selfdistributive Grupoids

DAVID STANOVSKÝ

Praha

Received December 9, 2010

The present paper is a comprehensive survey of non-indempotent left distributive left quasigroups. It contains several new results about free groupoids and normal forms of terms in certain subvarieties. It is a continuation of a series of papers on selfdistributive groupoids, started by [KepN,03].

1. Introduction

We consider *groupoids* (i.e., binary algebras, with the operation denoted usually multiplicatively), that are *left distributive*, it means they satisfy the identity

$$x(yz) \approx (xy)(xz) \tag{LD}$$

and *left quasigroups*, it means that

$$\text{for every } a, b \text{ there is a unique } c \text{ with } ac = b. \tag{LQ}$$

Equivalently, left distributive left quasigroups are groupoids, where all left translations are automorphisms.

This naturally defined class of algebras was studied by several authors, mostly in the idempotent case. It has appeared under various names, such as *pseudo-symmetric sets* [Nob,83], *quandles* [Joy,82b], *automorphic sets* [Bri,88], *racks* or *wracks* [FenR,92] [Ryd,95], *left-distributive algebras* [Lar,99] etc. The investigations include both theory and applications. Perhaps the most remarkable use of idempotent

Charles University in Prague, Faculty of Mathematics and Physics, Sokolovská 83, 18675 Praha, Czech Republic

The work is a part of the research project MSM 0021620839 financed by MŠMT ČR and partly supported by the grant GAČR 201/08/P056.

2000 Mathematics Subject Classification. 20N05

Key words and phrases. Groupoid, left distributive, left quasigroup, non-idempotent

E-mail address: stanovsk@karlin.mff.cuni.cz

left distributive left quasigroups is the construction of so called *knot quandle*, an invariant with respect to knot homotopy [Joy,82a] [Mat,84].

The research on non-idempotent left distributive left quasigroups was initiated by T. Kepka in [Kep,94] and continued by E. Jeřábek and the author in [JeřKS,05], [Sta,04a], [Sta,04b], [Sta,05] and [Sta,08]. The survey is based on these papers and several unpublished results that appeared in the author's PhD thesis [Sta,04a].

This is a continuation of a series of papers on selfdistributive groupoids, started by [KepN,03]. We refer the reader into the part A1 for all undefined terminology and notation, as well as for an introduction to general non-idempotent selfdistributive groupoids.

Contents

2. Basic examples

We start with a couple of examples that play a central role in the theory of LD left quasigroups.

Example 2.1 The groupoid C_n defined on the set $\{0, \ldots, n-1\}$ by $a \cdot b = b + 1 \pmod{n}$ is an LD left quasigroup.

Example 2.2 The groupoid C_∞ defined on the set of integers by $a \cdot b = b + 1$ is an LD left quasigroup.

By a *circle of length n* we mean any groupoid isomorphic to the groupoid C_n. Groupoids isomorphic to C_∞ will be called *infinite paths*, or, sometimes, circles of length ∞. In a sense, every LD left quasigroup can be built from an idempotent LD left quasigroup by replacing its elements with circles and infinite paths, see Proposition 4.5.

Example 2.3 Let G be a group and put

$$a * b = aba^{-1}$$

for every $a, b \in G$. It is easy to check that $G(*)$ is an LDI left quasigroup, called the *conjugation groupoid* of G.

Example 2.4 Let G be a group, u a central element in G and put

$$a *_u b = aba^{-1}u$$

for every $a, b \in G$. Again, it is easy to check that $G(*_u)$ is an LD left quasigroup. It is idempotent iff $u = 1$.

Example 2.5 Let G be a group, u a central involution in G and put

$$a *_u b = ab^{-1}au$$

for every $a, b \in G$. Then $G(*_u)$ is an LD left quasigroup. Moreover, it is left symmetric (see below). It is idempotent iff $u = 1$, and in this case it is called the *core* of G.

For more constructions of LSLD groupoids from groups see Section 8. We finish this section with a list of all (very) small LD left quasigroups. The classification of the non-idempotent ones follows easily from Proposition 4.5.

Example 2.6 Every 2-element LD left quasigroup is isomorphic to one of the following two groupoids:

	a	b
a	a	b
b	a	b

	a	b
a	b	a
b	b	a

Note that the former one is the conjugation groupoid of the group $\mathbb{Z}_2$ and the latter one is a circle of length 2.

Example 2.7 Every 3-element *idempotent* LD left quasigroup is isomorphic to one of the following three groupoids:

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

	a	b	c
a	a	c	b
b	a	b	c
c	a	b	c

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Example 2.8 Every 3-element *non-idempotent* LD left quasigroup is isomorphic to one of the following three groupoids:

	a	b	c
a	b	c	a
b	b	c	a
c	b	c	a

	a	b	c
a	a	c	b
b	a	c	b
c	a	c	b

	a	b	c
a	a	b	c
b	a	c	b
c	a	c	b

Note that the former one is a circle of length 3 and the latter two are formed from a circle of length one and a circle of length two.

Example 2.9 Every 4-element *non-idempotent* LD left quasigroup is isomorphic to one of the following eleven groupoids:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & b & c & d & a \\ b & b & c & d & a \\ c & b & c & d & a \\ d & b & c & d & a \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & a & x & y & z \\ b & a & c & d & b \\ c & a & c & d & b \\ d & a & c & d & b \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & b & a & u & v \\ b & b & a & u & v \\ c & r & s & d & c \\ d & r & s & d & c \end{array}$$

where $\{x, y, z\} = \{b, c, d\}$ and $\{r, s\} = \{a, b\}$ and $\{u, v\} = \{c, d\}$.

3. Notation and basic facts

Identities. We start with a list of several frequently used groupoid identities:

$$\begin{array}{ccc} \text{(LD)} & x(yz) \approx (xy)(xz) & \textit{left distributivity} \\ \text{(M)} & (xy)(uv) \approx (xu)(yv) & \textit{mediality} \\ \text{(LI)} & (xx)y \approx xy & \textit{left idempotency} \\ \text{(I)} & xx \approx x & \textit{idempotency} \\ (n\text{-LS}) & \underbrace{x(x(\cdots(x}_{n}y))) \approx y & \textit{left } n\textit{-symmetry} \end{array}$$

Left symmetry refers implicitly to left 2-symmetry. Left 1-symmetric groupoids are called *right zero bands*. Right distributivity, idempotency and symmetry are defined dually. We note that mediality and idempotency imply both left and right distributivity.

We say that a groupoid G is *left cancellative*, if all its left translations are injective, and *left divisible*, if all its left translations are onto. It is called a *left quasigroup*, if it is both left divisible and left cancellative. It means that the equation $ax = b$ has a unique solution x for any $a, b \in G$; such x is usually denoted $a\backslash b$.

Clearly, left n-symmetric groupoids are left quasigroups with

$$a\backslash b = \underbrace{a(a(\ldots(a}_{n-1}b))),$$

and every finite left quasigroup is left n-symmetric for some n. The class of left distributive left quasigroups is not closed on subalgebras (consider the infinite path), hence, it does not form a variety (not even a quasivariety). Left quasigroups however do form a variety when the left division is considered as basic operation; it is axiomatized by the identities $x(x\backslash y) \approx y$ and $x\backslash(xy) \approx y$.

Lemma 3.1 *Let G be a left distributive left quasigroup. Then*
(1) *G is left idempotent;*
(2) *if G is right cancellative, it is idempotent;*
(3) *if G is right distributive, it is idempotent.*

Proof. (1) $xy \approx x(x(x\backslash y)) \approx_{LD} (xx)(x(x\backslash y)) \approx (xx)y$.
(2) $x(xx) \approx_{LD} (xx)(xx)$ and use right cancellativity.
(3) $(xx)(xx) \approx_{RD} (xx)x$ and use left cancellativity. □

Since left distributive left quasigroups are left idempotent, we can (and will) apply the theory of LDLI groupoids developed by P. Jedlička [Jed,05] and the author [Sta,04b], [Sta,08], see Section 4.

Translations. Let G be a groupoid and $a \in G$. The *left translation* by a in G is a mapping $L_a : G \to G$, $x \mapsto ax$; right translations are defined dually and denoted R_a. For operations other than $\cdot$, for instance $*$, we use the notation L^*_a, R^*_a. For a left quasigroup G, we define the *left multiplication group* $\mathrm{LMlt}(G)$, to be a subgroup of the symmetric group over G generated by all L_a, $a \in G$.

It is often useful to translate identities into the language of translations.

Lemma 3.2 *Let G be a groupoid. Then*
(1) *G is left distributive, iff L_a is an endomorphism for every $a \in G$;*
(2) *G is a left quasigroup, iff L_a is a permutation for every $a \in G$;*
(3) *G is left n-symmetric, iff $(L_a)^n = id$ for every $a \in G$;*
(4) *G is left idempotent, iff $L_a = L_{aa}$ for every $a \in G$;*
(5) *G is medial, iff the mapping $G \times G \to G$, $(a, b) \mapsto ab$, is a homomorphism.*

Proof. Easy to check. □

Lemma 3.3 *Let G be a left distributive left quasigroup. Then*
(1) *$L_{\varphi(a)} = \varphi L_a \varphi^{-1}$ for every $a \in G$ and every automorphism φ of G;*
(2) *$L_{ab} = L_a L_b L_a^{-1}$ for every $a, b \in G$;*
(3) *the mapping $\lambda : a \mapsto L_a$ is a homomorphism of G into the conjugation groupoid of the left multiplication group of G.*

Proof. (1) Since $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in G$, we have $\varphi L_a = L_{\varphi(a)}\varphi$ and thus also $L_{\varphi(a)} = \varphi L_a \varphi^{-1}$ for every $a \in G$.
(2) follows from the fact that all left translations are automorphisms.
(3) According to (2), $\lambda(ab) = L_{ab} = L_a L_b L_a^{-1} = \lambda(a) * \lambda(b)$. □

Terms. In the present paper, we do not regard the left division \ as a basic operation. However, it often happens that there is a (multiplicative) term $t(x, y)$ such that $a\backslash b = t(a, b)$ for every a, b. In this case we say that the left quasigroup has *term-definable left division.*

In order to decrease the number of parentheses in terms, we assume implicitly that letters in terms are right associated, i.e. $xyz = x \cdot yz = x(yz)$. We define the n-th power by

$$x^n = \underbrace{x \cdots xx}_{n}.$$

Due to the following lemma, the other possible definitions of a power are obsolete.

Lemma 3.4 *Let G be an LI groupoid. Then*

(1) *G satisfies the identity $x^n y \approx xy$ for every $n \geq 1$;*
(2) *for every term t in a single variable x, G satisfies the identity $t \approx x^d$, where d is the right depth of t;*
(3) *G satisfies the identity $(x^m)^n \approx (x^n)^m \approx x^{m+n-1}$ for every $m, n \geq 1$.*

Proof. (1) The case $n = 1$ is trivial, the case $n = 2$ is LI and further we proceed by induction: if G satisfies $x^{n-1}y \approx xy$, then

$$x^n y = (xx^{n-1})y \approx (x^{n-1}x^{n-1})y \approx x^{n-1}y \approx xy.$$

(2) and (3) are obvious corollaries of (1). □

We say that a groupoid G (a variety $\mathcal{V}$ of groupoids, resp.) has *exponent n*, if n is the least positive integer such that the identity $x^{n+1} \approx x$ holds in G (in $\mathcal{V}$, resp.), provided such n exists. Otherwise, we define the exponent to be ∞. Note that every finite groupoid (or locally finite variety) has a finite exponent. The variety of n-LSLD groupoids has exponent n.

Further, we will denote $x^{[n]}y$ the term

$$\underbrace{x \cdots x}_{n} y = (L_x)^n(y).$$

Hence the left n-symmetry can be written as $x^{[n]}y \approx y$.

Finally, we will denote $\mathbf{F}_{\mathcal{V}}(X)$ the free groupoid over a set X in a variety $\mathcal{V}$ and assume its standard representation by terms modulo identities of $\mathcal{V}$.

Substructures. A subgroupoid of a left quasigroup is not necessarily a left quasigroup (it is indeed left cancellative, but not necessarily left divisible). We will thus use the notion of *left subquasigroup*. Subgroupoids of left quasigroups with term-definable left division are indeed left subquasigroups.

A non-empty subset I of a left quasigroup G is called a *left ideal*, if $a \in G$, $b \in I$ implies $ab \in I$ (in other words, if $GI \subseteq I$). I is called a *strong left ideal*, if $a \in G$, $b \in I$ implies $ab \in I$ and $a\backslash b \in I$. Clearly, if $I \subset G$ is a strong left ideal, then $G \smallsetminus I$ is also a strong left ideal. Ideals of left quasigroups with term-definable left division are always strong.

Definable sets. A subset S of a groupoid G is called *definable* in G if there exists a formula Φ with a single free variable such that $S = \{a \in G : \Phi(a)\}$. A relation α on G is called *definable*, if there exists a formula Φ with two free variables such that $\alpha = \{(a,b) \in G \times G : \Phi(a,b)\}$. A relation α is called *right stable*, if $(a,b) \in \alpha$ implies $(ac, bc) \in \alpha$ for every c.

Lemma 3.5 *Let G be an LD left quasigroup. Then*

(1) *every definable subset in G is either empty, or a strong left ideal;*
(2) *every definable right stable equivalence on G is a congruence of G.*

Proof. Indeed, for every automorphism α of G and every $a \in G$, $\Phi(a)$ holds iff $\Phi(\alpha(a))$ holds. Hence the claim follows from the fact that left translations and their inverses are automorphisms. $\square$

We use the lemma for the following observation: Note that the sets Ip_G of idempotent elements of an LD left quasigroup G and its complement $K_G = G \setminus Ip_G$ are either empty, or strong left ideals.

Remark 3.6 Lemma 3.5 holds also for sets and equivalences definable in the language $\{\cdot, \in J_1, \dots, \in J_n\}$, where $J_1, \dots, J_n$ are strong left ideals and $\in J_1, \dots, \in J_n$ the corresponding unary relational symbols. (We have no use for this more general statement.)

4. The smallest idempotent congruence

The main feature for investigation of non-idempotent LD left quasigroups is the fact that the smallest congruence with idempotent quotient, denoted by ip, has a very nice structure. This was first observed by P. Jedlička in [Jed,05], more generally for LDLI groupoids. We will need the following improvement of his result.

Let γ_k be the smallest congruence such that the corresponding factor satisfies the identity $x^{k+1} \approx x$. Indeed, $\gamma_k \subseteq \gamma_\ell$, iff $\ell \mid k$. Particularly, $\gamma_k \subseteq \gamma_1 = ip$ for every k.

Proposition 4.1 *Let G be an LDLI groupoid and $k \geq 1$. Then*

(1) *γ_k is the smallest equivalence on the set G containing all pairs (a, a^{k+1}), $a \in$ $\in G$;*

(2) *$\gamma_k = \{(a, b) \in G \times G : a^m = b^n$ for some m, n such that k divides $m - n\}$;*

(3) *if $(a, b) \in \gamma_k$, then $ac = bc$ holds for every $c \in G$.*

Proof. (1) Clearly γ_k must contain all pairs (a, a^{k+1}), $a \in G$. We prove that the equivalence α generated by these pairs is a congruence. So we need to check that $(ab, a^{k+1}b) \in \alpha$ and $(ba, ba^{k+1}) \in \alpha$ for every $a, b \in G$. The first claim follows from left idempotency, since $ab = a^{k+1}b$. For the second claim, using k-times left distributivity one obtains $ba^{k+1} = (ba)^{k+1}$. Consequently, $\alpha = \gamma_k$.

(2) First, let's assume that $(a, b) \in \gamma_k$ and we prove that $a^m = b^n$ for some m, n with $k \mid m - n$. Since γ_k is generated as an equivalence by the set $\{(a, a^{k+1}) : a \in G\}$, there are $c_0, \dots, c_\ell$ such that $a = c_0$, $b = c_\ell$ and either $c_i = c_{i+1}^{k+1}$, or $c_i^{k+1} = c_{i+1}$ for every $i = 0, \dots, \ell - 1$. We proceed by induction on ℓ. If $\ell = 0, 1$, it is trivial. So, assume that $a^m = c_{\ell-1}^n$ for some m, n with $k \mid m - n$. If $c_{\ell-1} = b^{k+1}$, then

$$b^{n+k} = (b^{k+1})^n = c_{\ell-1}^n = a^m$$

and $k \mid m - (n + k)$. If $c_{\ell-1}^{k+1} = b$, then

$$a^{m+k} = (a^m)^{k+1} = (c_{\ell-1}^n)^{k+1} = (c_{\ell-1}^{k+1})^n = b^n$$

and, again, $k \mid m + k - n$.

For the other inclusion, let's assume that $a^m = b^n$ for some m, n with $k \mid m-n$. Then also $a^{m+u} = (a^m)^{u+1} = (b^n)^{u+1} = b^{n+u}$ for every $u \geq 0$. Let m', n', q be such that $m = m'k + q$ and $n = n'k + q$. Since

$$(a, a^{k+1}) \in \gamma_k,\ (a^{k+1}, a^{2k+1}) \in \gamma_k,\ \ldots,\ (a^{m'k+1}, a^{(m'+1)k+1}) \in \gamma_k,$$

we have $(a, a^{m'k+k+1}) \in \gamma_k$ and similarly $(b, b^{n'k+k+1}) \in \gamma_k$. Since

$$a^{m'k+k+1} = a^{m+(k+1-q)} = b^{n+(k+1-q)} = b^{n'k+k+1},$$

we obtain $(a, b) \in \gamma_k$.

(3) If $(a, b) \in \gamma_k$, then $a^m = b^n$ for some m, n and thus $ac = a^m c = b^n c = bc$ for every $c \in G$ by left idempotency. □

Corollary 4.2 *Let G be an LDLI groupoid. Then*

(1) *ip is the smallest equivalence on the set G containing all pairs (a, aa), $a \in G$;*
(2) *$ip = \{(a, b) \in G \times G : a^m = b^n$ for some $m, n\}$;*
(3) *if $(a, b) \in ip$, then $ac = bc$ holds for every $c \in G$.*

Consequently, every block of ip is a subgroupoid of G satisfying the identity $xz \approx yz$ and it is term equivalent to a *connected* monounary algebra; the left translation is the corresponding unary operation.

Corollary 4.3 *Let G be an LD left quasigroup. Then every block of ip is either a circle, or an infinite path. If G has exponent n, then every block is a circle of length $k \mid n$.*

We define the *cycle type* of an LD left quasigroup G to be the set of all $k \in \mathbb{N} \cup \{\infty\}$ such that there is an ip-block isomorphic to C_k. Indeed, groupoids of exponent n have only divisors of n in its cycle type. For example, the cycle type contains 1 if and only if G has an idempotent element.

Note that the congruence lattice of C_n consists of the (pairwise different) congruences γ_k, $k \mid n$. Consequently, we have the following:

Corollary 4.4 *Circles of prime length are the only simple non-idempotent LD left quasigroups.*

Proposition 4.1 also yields the following description of LD left quasigroups. Let $H(*)$ be an LDI left quasigroup, B_u, $u \in H$, pairwise disjoint circles (of possibly infinite length), and $f_{u,v}$, $u, v \in H$, isomorphisms from B_v to B_{u*v}. Assume also that $f_{u,u}(a) = aa$ for every $u \in H$, $a \in B_u$. We define a groupoid $\mathrm{G}(H, f)$ as the disjoint union of all B_u, $u \in H$, together with the operation

$$a \cdot b = f_{u,v}(b) \qquad \text{for every } a \in B_u,\ b \in B_v.$$

Indeed, the sets B_u become blocks of the congruence ip.

Proposition 4.5 (1) *Let H, B_u and $f_{u,v}$ be as above. Then $\mathrm{G}(H, f)$ is a left quasigroup. It is left distributive, if and only if for every $u, v, w \in H$*

$$f_{u,v*w} \circ f_{v,w} = f_{u*v,u*w} \circ f_{u,w}.$$

(2) *Let G be an LD left quasigroup. Then G is isomorphic to* $\mathrm{G}(G/ip, f)$*, where* $f_{u/ip,v/ip}(a) = ua$ *for every* $u/ip, v/ip \in G/ip$ *and* $a \in v/ip$.

Proof. (1) The left quasigroup property is obvious from the construction. Since for every $a \in B_u$, $b \in B_v$, $c \in B_w$

$$a \cdot bc = f_{u,v*w}(f_{v,w}(c)) \quad \text{and} \quad ab \cdot ac = f_{u*v,u*w}(f_{u,w}(c)),$$

we get $a \cdot bc = ab \cdot ac$ iff $f_{u,v*w} \circ f_{v,w} = f_{u*v,u*w} \circ f_{u,w}$.

(2) According to Corollary 4.2(3), the mappings $f_{u/ip,v/ip}$ are well defined. And they are automorphisms, because the left translation L_u restricted to the subgroupoid v/ip is an automorphism. □

Later, we will use the following lemma, which is also an immediate consequence of Proposition 4.1.

Lemma 4.6 *Let G be an LD left quasigroup. Let t, s be terms such that the rightmost variables in t, s are distinct, but have the same depth. If G satisfies the identity $t \approx s$, then it is idempotent.*

Proof. Assume that there is $a \in G$ such that $aa \neq a$. Substitute aa for the rightmost variable of t and a for all other variables. Then the value of t is a square of the value of s and thus the equality fails. □

Remark 4.7 Proposition 4.1 and its corollaries originated in the paper of P. Jedlička [Jed,05] and were refined by the author in [Sta,08]. Proposition 4.5 is a particular case of Jedlička's description of LDLI groupoids in [Jed,05].

5. Varieties of LD left quasigroups

Let $\mathscr{I}$ denote the variety of idempotent groupoids. By the expression "variety of LD left quasigroups" we mean any variety of groupoids that contains only LD left quasigroups. For example, n-LSLD groupoids form a variety of LD left quasigroups (of exponent n).

Lemma 5.1 *Let G be an LD left quasigroup of exponent n, $n \in \mathbb{N} \cup \{\infty\}$. Then C_n is a homomorphic image of G, if and only if G is isomorphic to the direct product $C_n \times (G/ip)$.*

Proof. Pick a projection $g : G \to C_n$ and put $f(x) = (g(x), x/ip)$. Then $f : G \to$ $\to C_n \times (G/ip)$ is a homomorphism. Note that for k, ℓ finite, there is a homomorphism $C_k \to C_\ell$ iff $\ell \mid k$, and there is no homomorphism $C_k \to C_\infty$ for k finite. Hence every ip-block in G is isomorphic to C_n, since g restricted to any ip-block is a homomorphism. Consequently, g is bijective on every ip-block, because all endomorphisms of C_n are actually automorphisms, and thus f is an isomorphism. The other implication is clear. □

Lemma 5.2 *Let $\mathcal{V}$ be a variety of LD left quasigroups of exponent n.*

(1) *If n is finite, then $C_k \in \mathcal{V}$ iff $k \mid n$.*

(2) *If $n = \infty$, then $C_k \in \mathcal{V}$ for every $k \in \mathbb{N} \cup \{\infty\}$.*

Proof. Since C_k, k finite, is 1-generated, its presence in $\mathcal{V}$ is determined by the indentities of $\mathcal{V}$ in a single variable. By Lemma 3.4(2), every identity in a single variable x is $x^u \approx x^v$ for some $u, v \in \mathbb{N}$, and by cancellativity we get that it is equivalent to $x^m \approx x$ for $m = u - v + 1$. If $n = \infty$, then no such identity holds in $\mathcal{V}$ and thus all $C_k \in \mathcal{V}$. Otherwise, $n \mid m$ and (1) follows.

To finish the proof, we need to check that if $n = \infty$, then $C_\infty \in \mathcal{V}$ (note that C_∞ is not 1-generated!). Let $s \approx t$ be an identity satisfied in $\mathcal{V}$; we can assume that it contains at least two variables. Since $\mathcal{V}$ has no non-trivial identity in a single variable, the terms s, t have the same depth of the rightmost variables. Hence, since $\mathcal{V}$ is non-idempotent, by Lemma 4.6 the rightmost variables of s, t are equal. Consequently, C_∞ satisfies $s \approx t$. □

Corollary 5.3 *Every variety of LD left quasigroups has finite exponent.*

Proof. If not, it contains the groupoid C_∞, which has a subgroupoid that is not left divisible. □

Theorem 5.4 *Let $\mathcal{V}$ be a variety of LD left quasigroups of exponent n. Then $\mathbf{F}_{\mathcal{V}}(X)$ is isomorphic to $C_n \times \mathbf{F}_{\mathcal{V}\cap\mathcal{I}}(X)$. Consequently, the variety $\mathcal{V}$ is generated by $(\mathcal{V} \cap \mathcal{I}) \cup \{C_n\}$.*

Proof. Since $C_n \in \mathcal{V}$, it is a homomorphic image of $\mathbf{F}_{\mathcal{V}}(X)$. Hence, by Lemma 5.1, $\mathbf{F}_{\mathcal{V}}(X) \simeq C_n \times H$, where $H = \mathbf{F}_{\mathcal{V}}(X)/ip$. It is easy to see that $H \simeq \mathbf{F}_{\mathcal{V}\cap\mathcal{I}}(X)$, because ip is the smallest idempotent congruence. □

Now, we can describe the lattice of varieties of LD left quasigroups.

Theorem 5.5 *Let $\mathcal{V}$ be a variety of LD left quasigroups of exponent n. Let L denote the lattice of subvarieties of $\mathcal{V} \cap \mathcal{I}$, K its sublattice of varieties containing right zero bands and N the lattice of positive integer divisors of n. The lattice of subvarieties of the variety $\mathcal{V}$ is isomorphic to the lattice*

$$(L \times \{1\}) \cup (K \times (N \setminus \{1\}))$$

(regarded as a subposet of $L \times N$), mapping a variety $\mathcal{U}$ of exponent m to the pair $\Phi(\mathcal{U}) = (\mathcal{U} \cap \mathcal{I}, m)$.

Proof. First, we check that the mapping Φ is well-defined: the exponent m of a subvariety $\mathcal{U}$ is clearly a divisor of n and since $\mathcal{U}$ contains C_m, it contains the right zero band $(C_m \times C_m)/ip$ and thus it contains the whole variety of right zero bands (because it is minimal). Next, Φ is injective: if $\mathcal{U}_1$ and $\mathcal{U}_2$ are distinct varieties of exponent m, then $\mathcal{U}_1 \cap \mathcal{I}$ and $\mathcal{U}_2 \cap \mathcal{I}$ are distinct, because $\mathcal{U}_i$ is generated by $(\mathcal{U}_i \cap \mathcal{I}) \cup \{C_m\}$, $i = 1, 2$. The mapping Φ is onto, a pair $(\mathcal{W}, m)$ is the image of the variety generated by $\mathcal{W} \cup \{C_m\}$. Indeed, let G be an idempotent groupoid in the variety generated by $\mathscr{W} \cup \{C_m\}$ and we show that $G \in \mathscr{W}$. The case $m = 1$ is trivial, so let $m > 1$. By Birkhoff's HSP theorem, there are $H \in \mathscr{W}$, $K \leq H \times C_m^k$ (for some k) and an onto homomorphism $\varphi : K \to G$. Since ip is the smallest idempotent congruence of K and G is idempotent, there is an onto homomorphism $\psi : K/ip \to G$. Further, $K/ip \leq (K \times C_m^k)/ip \simeq K \times (C_m^k/ip)$. However, C_m^k/ip is a right zero band and thus it is in $\mathscr{W}$. Consequently, G is a homomorphic image of a subgroupoid of a groupoid from $\mathscr{W}$, thus it is in $\mathscr{W}$. Finally, Φ clearly preserves the order and it follows from Theorem 5.4 that also Φ^{-1} preserves the order. Consequently, Φ is a lattice isomorphism. □

Example 5.6 B. Roszkowska proved in [Ros,87] that the lattice of subvarieties of left symmetric medial idempotent (LSMI) groupoids is isomorphic to the lattice of positive integers ordered by divisibility with an additional top element. A number n corresponds to the variety based by $w_n(x, y) \approx y$ (relatively to LSMI), where

$$w_n(x, y) = \underbrace{xyxyxy\dots}_{n}$$

Note that right zero bands satisfy $w_n(x, y) \approx y$ iff n is even. Thus, using Theorem 5.5, it is easy to describe bases of all proper subvarieties of left symmetric left distributive medial groupoids (relatively to LSLDM):

- $xx \approx x$;
- $w_n(x, y) \approx y$ and $xx \approx x$, for every n;
- $w_n(x, y) \approx y$, for every n even.

(Note that mediality and idempotency imply left distributivity, however, non-idempotent medial groupoids are not necessarily left distributive.)

Example 5.7 J. Płonka [Pło,85] investigated n-LSLDI groupoids satisfying

$$x(yz) \approx y(xz) \quad \text{and} \quad xz \approx (yx)z$$

and called them *n-cyclic groupoids*. Płonka proved that the only non-trivial subvarieties of n-cyclic groupoids are m-cyclic groupoids for $m \mid n$. One can thus use Theorem 5.5 to describe the subvarieties of the non-idempotent generalization of n-cyclic groupoids. Every non-trivial one is generated by idempotent m-cyclic groupoids and the groupoid C_k, for some divisors m, k of n; hence there are exactly $q^2 + 1$ such subvarieties, where q is the number of divisors of n.

Remark 5.8 A similar approach works for an arbitrary variety of LDLI groupoids of finite exponent. For details, see [Sta,04b]. A particular case of Theorems 5.4 and 5.5 was proved by T. Kepka [Kep,94] for the variety of LSLD groupoids.

6. Normal form of terms for n-LSLD groupoids

It is important to observe that any LD left quasigroup G satisfies the identity

$$xy \cdot z \approx xy \cdot x(x\backslash z) \approx_{LD} xy(x\backslash z).$$

If G is left n-symmetric, it reads

$$xy \cdot z \approx xyx^{[n-1]}z,$$

and similarly, by induction,

$$(x_1x_2\cdots x_m)\cdot z \approx x_1x_2\cdots x_{m-1}x_mx_{m-1}^{[n-1]}\cdots x_2^{[n-1]}x_1^{[n-1]}z.$$

Consequently, using LD and n-LS, one can transform any term over an alphabet X into an equivalent term of the form $x_1x_2\cdots x_{m-1}x_m$, for some m and $x_1,\dots,x_m \in X$. The following theorem shows a useful normal form for terms in the variety of n-LSLDI and n-LSLD groupoids, and a description of free groupoids.

Theorem 6.1 *Let X be a non-empty set and $n \geq 2$. Denote G_X the free product of $|X|$ copies of the cyclic group $\mathbb{Z}_n$ and H_X the direct product $G_X \times \mathbb{Z}_n$. We identify each element of X with a generator of the respective copy of $\mathbb{Z}_n$.*

(1) *The free n-LSLDI groupoid over X is isomorphic to the subgroupoid generated by X in the conjugation groupoid $G_X(*)$. Every term over X is n-LSLDI-equivalent to a* unique *term of the form*

$$x_1^{[k_1]}x_2^{[k_2]}\cdots x_{m-1}^{[k_{m-1}]}x_m,$$

where $x_1,\dots,x_m \in X$, $x_i \neq x_{i+1}$ and $k_i \in \{1,\dots,n-1\}$ for every $i \leq m-1$.

(2) *The free n-LSLD groupoid over X is isomorphic to the subgroupoid generated by X in the groupoid $H_X(*_{(e,u)})$, where e is the unit of G_X and u a generator of $\mathbb{Z}_n$. Every term over X is n-LSLD-equivalent to a* unique *term of the form*

$$x_1^{[k_1]}x_2^{[k_2]}\cdots x_{m-1}^{[k_{m-1}]}x_m^{k_m},$$

where $x_1,\dots,x_m \in X$, $x_i \neq x_{i+1}$ and $k_i \in \{1,\dots,n-1\}$ for every $i \leq m-1$ and $k_m \in \{1,\dots,n\}$.

Proof. We have just seen that every term is n-LSLD-equivalent to a term $x_1x_2\cdots x_m$, $x_i \in X$. Collect equal neighbours and use left n-symmetry (or idempotency in (1), for the rightmost variable) to decrease the exponents below n. The resulting term has the described form. Now we prove its uniqueness.

(1) Denote F_X the subgroupoid generated by X in $G_X(*)$. Clearly, F_X is left n-symmetric, because it is generated by elements of order n. Consider a term $t = x_1^{[k_1]}x_2^{[k_2]}\cdots x_{m-1}^{[k_{m-1}]}x_m$, $x_1,\dots,x_m \in X$, $x_i \neq x_{i+1}$ and $k_i \in \{1,\dots,n-1\}$ for every i. The value of t in F_X (when variables are identified with the respective generators) is

$$x_1^{[k_1]} * x_2^{[k_2]} * \cdots * x_{m-1}^{[k_{m-1}]} * x_m = x_1^{k_1}x_2^{k_2}\dots x_{m-1}^{k_{m-1}}x_mx_{m-1}^{-k_{m-1}}\dots x_2^{-k_2}x_1^{-k_1}.$$

This word is irreducible in the free product G_X, so for different t's we get different values.

(2) Denote F_X the subgroupoid generated by X in $H_X(*_{(e,u)})$. Clearly, F_X is left n-symmetric, because it is generated by elements of order n. Consider a term $t = x_1^{[k_1]}x_2^{[k_2]}\cdots x_{m-1}^{[k_{m-1}]}x_m^{k_m}$, $x_1,\dots,x_m \in X$, $x_i \neq x_{i+1}$ and $k_i \in \{1,\dots,n-1\}$ for every $i \leq m-1$ and $k_m \in \{1,\dots,n\}$. The value of t in F_X is

$$x_1^{[k_1]} * x_2^{[k_2]} * \cdots * x_{m-1}^{[k_{m-1}]} * x_m^{k_m} = x_1^{k_1}x_2^{k_2}\dots x_{m-1}^{k_{m-1}}x_mx_{m-1}^{-k_{m-1}}\dots x_2^{-k_2}x_1^{-k_1}e^{-1+\sum_i k_i}.$$

Again, for different t's we get different values. $\square$

Remark 6.2 Note that the normal form of (2) follows directly from (1) and Theorem 5.4.

Remark 6.3 The free groupoid over X in the variety *generated by* LDI left quasigroups is isomorphic to the subgroupoid generated by X in the conjugation groupoid of the free group over X (see [DráKM,94]). Consequently, for the variety of LDI left quasigroups in the language $\{\cdot, \backslash\}$, there is a normal form similar to that from Theorem 6.1.

An important particular case is the variety of (2-)LSLD groupoids, and also its medial subvariety [Ros,87].

Theorem 6.4 *Let X be a non-empty set and $(X, \leq)$ a linear order.*
(1) *Every term over X is LSLDI-equivalent to a* unique *term of the form*

$$x_1 x_2 \cdots x_{n-1} x_n, \qquad x_1, \dots, x_n \in X \text{ and } x_i \neq x_{i+1},\ i = 1, \dots, n-1.$$

(2) *Every term over X is LSLD-equivalent to a* unique *term of the form*

$$x_1 x_2 \cdots x_{n-1} x_n, \qquad x_1, \dots, x_n \in X \text{ and } x_i \neq x_{i+1},\ i = 1, \dots, n-2.$$

(3) *Every term over X is LSMI-equivalent to a* unique *term of the form*

$$\begin{aligned} x_1 x_2 \dots x_{n-1} x, \qquad & x_1, \dots, x_{n-1}, x \in X, \\ & \{x_1, x_3, \dots\} \cap \{x_2, x_4, \dots\} = \emptyset,\ x \notin \{x_{n-1}, x_{n-3}, \dots\} \\ & x_1 \leq x_3 \leq x_5 \dots \text{ and } x_2 \leq x_4 \leq x_6 \dots \end{aligned}$$

(4) *Every term over X is LSLDM-equivalent to a* unique *term of the form*

$$\begin{aligned} x_1 x_2 \dots x_{n-1} x, \qquad & x_1, \dots, x_{n-1}, x \in X, \\ & \{x_1, x_3, \dots\} \cap \{x_2, x_4, \dots\} = \emptyset, \\ & x_1 \leq x_3 \leq x_5 \dots \text{ and } x_2 \leq x_4 \leq x_6 \dots \end{aligned}$$

Proof. (1) and (2) are immediate corollaries of Theorem 6.1. For (3) and (4), note that mediality is equivalent to the identity

$$xyzu \approx zyxu$$

and consider the subgroupoid F generated by X in the core of the free abelian group over X. It is easy to check that different terms in the described form have different values in F. $\square$

Remark 6.5 Theorem 6.1 appeared in the author's PhD thesis [Sta,04a] and has never been published. Its partial corollary, Theorem 6.4, appeared in [Sta,05]. The representation of free groupoids presented in the original paper was different: free LSLDI groupoids were represented as subgroupoids of the core of free groups, see Section 8.

7. An alternative description of n-LSLD groupoids

We show an equivalence of the variety of n-LSLD groupoids and the variety $\mathscr{A}_n$ of algebras $A(\circ, f)$ satisfying the following conditions:

(1) $A(\circ)$ is an idempotent n-LSLD groupoid;

(2) f is an automorphism of $A(\circ)$ of order n;

(3) $f(x) \circ y \approx x \circ y$ holds in A.

Note that (2) can be expressed by the identities

$$\underbrace{f \dots f}_{n}(x) \approx x \quad \text{and} \quad f(x \circ y) \approx f(x) \circ f(y),$$

hence $\mathscr{A}_n$ is a variety.

Lemma 7.1 *Let G be an n-LSLD groupoid and put*

$$x \circ y = xy^n \quad \textit{and} \quad f(x) = xx.$$

Then $G(\circ, f) \in \mathscr{A}_n$.

Proof. Note that $xy^n \approx (xy)^n$ and recall Lemma 3.4(3) saying that LI implies $(x^m)^n \approx x^{m+n-1}$. Hence $A(\circ)$ satisfies LD, since $x \circ (y \circ z) = x(yz^n)^n \approx x \cdot y((z)^n)^n \approx$ $\approx x \cdot yz^{2n-1} \approx_{LI} x \cdot y^n z^{2n-1} \approx_{LD} (xy^n)(xz^{2n-1}) \approx (xy^n)(xz^n)^n = (x \circ y) \circ (x \circ z)$. Also, $A(\circ)$ satisfies n-LS, because $x \circ \dots \circ x \circ y = x(x(\dots(xy)^n \dots)^n)^n \approx x \dots x(((y^n)^n) \dots)^n \approx$ $\approx x \dots xy^{n^2-n+1} \approx_{LS} y^{n^2-n+1} \approx_{LS} y$. Clearly, $A(\circ)$ is idempotent, since $x \circ x = x^{n+1} \approx x$.

Now, $f^n(x) \approx_{LI} x^{n+1} \approx_{LS} x$, so f is a permutation of order n and $f(x \circ y) =$ $= (xy^n)(xy^n) \approx_{LD} x(y^n y^n) \approx_{LI} (xx)(y^{2n-1}) \approx (xx)(yy)^n = f(x) \circ f(y)$ shows that it is a homomorphism. Finally, $f(x) \circ y = (xx)y^n \approx_{LI} xy^n = x \circ y$. □

Lemma 7.2 *Let $A(\circ, f) \in \mathscr{A}_n$ and put*

$$x \bullet y = f(x \circ y).$$

Then $A(\bullet)$ is an n-LSLD groupoid.

Proof. We have $x \bullet (y \bullet z) = f(x \circ f(y \circ z)) \approx_{(2)} f(x) \circ (f^2(y) \circ f^2(z)) \approx_{(3)} f^2(x) \circ (f^2(y) \circ$ $\circ f^2(z)) \approx_{LD} (f^2(x) \circ f^2(y)) \circ (f^2(x) \circ f^2(z)) \approx_{(2)} f(f(x \circ y) \circ f(x \circ z)) = (x \bullet y) \bullet (x \bullet z)$ and $x \bullet \dots \bullet x \bullet y \approx_{(2)} f(x) \circ f^2(x) \circ \dots \circ f^n(x) \circ f^n(y) \approx_{(3)} f(x) \circ f(x) \circ \dots \circ f(x) \circ f^n(y) \approx$ $\approx_{LS} f^n(y) \approx_{(2)} y$. □

Lemma 7.3 *Let G be an n-LSLD groupoid and $G(\bullet)$ the groupoid that results through application first the construction from Lemma 7.1 and then that of Lemma 7.2. Then $G(\bullet) = G$.*

Proof. Straightforward computation. □

Lemma 7.4 *Let $A(\circ, f) \in \mathscr{A}_n$ and let $A(*, g)$ be the algebra that results through application first the construction from Lemma 7.2 and then that of Lemma 7.1. Then $A(*, g) = A(\circ, f)$.*

Proof. Straightforward computation. □

Theorem 7.5 *The varieties of n-LSLD groupoids and $\mathscr{A}_n$ are term equivalent.*

Proof. It follows immediately from the preceding lemmas. □

Remark 7.6 The contents of this section are based on an idea of T. Kepka and have never been published before.

8. LSLD operations on groups

Let G be a group, f an involutory automorphism of G, g an involutory *antiautomorphism* g of G (i.e., a permutation of G such that $g(xy) = g(y)g(x)$) and $u \in G$. We put for all $a, b \in G$

$$a *_g b = ag(b)a \tag{a}$$

$$a \circ_{f,u} b = af(a^{-1}bu) \tag{b}$$

$$a \diamond_{f,u} b = af(ba^{-1}u) \tag{c}$$

$$a \odot_u b = aua^{-1}b \tag{d}$$

Let $\mathrm{Z}(G)$ denote the center of the group G.

Lemma 8.1 *Let G be a group.*

(a) *$G(*_g)$ is an LSLD groupoid, provided that $a^2g(a^2) = 1$ and $ag(a) \in \mathrm{Z}(G)$ for all $a \in G$;*
(b) *$G(\circ_{f,u})$ is an LSLD groupoid, provided that $u^2 = 1$ and $f(u) = u$;*
(c) *$G(\diamond_{f,u})$ is an LSLD groupoid, provided that $u^2 = 1$, $ua = af(u)$ and $af(a) \in \in \mathrm{Z}(G)$ for all $a \in G$;*
(d) *$G(\odot_u)$ is an LSLD groupoid, provided that $u^2 = 1$.*

Moreover, $G(\diamond_{f,e})$ is medial. If G is abelian, then both $G(*_g)$ and $G(\circ_{f,e}) = G(\diamond_{f,e})$ are medial too.

Proof. Straightforward coputation. □

Note that

(a) $a *_g a = a$ iff $g(a) = a^{-1}$; hence $G(*_g)$ is idempotent iff g is the inverse operation of the group G; in this case, $G(*_g)$ is the *core* of G.
(b) $G(\circ_{f,u})$ is idempotent iff $u = 1$; otherwise, it contains no idempotent elements.
(c) $G(\diamond_{f,u})$ is idempotent iff $u = 1$; otherwise, it contains no idempotent elements.
(d) $G(\odot_u)$ is idempotent iff $u = 1$; in this case, it is a right zero band; otherwise, it contains no idempotent elements.

Let $\mathscr{A}$ ($\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ resp.) denote the variety generated by all $G(*_g)$ ($G(\circ_{f,e})$, $G(\diamond_{f,e})$, $G(\odot_e)$, resp.). Let $\mathscr{A}_{\mathrm{ab}}$ denote the variety generated by all $G(*_g)$, G an *abelian* group.

Theorem 8.2 (1) *The varieties $\mathscr{A} \cap \mathscr{I}$ and $\mathscr{B} \cap \mathscr{I}$ coincide with the variety of LSLDI groupoids.*
(2) *The varieties $\mathscr{A}_{\mathrm{ab}} \cap \mathscr{I}$ and $\mathscr{C} \cap \mathscr{I}$ coincide with the variety of LSMI groupoids.*

(3) *The varieties $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{D}$ coincide with the variety of LSLD groupoids.*
(4) *The varieties $\mathscr{A}_{\mathrm{ab}}$ and $\mathscr{C}$ coincide with the variety of LSLDM groupoids.*

Proof. (1),(2) Let X be a set. It is sufficient to find in each of the varieties a groupoid, where all terms over X in the normal form from Theorem 6.4 have different values. Let G_X denote the free group over X and A_X the free abelian group over X. It is straightforward to check that the following groupoids have the property:

(a) $G_X(*_\iota)$, where ι stands for the inverse operation;
(b) $G_{X\cup\bar{X}}(*_{f,1})$, where $\bar{X}$ is a copy of X and f is the ivolution that maps $x \leftrightarrow \bar{x}$ for every $x \in X$.
(c) $A_X(*_\iota) = A_X(\diamond_{\iota,1})$, where ι stands for the inverse operation.

(3),(4) First, let's prove the case of the variety $\mathscr{D}$. Similarly as above, it is easy to check that the free product of G_X and $\mathbb{Z}_2$ equipped with the operation $\odot_u$, where u is the nonzero element of $\mathbb{Z}_2$, has the property that all terms over X in the normal form from Theorem 6.4 have different values. To prove the rest, just notice that the circle C_2 is in any non-idempotent LSLD variety (see Corollary 4.3) and thus one can use Theorem 5.4. □

Remark 8.3 One can consider a different non-idempotent generalization of the core. Put

$$a \star_f b = af(b)a$$

for all $a, b \in G$. Then $G(\star_f)$ is an LSLD grupoid, provided that f is an involutory automorphism of G and it satisfies $a^2 f(a^2) = 1$ and $af(a) \in Z(G)$ for every $a \in G$. Clearly, $G(\star_f)$ is idempotent iff f is the inverse operation. However, in this case, G is abelian. Note that the variety generated by all $G(\star_f)$, G a group, is a proper subvariety of LSLD groupoids, because it satisfies the equation

$$xyzxyzuzyxzyxv \approx uv.$$

Remark 8.4 In this section, we gathered entirely from the paper [Sta,05].

9. Subdirectly irreducible non-idempotent LD left quasigroups

We recall that, according to Corollary 4.4, there are very few simple non-idempotent LD left quasigroups. However, considering the subdirectly irreducible ones, we find a relatively rich, though rather tame, structure. Its description is the topic of the present section. The results were published recently as a (rather long) selfcontained paper [Sta,08]. For this reason, we omit many details here, and some proofs are rather sketchy. For a complete account, see the original paper.

We recall that a groupoid G is *subdirectly irreducible*, if it has the smallest nontrivial congruence, called the *monolith* and denoted μ_G. We also recall that

$$Ip_G = \{a \in G : aa = a\} \quad \text{and} \quad K_G = \{a \in G : aa \neq a\}$$

are either empty, or strong left ideals in an LD left quasigroup G.

Lemma 9.1 *Let G be a non-idempotent subdirectly irreducible LD left quasigroup. Then K_G contains no proper strong left ideal. Consequently, it contains no definable proper subset.*

Proof. Let $I \subset K_G$ be a proper strong left ideal in K_G and denote ρ_I the set of all $(a, b) \in ip$ such that $a = b$ or $a, b \in I$. This equivalence is a non-trivial congruence of G. Now, apply the same to the strong left ideal $J = K_G \setminus I$, obtain ρ_J and note that ρ_I and ρ_J have trivial intersection, contradicting subdirect irreducibility of G. The second statement follows from Lemma 3.5. □

Theorem 9.2 *Let G be a non-idempotent subdirectly irreducible LD left quasigroup. Then there is a prime p and a number r such that the cycle type of G is $\{p^r\}$ or $\{1, p^r\}$. Consequently, the monolith of G is below $\gamma_{p^{r-1}}$.*
Moreover, if G has term-definable left division, the monolith of G is $\gamma_{p^{r-1}}$.

Proof. First, assume that all non-trivial ip-blocks are infinite. Then $\gamma_k \neq \gamma_l$ for every $k \neq l$, and so there is an infinite decreasing sequence

$$\gamma_2 \supset \gamma_4 \supset \cdots \supset \gamma_{2^k} \supset \cdots$$

with trivial intersection, a contradiction. So, let n be the least number ≥ 2 that appears in the cycle type of G. Then $K_n = \{a \in G : a^{n+1} = a\}$ is a strong left ideal and thus $K_n = K_G$. It means that the cycle type is $\{n\}$ or $\{1, n\}$. If $n = kl$ for some relatively prime k, l, then γ_k and γ_l are non-trivial congruences with trivial intersection, a contradiction. Hence n is a prime power and $\mu_G \subseteq \gamma_{p^{r-1}}$.

If G has term-definable left division, consider the strong left ideal

$$I = \{a \in G : (a, b) \in \mu_G \text{ for some } b \neq a\}.$$

According to Lemma 9.1, $I = K_G$ and thus $\mu_G = \gamma_{p^{r-1}}$. □

Let $\mathrm{Aut}(G)$ denote the automorphism group of a groupoid G and let

$$\mathrm{Aut}_n(G) = \{\varphi \in \mathrm{Aut}(G) : \varphi^n = id\}.$$

It is easy to check that $\mathrm{Aut}_n(G)$ is a left n-symmetric subgroupoid of the conjugation groupoid of $\mathrm{Aut}(G)$.

Lemma 9.3 *Let K be an idempotent-free LD left quasigroup and I be a left subquasigroup of the conjugation groupoid of* $\mathrm{Aut}(K)$. *Let G be the disjoint union of I and K. Then the following conditions are equivalent.*

(1) *The operations of I and K can be extended onto G so that G becomes an LD left quasigroup with*

$$\varphi \cdot u = \varphi(u)$$

for all $\varphi \in I$, $u \in K$.

(2) *$L^K_u \varphi (L^K_u)^{-1} \in I$ and $(L^K_u)^{-1} \varphi L^K_u \in I$ for all $\varphi \in I$, $u \in K$; here L^K_u denotes the left translation of u in K.*

If the conditions are satisfied, the operation on G is uniquely determined and

$$u\cdot\varphi = L_u^K\varphi(L_u^K)^{-1} = L_u^K * \varphi$$

for all $\varphi\in I$, $u\in K$.
Moreover, G is n-LS, if and only if K is n-LS and $\varphi^n = id$ for every $\varphi\in I$.

Proof. This is a rather long but straightforward computation. □

We will denote the groupoid G from Lemma 9.3 by $I\sqcup K$ and call it the *extension* of K by I.

$I\sqcup K$	ψ	v
φ	$\varphi\psi\varphi^{-1}$	$\varphi(v)$
u	$L_u^K\psi(L_u^K)^{-1}$	uv

The *full extension* of K is the extension $\mathrm{Full}(K) = \mathrm{Aut}(K)\sqcup K$ and the *full n-extension* of K is $\mathrm{Full}_n(K) = \mathrm{Aut}_n(K)\sqcup K$.

We are ready to prove the main theorem, describing the structure of non-idempotent subdirectly irreducible LD left quasigroups.

Theorem 9.4 *Let G be a non-idempotent subdirectly irreducible LD left quasigroup. Then G embeds into* $\mathrm{Full}(K_G)$ *by an injective homomorphism* Φ *defined*

$$\Phi(u) = u \text{ for } u\in K_G \quad\text{ and }\quad \Phi(a) = L_a|_{K_G} \text{ for } a\in Ip_G.$$

Moreover, if G is n-LS, then it embeds by Φ *into* $\mathrm{Full}_n(K_G)$.

Proof. Using Lemma 3.2, it is easy to check that the mapping Φ is a homomorphism. To prove injectivity, we define an equivalence α on G by setting $(a,b)\in\alpha$ iff $a=b$ or a,b are idempotent and $L_a|_{K_G} = L_b|_{K_G}$. This is a congruence of G and the intersection of α and ip is trivial. Hence α is trivial and $L_a|_{K_G}\neq L_b|_{K_G}$ for all idempotent elements $a\neq b$.
Finally, if G is n-LS, then it embeds into $\mathrm{Full}_n(K_G)$, because $(L_a)^n = id$. □

It follows that every subdirectly irreducible LD left quasigroup is isomorphic to some $I\sqcup K$, where K is an idempotent-free LD left quasigroup and I is a left subquasigroup of the conjugation groupoid of $\mathrm{Aut}(K)$. In further text, we will address this situation by saying that $G = I\sqcup K$.

Let k,ℓ be positive integers. Denote $C(k,\ell)$ the set

$$\{0,\dots,k-1\}\times\{0,\dots,\ell-1\},$$

$P(k,\ell)$ the group of all permutations π on the set $C(k,\ell)$ such that

$$\pi(i,a) = (j,b) \text{ implies } \pi(i,a+1) = (j,b+1)$$

(here addition means mod ℓ) and

$$P_n(k,\ell) = \{\pi\in P(k,\ell) : \pi^n = id\}.$$

Corollary 9.5 *Let G be a non-idempotent subdirectly irreducible LD left quasigroup of cycle type $\{1, p^r\}$ with k non-trivial ip-blocks. Then*

$$|G| \leq kp^r + |P(k, p^r)| = kp^r + k!(p^r)^k.$$

Moreover, if G is n-LS, then

$$|G| \leq kp^r + |P_n(k, p^r)|.$$

The upper bound on the number of idempotent elements is optimal. For every k and p^r, there is a subdirectly irreducible LD left quasigroup G of cycle type $\{1, p^r\}$ with k non-trivial ip-blocks such that $|G| = kp^r + |P(k, p^r)|$. The bound is optimal also in the case of n-LSLD groupoids, provided n has a proper divisor not greater than k (and indeed $p^r \mid n$, for otherwise there is no such n-LSLD groupoid).

Example 9.6 Let $K = C(k, p^r)$ and put

$$(i, a) \cdot (j, b) = (j, b + 1)$$

for every $0 \leq i, j < k$ and $0 \leq a, b < p^r$. Easily, K is an LD left quasigroup and $\mathrm{Aut}(K) = P(k, p^r)$. Moreover, K is n-LS iff $p^r \mid n$, and $\mathrm{Aut}_n(K) = P_n(k, p^r)$. Thus $|\mathrm{Full}(K)|$ and $|\mathrm{Full}_n(K)|$ attain the upper bound from Corollary 9.5. One can prove that $\mathrm{Full}(K)$ and $\mathrm{Full}_n(K)$ are subdirectly irreducible whenever n is divisible by p^r and by some number q with $1 < q \leq k$ (use Theorem 9.10).

The upper bound for n-LSLD groupoids is not necessarily reached when k is too small, i.e. when no $1 < q \leq k$ divides n. For example, we prove that there is no subdirectly irreducible 3-LSLD groupoid (of cycle type $\{1,3\}$) with two non-trivial ip-blocks, regardless the number of idempotent elements. First, note that $P_3(2, 3)$ is not transitive and the sets $\{i\} \times \{0, 1, 2\}$, $i = 0, 1$ are its orbits. Second, note that there is only one (up to isomorphism) two-element idempotent LD left quasigroup (see Example 2.6) and thus every idempotent-free 3-LSLD groupoid K with two ip-blocks contains two proper (strong) left ideals (namely, each of the two ip-blocks) and so does $\mathrm{Full}_3(K)$. Hence, according to Lemma 9.1, $\mathrm{Full}_3(K)$ is not subdirectly irreducible.

We also note that the above considerations are not limited to finite groupoids; if k is an infinite cardinal number, then the upper bound on the number of idempotents in a subdirectly irreducible LD left quasigroup with k (finite) blocks is 2^k and this bound is reached by a simple modification of Example 9.6. (The condition "$q \mid n$ for some $1 < q \leq k$" becomes trivial here.)

We finish the section with a couple of properties of subdirectly irreducible LD left quasigroups. Proofs are omitted, see [Sta,08].

Let G be an LD left quasigroup and $k \geq 1$. We define a mapping

$$\rho_k : G \to G, \quad \rho_k(a) = a^k.$$

This is an automorphism of G, because $(xy)^k \approx_{LD} xy^k \approx_{LI} x^k y^k$. Moreover, ρ_k commutes with any automorphism φ of G, because $\varphi(a^k) = \varphi(a)^k$ for every $a \in G$. Consequently, $\{\rho_k\}$ is a one-element strong left ideal in $\mathrm{Full}(K)$ for any idempotent-free

LD left quasigroup K. If G is a subgroupoid of $\mathrm{Full}(K)$, we will denote G^- the subgroupoid $G \setminus \{\rho_k : k \in \mathbb{N}\}$.

Proposition 9.7 *Let $G = I \sqcup K$. Then G is subdirectly irreducible, if and only if G^- is subdirectly irreducible.*

Proposition 9.8 *Let $G_1 = I_1 \sqcup K$ and $G_2 = I_2 \sqcup K$ be non-idempotent LD left quasigroups with $I_1 \subseteq I_2$. If G_1 is subdirectly irreducible, then G_2 is so.*

The following theorems settle conditions, when an idempotent-free LD left quasigroup possesses a subdirectly irreducible extension.

Theorem 9.9 *Let K be an idempotent-free LD left quasigroup. The following statements are equivalent:*

(1) *There is a subdirectly irreducible LD left quasigroup G with $K_G = K$.*
(2) $\mathrm{Full}(K)$ *is subdirectly irreducible.*
(3) $\mathrm{Full}(K)^-$ *is subdirectly irreducible.*

The three statements are implied by

(4) *There is a congruence ν of K such that every non-trivial* $\mathrm{Aut}(K)$*-invariant congruence of K contains ν.*

Moreover, if K has term-definable left division and cycle type $\{p^r\}$, then each of the four statements is equivalent to

(5) *Every non-trivial* $\mathrm{Aut}(K)$*-invariant congruence of K contains $\gamma_{p^{r-1}}$.*

Theorem 9.10 *Let K be an idempotent-free n-LSLD groupoid of cycle type $\{p^r\}$. The following statements are equivalent:*

(1) *There is a subdirectly irreducible n-LSLD groupoid G with $K_G = K$.*
(2) $\mathrm{Full}_n(K)$ *is subdirectly irreducible.*
(3) $\mathrm{Full}_n(K)^-$ *is subdirectly irreducible.*
(4) *Every non-trivial* $\mathrm{Aut}_n(K)$*-invariant congruence of K contains $\gamma_{p^{r-1}}$.*

Remark 9.11 The study of subdirectly irreducible LSLD groupoids was initiated by T. Kepka [Kep,94] and the restriction of the results of this section to LSLD groupoids was published by T. Kepka, E. Jeřábek and the author in [JeřKS,05], together with many examples. The results were generalized to the present form in [Sta,08], where one also finds a list of small subdirectly irreducible LD left quasigroups.

10. Further results

We say that a left quasigroup G has the *left inverse property*, if for every $a \in G$ there is an element $b \in G$ such that $(L_a)^{-1} = L_b$. In other words, such that $a \backslash u = bu$ for every $u \in G$. (Hence b may be considered as "a^{-1}".)

Proposition 10.1 *Every LD left quasigroup G can be embedded into an LD left quasigroup H such that for every integer n and for every $a \in H$, there is $b \in H$ with $(L_a)^n = L_b$.*

Proof. Let $H = G \times \mathbb{Z}$ and put $(a,k) \cdot (b,l) = (a^{[k]}b, l)$. Clearly, $a \mapsto (a,1)$ is an embedding of G into H. The groupoid H is a left quasigroup, since any left translation $L_{(a,k)}$ acts on each $G \times \{l\}$ like the permutation $(L_a)^k$. It is left distributive, since $(a,k)\cdot((b,l)\cdot(c,m)) = (a,k)\cdot(b^{[l]}c,m) = (a^{[k]}b^{[l]}c,m) =_{LD} ((a^{[k]}b)^{[l]}c,m) = (a^{[k]}b,l)\cdot(a^{[k]}c,m) = ((a,k)\cdot(b,l))\cdot((a,k)\cdot(c,m))$. And for every integer n and $(a,k) \in H$, we claim that $(L_{(a,k)})^n = L_{(a,kn)}$. Indeed, $(a,k)^{[n]} \cdot (b,l) = (a^{[kn]}b,l) = (a,kn)\cdot(b,l)$. □

Corollary 10.2 *Every LD left quasigroup can be embedded into an LD left quasigroup with the left inverse property.*

In fact, to obtain just the left inverse property, one can do slightly better: restrict the second coordinate to ± 1 and obtain a smaller $H = G \times \{1,-1\}$.

There is a connection between congruences of an LD left quasigroup and normal subgroups of its left multiplication group.

Proposition 10.3 *Let G be an LD left quasigroup.*

(1) *For a congruence ρ of G, put $N_\rho = \langle L_a L_b^{-1} : (a,b) \in \rho\rangle$. Then N_ρ is a normal subgroup of* $\mathrm{LMlt}(G)$.

(2) *For a normal subgroup N of* $\mathrm{LMlt}(G)$, *put $(a,b) \in \rho_N$, iff there is $\varphi \in N$ such that $\varphi(a) = b$. Then ρ_N is a congruence of G.*

Proof. (1) It is sufficient to show that $L_c L_a L_b^{-1} L_c^{-1} \in N_\rho$ for every $(a,b) \in \rho$ and $c \in G$. Due to Lemma 3.3(2), this expression is equal to $L_{ca}L_{cb}^{-1}$ and indeed $(ca,cb) \in \rho$.

(2) Clearly, ρ_N is an equivalence on G. Let $(a,b) \in \rho_N$ and $c \in G$. We have $bc = \varphi(a)c = \varphi(a)(a\backslash ac) = L_{\varphi(a)}L_a^{-1}(ac) = \varphi L_a \varphi^{-1} L_a^{-1}(ac)$, and so $(ac,bc) \in \rho_N$, because $\varphi \in N$ and, by normality of N, also $L_a\varphi^{-1}L_a^{-1} \in N$. Further, $cb = c\varphi(a) = \varphi(\varphi^{-1}(c)a) = \varphi(\varphi^{-1}(c)(c\backslash ca)) = \varphi L_{\varphi^{-1}(c)} L_c^{-1}(ca) = L_c \varphi L_c^{-1}(ca)$, and so $(ca,cb) \in \rho_N$, because $L_c\varphi L_c^{-1} \in N$. □

Remark 10.4 Both statements appeared in the author's PhD thesis [Sta,04a]. Proposition 10.3 is a straightforward generalization of observations of H. Nagao [Nag,79] for LSLD groupoids.

References

[Bri,88] Brieskorn, E.: *Automorphic sets and singularities,* Contemp. Math. **78** (1988), 45–115.

[DráKM,94] Drápal, A., Kepka, T., Musílek, M.: *Group conjugation has non-trivial LD-identities,* Comment. Math. Univ. Carolinae, **35/2** (1994), 219–222.

[FenR,92] Fenn, R., Rourke, C.: *Racks and links in codimension two,* J. Knot Theory Ramifications **1** (1992), 343–406.

[Jed,05] Jedlička, P.: *On left distributive left idempotent groupoids,* Comment. Math. Univ. Carolinae **46/1** (2005), 15–20.

[JeřKS,05] Jeřábek, E., Kepka, T., Stanovský, D.: *Subdirectly irreducible non-idempotent left symmetric left distributive groupoids,* Discussiones Math. – General Algebra and Appl. **25/2** (2005), 235–257.

[Joy,82a] Joyce, D.: *A classifying invariant of knots, the knot quandle,* J. Pure Appl. Alg. **23** (1982), 37–66.

[Joy,82b] Joyce, D.: *Simple quandles,* J. Alg. **79** (1982), 307–318.

[Kep,94] Kepka, T.: *Non-idempotent left symmetric left distributive groupoids,* Comment. Math. Univ. Carolinae **35/1** (1994), 181–186.

[KepN,03] Kepka, T., Němec, P.: *Selfdistributive groupoids. A1. Non-indempotent left distributive groupoids,* Acta Univ. Carolin. Math. Phys. **44/1** (2003), 3–94.

[Lar,99] Larue, D.: *Left-distributive idempotent algebras,* Commun. Alg., **27/5** (1999), 2003–2009.

[Mat,84] Matveev, S. V.: *Distributive groupoids in knot theory,* Math. USSR Sbornik, **47/1** (1984), 73–83. (Mat. Sbornik 119/1 (1982) in Russian)

[Nag,79] Nagao, H.: *A remark on simple symmetric sets,* Osaka J. Math. **16** (1979), 349–352.

[Nob,83] Nobusawa, N.: *Some structure theorems on pseudo-symmetric sets,* Osaka J. Math. **20** (1983), 727–734.

[Pło,85] Płonka, J.: *On k-cyclic groupoids,* Mat. Japonica **30** (1985), 371–382.

[Ros,87] Roszkowska, B.: *The lattice of varieties of symmetric idempotent entropic groupoids,* Demonstratio Math. **20** (1987), 259–275.

[Ryd,95] Ryder, H.: *The congruence structure of racks,* Commun. Alg. **23** (1995), 4971–4989.

[Sta,04a] Stanovský, D.: *Left distributive left quasigroups,* PhD Thesis, Charles University in Prague, 2004.

[Sta,04b] Stanovský, D.: *On varieties of left distributive left idempotent groupoids,* Discussiones Math. – General Algebra and Appl. **24/2** (2004), 267–275.

[Sta,05] Stanovský, D.: *Left symmetric left distributive operations on a group,* Algebra Universalis **54/1** (2005), 97–103.

[Sta,08] Stanovský, D.: *Subdirectly irreducible non-idempotent left distributive left quasigroups,* Commun. Algebra, **36/7** (2008), 2654–2669.