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Stochastic Bilinear Equations with Fractional Gaussian Noise in Hilbert Space

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A fractional Gaussian noise is a formal derivative of a fractional Brownian motion. An explicit formula for a weak solution to the stochastic billinear equation in a separable Hilbert space with fractional Gaussian noise in the singular case $H < 1/2$ is given. The stochastic integral is understood in the Skorokhod sense.

1. Introduction

It is well known that the unique solution to a stochastic bilinear equation

$$\begin{aligned} \mathrm{d}X(t) &= A(t)X(t)\mathrm{d}t + BX(t)\mathrm{d}W(t), \\ X(0) &= x_0, \end{aligned} \tag{1.1}$$

where A is a real-valued bounded Borel function, $B, x_0 \in \mathbb{R}$, and W is a standard Brownian motion (Wiener process), is given by an explicit formula

$$X(t) = \exp\{BW(t)\} \exp\Big\{\int_0^t A(u)\,\mathrm{d}u - \frac{1}{2}B^2 t\Big\} x_0. \tag{1.2}$$

Denoting by U the fundamental solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \Big(A(t) - \frac{1}{2}B^2 t\Big) x$$

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and by S_B the group generated by B, $S_B(t) = \exp\{Bt\}$, we may rewrite (1.2) as

$$X(t) = S_B(W(t))U(t,0)x_0. \tag{1.3}$$

In this form, the result remains valid for the multidimensional generalizations of (1.1), when $A(t)$ and B are commuting matrices, and for bilinear stochastic evolution equations in a Hilbert space (see Chapter 6 in [4] for a thorough discussion).

In the paper [5], a formula of the type (1.3) was established for solutions to a stochastic bilinear equation in a Hilbert space, driven by a fractional Brownian motion. Recall that a fractional Brownian motion on an interval $[0,T]$ with a Hurst parameter $H \in (0,1)$ is a real-valued centered Gaussian process $\{B^H(t), t \in [0,T]\}$, the covariance of which is given by

$$\mathbb{E}\left[B^H(s)B^H(t)\right] = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \;\; s,t \in [0,T].$$

Let V be a separable Hilbert space, $A(t) : \mathrm{Dom}(A(t)) \to V,\; t \in [0,T]$, closed linear operators generating on evolution system on V and $B : \mathrm{Dom}(B) \to V$ a generator of a strongly continuous group S_B on V, commuting with $A(t)$. Let us consider an equation

$$\begin{aligned} \mathrm{d}X(t) &= A(t)X(t)\,\mathrm{d}t + BX(t)\,\mathrm{d}B^H(t) \\ X(0) &= x_0 \end{aligned} \tag{1.4}$$

in V. If $H > 1/2$, it is shown in [5] that (under some additional hypothesis upon $A(t)$ and B) the weak solution of (1.4) has again the form (1.3), where now U denotes the evolution system generated by $\{A(t) - Ht^{2H-1}B^2, t \in [0,T]\}$.

We aim at extending the result from [5] to the singular case $H < 1/2$. In this case, one faces at least two problems. First, stochastic integrals with respect to a fractional Brownian motion with $H < 1/2$ behave much less regularly than those for $H > 1/2$. In our paper, we use the Skorokhod-type stochastic integral introduced in [2], and the corresponding change of variables formula ([2], Corollary 4.8). Secondly, the function $t \mapsto Ht^{2H-1}$ blows up as $t \to 0+$ if $H < 1/2$, so it is not obvious, whether $\{A(t) - Ht^{2H-1}B^2, t \in [0,T]\}$ still generates an evolution system, even if the operators $A(t)$ and B are "nice". The corresponding evolution system U is constructed in Section 2 of the paper. We do not know if U is smooth enough, so one cannot apply the Itô formula directly to the right-hand side of (1.3) and one has to resort to a suitable approximation procedure; this is done in Section 3. In Section 4, two illustrative examples are given.

Finally, let us note that two particular cases of our result have been already studied. In [11], a one-dimensional space V is dealt with. In [12], the space V may be infinite-dimensional, but $A(t)$ and B must be bounded.

2. Deterministic equations

We would like to use the methods from [5] where the system of linear operators $\{A(t) - Ht^{2H-1}B^2, t \in [0,T]\}$ (under additional assumptions) is well-defined and generates a strongly continuous evolution system and the standard one-dimensional Itô formula for a fractional Brownian motion can be applied. But in the case $H < \frac{1}{2}$ the system of operators $\{A(t) - Ht^{2H-1}B^2, t \in [0,T]\}$ has a singularity at $t = 0$ because

$$t^{2H-1} \longrightarrow +\infty \quad \text{as} \quad t \to 0+$$

so we use the approximating sequence $\{u_n, n \in \mathbb{N}\}$ of the function $u(t) = t^{2H-1}, t > 0$, defined as

$$u_n(t) = \begin{cases} t^{2H-1} & , \quad t > \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{2H-1} & , \quad 0 \le t \le \frac{1}{n}. \end{cases}$$

and pass to the limit in an appropriate sense.

The approximating sequence $\{u_n, n \in \mathbb{N}\}$ has the following important properties

(U1) for all $n \in \mathbb{N}$ the function u_n is Lipschitz continuous on the interval $[0,T]$

(U2) u_n converges to u in the space $L^1([0,T])$

(U3) for all $n \in \mathbb{N}$ and $t > 0 \quad 0 \le u_n(t) \le u(t)$.

We have to assume that the system of linear operators $\{A(t), t \in [0,T]\}$ on V satisfies

(A1) for all $t \in [0,T]$ the operators $A(t)$ are closed and densely defined with the domain $D := \mathrm{Dom}(A(t))$ independent of t

(A2) the resolvent set contains all $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \ge \omega$ for some fixed $\omega \in \mathbb{R}$ and for some constant $M > 0$ independent of t the resolvent $R(\lambda, A(t))$ satisfies

$$\|R(\lambda, A(t))\|_{\mathscr{L}(V)} \le \frac{M}{|\lambda - \omega| + 1}$$

for all $\lambda \in \mathbb{C}, \Re(\lambda) \ge \omega, t \in [0,T]$.

(A3) there exist constants $L > 0$ and $0 < \gamma \le 1$ such that

$$\|A(t) - A(s)\|_{\mathscr{L}(D;V)} \le L|t - s|^\gamma,$$

where the space D is equipped with the graph norm generated by the operator $A(0) - \omega I$, i.e.

$$\|x\|_V + \|(A(0) - \omega I)x\|_V.$$

These conditions (A1), (A2), (A3) imply that the system of operators $\{A(t), t \in [0,T]\}$ generates a strongly continuous evolution system $\{U_A(t,s), 0 \le s \le t \le T\}$ satisfying (see e.g. [13], Theorem 5.2.1.)

$$\mathrm{Im}(U_A(t,s)) \subset D, \tag{2.1}$$

$$\|U_A(t,s)\|_{\mathscr{L}(V)} \le C, \tag{2.2}$$

$$\left\|\frac{\partial}{\partial t} U_A(t,s)\right\|_{\mathscr{L}(V)} = \|A(t)U_A(t,s)\|_{\mathscr{L}(V)} \le \frac{C}{t-s}, \tag{2.3}$$

$$\|A(t)U_A(t,s)(A(s) - \omega I)^{-1}\|_{\mathscr{L}(V)} \le C \tag{2.4}$$

for some constant $C > 0$ and any $0 \leq s < t \leq T$.
For any $n \in \mathbb{N}$ we define the system of linear operators $\{A_n(t), t \in [0,T]\}$ on V with the domain D by

$$A_n(t) = A(t) - Hu_n(t)B^2, t \in [0,T],$$

and we will show that this system generates a strongly continuous evolution system on V. For simplicity we can assume that $\omega < 0$. Let us remind that since the operator $-A(0)$ is sectorial, the fractional powers $(-A(0))^\alpha$ for $\alpha \in (0,1]$ are well-defined (see e.g. [9]). Since the graph norms $\|x\|_V + \|(A(t) - \omega_0 I)x\|_V, t \in [0,T], \omega_0 \geq \omega$, generated by operators $A(t) - \omega_0 I, t \in [0,T], \omega_0 \geq \omega$, are equivalent, we can choose one fixed norm

$$\|x\|_D = \|x\|_V + \|A(0)x\|_V$$

on D.

Proposition 2.1 *Assume that the conditions* (A1), (A2), (A3) *are satisfied for the system* $\{A(t), t \in [0,T]\}$. *Let* $B : \mathrm{Dom}(B) \to V$ *be a linear densely defined operator such that* B^2 *is closed and* $\mathrm{Dom}(B^2) \supset \mathrm{Dom}((-A(0))^\alpha)$ *for some* $\alpha \in (0,1)$.
Then the conditions (A1), (A2), (A3) *are satisfied for the system* $\{A_n(t), t \in [0,T]\}$ *with* $\mathrm{Dom}(A_n(t)) = D$ *and any fixed* $n \in \mathbb{N}$. *Thus the system of operators* $\{A_n(t), t \in [0,T]\}$ *generates strongly continuous evolution systems* $\{U_n(t,s), 0 \leq s \leq t \leq T\}$ *on* V.

Proof. First note that the assumption (A3) is equivalent to

$$\left\|(A(t) - A(s))A^{-1}(0)\right\|_{\mathscr{L}(V)} \leq L|t-s|^\gamma \tag{2.5}$$

which implies that there exists a constant $C_0 > 0$ independent of t such that

$$\|A(0)x\|_V \leq C_0\|A(t)x\|_V \tag{2.6}$$

for all $t \in [0,T]$ and $x \in D$.
Indeed, (2.5) is equivalent to

$$\left\|A(0)(A^{-1}(t) - A^{-1}(s))\right\|_{\mathscr{L}(V)} \leq \tilde{L}|t-s|^\gamma$$

for some constants $\tilde{L} > 0$ and $0 < \gamma \leq 1$ (see [3], p. 32). Thus for $s = 0$ we get

$$\|A(0)A^{-1}(t) - I\|_{\mathscr{L}(V)} \leq \tilde{L}T^\gamma,\ 0 \leq t \leq T,$$

so

$$\|A(0)A^{-1}(t)\|_{\mathscr{L}(V)} \leq 1 + \tilde{L}T^\gamma,\ 0 \leq t \leq T,$$

which is equivalent to (2.6).
Now we can use (2.6) and (A2) to get

$$\|A(0)R(\lambda, A(t))x\|_V \leq C_0\|A(t)R(\lambda, A(t))x\|_V \leq C_0(M(1+\omega)+1)\|x\|_V \tag{2.7}$$

for any $x \in V$ and $\lambda \in \mathbb{C},\ \Re(\lambda) \geq \omega$. By the Corollary 2.6.11 from [9] there exists a constant $C_{A(0)} > 0$ depending on $A(0)$ such that for any $\rho > 0$ and $x \in V$

$$\|B^2 R(\lambda, A(t))x\|_V \leq C_{A(0)}[\rho^\alpha\|R(\lambda, A(t))x\|_V + \rho^{\alpha-1}\|A(0)R(\lambda, A(t))\|_V].$$

Using (A2) and (2.7)

$$\|B^2R(\lambda, A(t))x\|_V \leq C_{A(0)}[\rho^\alpha \frac{M}{1+|\lambda-\omega|}\|x\|_V + \rho^{\alpha-1}C_0(M(1+\omega)+1)\|x\|_V].$$

Thus

$$\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathscr{L}(V)} \leq H\|u_n\|_{\mathscr{C}([0,T])}C_{A(0)}[\rho^\alpha \frac{M}{1+|\lambda-\omega|} + \rho^{\alpha-1}C_0(M(1+\omega)+1)].$$

For $\rho > 0$ enough large we get

$$H\|u_n\|_{\mathscr{C}([0,T])}C_{A(0)}\rho^{\alpha-1}C_0(M(1+\omega)+1) < \frac{1}{2}$$

hence

$$\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathscr{L}(V)} \leq H\|u_n\|_{\mathscr{C}([0,T])}C_{A(0)}\rho^\alpha \frac{M}{1+|\lambda-\omega|} + \frac{1}{2}.$$

If we now choose some $\omega_1 \geq \omega$ such that for all $\lambda \in \mathbb{C}$, $\Re(\lambda) \geq \omega_1$ and

$$2H\|u_n\|_{\mathscr{C}([0,T])}C_{A(0)}\rho^\alpha M - 1 + \omega < \Re(\lambda)$$

then

$$\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathscr{L}(V)} \leq K < 1$$

for all $t \in [0, T]$, where $K > 0$ is a constant strictly smaller than 1.
Therefore

$$\begin{aligned}
\|R(\lambda, A_n(t))\|_{\mathscr{L}(V)} &= \|(\lambda I - A(t) + Hu_n(t)B^2)^{-1}\|_{\mathscr{L}(V)} \\
&= \left\|\left[I(\lambda I - A(t)) + Hu_n(t)B^2R(\lambda, A(t))(\lambda I - A(t))\right]^{-1}\right\|_{\mathscr{L}(V)} \\
&= \left\|\left\{[I + Hu_n(t)B^2R(\lambda, A(t))](\lambda I - A(t))\right\}^{-1}\right\|_{\mathscr{L}(V)} \\
&= \left\|R(\lambda, A(t))\left[I - (-Hu_n(t)B^2R(\lambda, A(t)))\right]^{-1}\right\|_{\mathscr{L}(V)} \\
&\leq \frac{M}{1+|\lambda-\omega|} \times \frac{1}{1-K} \times \frac{1+|\lambda-\omega_1|}{1+|\lambda-\omega_1|} \\
&\leq \frac{M}{1-K} \times \frac{1}{1+|\lambda-\omega_1|} \times \left(\frac{1}{1+|\lambda-\omega|} + \frac{|\lambda-\omega|}{1+|\lambda-\omega|} + \frac{|\omega-\omega_1|}{1+|\lambda-\omega|}\right) \\
&\leq \frac{M(2+|\omega_1-\omega|)}{1-K} \times \frac{1}{1+|\lambda-\omega_1|}
\end{aligned}$$

which is (A2) for the system of operators $\{A_n(t), t \in [0, T]\}$.
From (A3) and (U1) we have

$$\begin{aligned}
\|A(t) - A(s)\|_{\mathscr{L}(D;V)} &\leq L|t-s|^\gamma, \\
|u_n(t) - u_n(s)| &\leq L_u|t-s|^\gamma
\end{aligned}$$

for some constants $L, L_u > 0$. Note that the norm $\|x\|_V + \|(A_n(t) - \omega_1 I)x\|_V$ is dominated by the norm $\|x\|_D$. Thus

$$\begin{aligned}\|A_n(t) - A_n(s)\|_{\mathscr{L}(D;V)} &\le \|A(t) - A(s)\|_{\mathscr{L}(D;V)} + H|u_n(t) - u_n(s)|\|B^2\|_{\mathscr{L}(D;V)} \\ &\le L|t-s|^\gamma + HL_u|t-s|^\gamma \|B^2\|_{\mathscr{L}(D;V)} \le L_{A_n}|t-s|^\gamma\end{aligned}$$

for some finite constant $L_{A_n} > 0$ because the operators $B^2 A^{-1}(0) \in \mathscr{L}(V)$ by the closed graph theorem, so (A3) is satisfied for the system of operators $\{A_n(t), t \in \in [0, T]\}$. □

Since $\{U_n(t, s), 0 \le s \le t \le T\}$ is a strongly continuous evolution system for any $n \in \mathbb{N}$ it satisfies the equations

$$\frac{\partial}{\partial t} U_n(t, s)x = (A(t) - Hu_n(t)B^2)U_n(t, s)x$$

and

$$U_n(t, s)x = U_A(t, s)x - \int_s^t Hu_n(r)U_A(t, r)B^2 U_n(r, s)x \,\mathrm{d}r$$

for any $x \in V$ and $0 \le s \le t \le T$.

Proposition 2.2 *Let* $\{U_A(t, s), 0 \le s \le t \le T\}$ *be a strongly continuous evolution system and* $B : \mathrm{Dom}(B) \to V$ *be a linear densely defined operator such that* B^2 *is closed and* $\mathrm{Dom}(B^2) \supset D$*. Moreover, assume that*

$$\|U_A(t, s)B^2\|_{\mathscr{L}(V)} \le \frac{C_A}{(t-s)^\beta} \tag{2.8}$$

for some constants $C_A > 0,\ 0 < \beta < 2H$ *and* $0 \le s < t \le T$.
Then for any $x \in V$ *there exists unique continuous solution* $\{U(t, 0)x, 0 \le t \le T\}$ *to the equation*

$$y(t) = U_A(t, 0)x - \int_0^t Hr^{2H-1} U_A(t, r)B^2 y(r) \,\mathrm{d}r \tag{2.9}$$

on the interval $[0, T]$.

Proof. Fix $x \in V$. We show that the mapping

$$(\Phi(y))(t) = U_A(t, 0)x - \int_0^t Hr^{2H-1} U_A(t, r)B^2 y(r) \,\mathrm{d}r$$

is continuous from $\mathscr{C}([0, T]; V)$ into $\mathscr{C}([0, T]; V)$ (we denote by $\mathscr{C}([0, T]; V)$ the space of all continuous functions from the interval $[0, T]$ to the space V) and that Φ is a contraction mapping.

Take $y \in \mathscr{C}([0,T];V)$ and $t_1, t_2 \in [0,T], t_1 < t_2$. Then

$$\begin{aligned}
&\left\|(\Phi(y))(t_2) - (\Phi(y))(t_1)\right\|_V \leq \|U_A(t_2,0)x - U_A(t_1,0)x\|_V \\
&\quad + \left\| \int_0^{t_2} Hr^{2H-1} U_A(t_2,r)B^2 y(r)\,\mathrm{d}r - \int_0^{t_1} Hr^{2H-1} U_A(t_1,r)B^2 y(r)\,\mathrm{d}r \right\|_V \\
&\quad \leq \|U_A(t_2,0)x - U_A(t_1,0)x\|_V + \left\| \int_0^{t_1} Hr^{2H-1}(U_A(t_2,r) - U_A(t_1,r))B^2 y(r)\,\mathrm{d}r \right\|_V \\
&\quad + \left\| \int_{t_1}^{t_2} Hr^{2H-1} U_A(t_2,r)B^2 y(r)\,\mathrm{d}r \right\|_V = T_1 + T_2 + T_3.
\end{aligned}$$

Since $t \mapsto U_A(t,0)x$ is continuous for any $x \in V$ we have

$$T_1 = \|U_A(t_2,0)x - U_A(t_1,0)x\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$.
Since for any $0 < r < t_1$ and some $r < t_3 < t_1$

$$\begin{aligned}
&\left\|Hr^{2H-1}(U_A(t_2,r) - U_A(t_1,r))B^2 y(r)\right\|_V \\
&\quad = \left\|Hr^{2H-1}(U_A(t_2,t_3)U_A(t_3,r) - U_A(t_1,t_3)U_A(t_3,r))B^2 y(r)\right\|_V \\
&\quad \leq Hr^{2H-1}\|U_A(t_2,t_3) - U_A(t_1,t_3)\|_{\mathscr{L}(V)}\|U_A(t_3,r)B^2 y(r)\|_V \longrightarrow 0
\end{aligned}$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$ and by (2.8)

$$\begin{aligned}
&\left\| \int_0^{t_1} Hr^{2H-1}(U_A(t_2,r) - U_A(t_1,r))B^2 y(r)\,\mathrm{d}r \right\|_V \\
&\quad \leq H\|y\|_{\mathscr{C}([0,T];V)} \int_0^{t_1} r^{2H-1}[\|U_A(t_2,r)B^2\|_{\mathscr{L}(V)} + \|U_A(t_1,r)B^2\|_{\mathscr{L}(V)}]\,\mathrm{d}r \\
&\quad \leq H\|y\|_{\mathscr{C}([0,T];V)} C_A \int_0^{t_1} \left[\frac{r^{2H-1}}{(t_2-r)^\beta} + \frac{r^{2H-1}}{(t_1-r)^\beta} \right] \mathrm{d}r \leq 2H\|y\|_{\mathscr{C}([0,T];V)} C_A \int_0^{t_1} \frac{r^{2H-1}}{(t_1-r)^\beta}\,\mathrm{d}r \\
&\quad = 2H\|y\|_{\mathscr{C}([0,T];V)} C_A t_1^{2H-\beta} \int_0^1 r^{2H-1}(1-r)^{-\beta}\,\mathrm{d}r \\
&\quad \leq 2H\|y\|_{\mathscr{C}([0,T];V)} C_A T^{2H-\beta} \mathrm{B}(2H, 1-\beta) < +\infty,
\end{aligned}$$

thus

$$T_2 = \left\| \int_0^{t_1} Hr^{2H-1}(U_A(t_2,r) - U_A(t_1,r))B^2 y(r)\,\mathrm{d}r \right\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$ by the Lebesgue dominated convergence theorem. Recall that $\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du,\ a > 0, b > 0$, denotes the Beta function.
By (2.8) we get

$$\begin{aligned}
T_3 &= \left\| \int_{t_1}^{t_2} Hr^{2H-1} U_A(t_2,r)B^2 y(r)\,\mathrm{d}r \right\|_V \leq H\|y\|_{\mathscr{C}([0,T];V)} C_A \int_{t_1}^{t_2} \frac{r^{2H-1}}{(t_2-r)^\beta}\,\mathrm{d}r \\
&= H\|y\|_{\mathscr{C}([0,T];V)} C_A t_2^{2H-\beta} \int_{\frac{t_1}{t_2}}^1 r^{2H-1}(1-r)^{-\beta}\,\mathrm{d}r \longrightarrow 0
\end{aligned}$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$. Therefore

$$\left\|(\Phi(y))(t_2) - (\Phi(y))(t_1)\right\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$ and the function $t \mapsto (\Phi(y))(t)$ is continuous on $[0, T]$ for any $y \in \mathscr{C}([0, T]; V)$.
For any $y_1, y_2 \in \mathscr{C}([0, T]; V)$, $t \in [0, T]$ and $T > 0$ small enough there exists a constant $0 < L_T < 1$ depending only on A, B, T, H such that

$$\begin{aligned}
&\left\|(\Phi(y_2))(t) - (\Phi(y_1))(t)\right\|_V = \left\| \int_0^t H r^{2H-1} U_A(t, r) B^2 (y_2(r) - y_1(r))\,\mathrm{d}r \right\|_V \\
&\leq H \|y_1 - y_2\|_{\mathscr{C}([0,T];V)} C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta}\,\mathrm{d}r \leq H \|y_1 - y_2\|_{\mathscr{C}([0,T];V)} C_A T^{2H-\beta} \mathrm{B}(2H, 1-\beta) \\
&\leq L_T \|y_1 - y_2\|_{\mathscr{C}([0,T];V)}
\end{aligned}$$

holds so Φ is a contraction. Hence, by the Banach fixed-point theorem there exists a unique solution to the equation (2.9) for T enough small. Applying standard methods we get a unique continuous solution $(U(t, 0)x, t \in [0, T])$ to (2.9) for any $T > 0$. □

The next proposition describes the relation between U_n and U.

Proposition 2.3 *Let $\{U_n(t, s), 0 \leq s \leq t \leq T\}$ be strongly continuous evolution systems on V associated with the operators $\{A_n(t), t \in [0, T]\}$. Suppose that the assumptions of Proposition 2.2 are satisfied. Then for any $x \in V$ there exists a constant $K_U > 0$ depending only on H, A, B and T such that*

$$\sup \{\|U_n(t, 0)x\|_V;\ n \in \mathbb{N},\ 0 \leq t \leq T\} \leq K_U \|x\|_V. \tag{2.10}$$

Moreover, the convergence

$$\|U_n(\,.\,, 0)x - U(\,.\,, 0)x\|_{\mathscr{C}([0,T];V)} \xrightarrow[n \to +\infty]{} 0 \tag{2.11}$$

holds for any $x \in V$.

Proof. Fix $x \in V$. For any $n \in \mathbb{N}$ and $t \in [0, T]$ using (2.2), (2.8) we obtain

$$\begin{aligned}
\|U_n(t, 0)x\|_V &\leq \|U_A(t, 0)x\| + \left\| \int_0^t H u_n(r) U_A(t, r) B^2 U_n(r, 0)x\,\mathrm{d}r \right\|_V \\
&\leq C\|x\|_V + H C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta} \|U_n(r, 0)x\|_V\,\mathrm{d}r.
\end{aligned}$$

The generalized Gronwall inequality (see [8], Lemma 7.1.2) yields

$$\|U_n(t, 0)x\|_V \leq K_U \|x\|_V$$

for some finite constant $K_U > 0$ independent of n, t and the first part of the statement holds.

It remains to prove the second part. For any $x \in V$ and $t \in [0, T]$ using (2.10) and (2.8) we get

$$\begin{aligned}
&\|U_n(t,0)x - U(t,0)x\|_V \\
&\quad = \left\| \int_0^t H u_n(r) U_A(t,r) B^2 U_n(r,0)x \, \mathrm{d}r - \int_0^t H r^{2H-1} U_A(t,r) B^2 U(r,0)x \, \mathrm{d}r \right\|_V \\
&\quad \le \left\| \int_0^t H(u_n(r) - r^{2H-1}) U_A(t,r) B^2 U_n(r,0)x \, \mathrm{d}r \right\|_V \\
&\quad + \left\| \int_0^t H r^{2H-1} U_A(t,r) B^2 (U_n(r,0)x - U(r,0)x) \, \mathrm{d}r \right\|_V \\
&\quad \le H C_A K_U \|x\|_V \int_0^t \frac{r^{2H-1} - u_n(r)}{(t-r)^\beta} \, \mathrm{d}r + H C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta} \|U_n(r,0)x - U(r,0)x\|_V \, \mathrm{d}r.
\end{aligned}$$

If we use the definition of $\{u_n, n \in \mathbb{N}\}$ we obtain the inequality

$$\int_0^t \frac{r^{2H-1} - u_n(r)}{(t-r)^\beta} \, \mathrm{d}r \le \left(\frac{1}{n}\right)^{2H-\beta} \mathrm{B}(2H, 1-\beta)$$

and hence

$$\begin{aligned}
&\|U_n(t,0)x - U(t,0)x\|_V \\
&\quad \le H C_A K_U \|x\|_V \left(\frac{1}{n}\right)^{2H-\beta} \mathrm{B}(2H, 1-\beta) + H C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta} \|U_n(r,0)x - U(r,0)x\|_V \, \mathrm{d}r.
\end{aligned}$$

Using again the generalized Gronwall inequality ([8], Lemma 7.1.2) we get

$$\|U_n(t,0)x - U(t,0)x\|_V \le H C_A K_U \|x\|_V \mathrm{B}(2H, 1-\beta) \left(\frac{1}{n}\right)^{2H-\beta} K_T,$$

where $K_T > 0$ is a finite constant independent of n, t, therefore

$$\|U_n(\,.\,,0)x - U(\,.\,,0)x\|_{\mathscr{C}([0,T];V)} \xrightarrow[n \to +\infty]{} 0.$$

□

3. Stochastic bilinear equation

Throughout this section we assume that the hypothesis (A1), (A2), (A3), (2.8) and $\mathrm{Dom}(B^2) \supset \mathrm{Dom}((-A(0))^\alpha)$ for some $\alpha \in (0, 1)$ are satisfied. Also let $A^*(t)$ be the adjoint operator to the operator $A(t)$ for each $t \in [0, T]$. Assume that the domain $\mathrm{Dom}(A^*(t)) = D^*$ of the operator $A^*(t)$ is independent of t. Moreover, assume that

(B1) $D^* \subset \mathrm{Dom}((B^*)^2)$

(B2) linear operator B on V is closed and densely defined and generates a strongly continuous group $\{S_B(t), t \in \mathbb{R}\}$

and

(AB) the operators $A(t)$ and $\{S_B(u), u \in \mathbb{R}\}$ commute on the domain D for all $t \in \in [0, T]$

It is well known that (B2) yields an existence of constants $M_B \geq 1, \omega_B \geq 0$ such that the inequality

$$\|S_B(u)\|_{\mathscr{L}(V)} \leq M_B \exp\{\omega_B |u|\} \tag{3.1}$$

holds for each $u \in \mathbb{R}$.

An explicit formula for the weak solution to the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= A(t)X(t)\mathrm{d}t + BX(t)\mathrm{d}B^H(t), \\ X(0) &= x_0, \end{aligned} \tag{3.2}$$

on the interval $[0, T]$ is given in this section, where $x_0 \in V$ is a deterministic initial value and $\{B^H(t), t \in [0, T]\}$ is a one-dimensional real-valued fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ on the interval $[0, T]$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition 3.1 *A* $(\mathscr{B}([0, T]) \otimes \mathscr{F})$*-measurable stochastic process* $\{X(t), t \in [0, T]\}$ *is said to be*

(I) *a* ***strong solution*** *to the equation* (3.2) *if* $X(t) \in D$ $\mathbb{P}$*-a.s. for all* $t \in [0, T]$ *and*

$$X(t) = x_0 + \int_0^t A(r)X(r)\,\mathrm{d}r + \int_0^t BX(r)\,\mathrm{d}B^H(r) \quad \mathbb{P} - \text{a.s.}$$

for all $t \in [0, T]$.

(II) *a* ***weak solution*** *to the equation* (3.2) *if for any* $y \in D^*$

$$\langle X(t), y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \langle X(r), A^*(r)y\rangle_V\,\mathrm{d}r + \int_0^t \langle X(r), B^*y\rangle_V\,\mathrm{d}B^H(r) \quad \mathbb{P} - \text{a.s.}$$

for all $t \in [0, T]$.

Let U_n be the strongly continuous evolution system associated with the system of operators $\{A_n(t), t \in [0, T]\}$ constructed in Proposition 2.1. Define approximating processes $\{X_n(t), t \in [0, T]\}, n \in \mathbb{N}$, as

$$X_n(t) = S_B(B^H(t))U_n(t, 0)x_0, \ t \in [0, T].$$

Proposition 3.2 *If* $x_0 \in D$ *then the process* $\{X_n(t), t \in [0, T]\}$ *is a strong solution to the equation*

$$\begin{aligned} \mathrm{d}X_n(t) &= \Big(A(t) + H(t^{2H-1} - u_n(t))B^2\Big)X_n(t)\mathrm{d}t + BX_n(t)\mathrm{d}B^H(t), \\ X_n(0) &= x_0. \end{aligned} \tag{3.3}$$

If $x_0 \in V$ *and for some constant* $C_0^* > 0$ *independent of* t

$$\|A^*(t)x\|_V \leq C_0^* \|A^*(0)x\|_V \tag{3.4}$$

holds for each $x \in D^*$ *then the process* $\{X_n(t), t \in [0, T]\}$ *is a weak solution to the equation (3.3).*

Proof. Fix $y \in \mathrm{Dom}((B^*)^2)$. An idea of the proof is to apply the one-dimensional Itô formula for a fractional Brownian motion (see [2], Corollary 4.8) to the function

$$f(t,x) := \langle S_B(x)U_n(t,0)x_0, y\rangle_V = \langle U_n(t,0)x_0, S_B^*(x)y\rangle_V,\ t \geq 0,\ x \in \mathbb{R}.$$

Clearly, $f \in \mathscr{C}^{1,2}([0,T] \times \mathbb{R})$,

$$\begin{aligned}
\frac{\partial}{\partial t} f(t,x) &= \langle (A(t) - Hu_n(t)B^2)U_n(t,0)x_0, S_B^*(x)y\rangle_V,\\
\frac{\partial}{\partial x} f(t,x) &= \langle U_n(t,0)x_0, S_B^*(x)B^*y\rangle_V,\\
\frac{\partial^2}{\partial x^2} f(t,x) &= \langle U_n(t,0)x_0, S_B^*(x)(B^*)^2 y\rangle_V.
\end{aligned}$$

We have to check that

$$\max\left\{\left|\frac{\partial}{\partial t} f(t,x)\right|, \left|\frac{\partial^2}{\partial x^2} f(t,x)\right|\right\} \leq C_f \mathrm{e}^{\lambda x^2} \tag{3.5}$$

for some constants $C_f > 0$ and $0 < \lambda < 1/4T^{2H}$.
Note that for all $b \in \mathbb{R}$ the inequality

$$\exp\{bx\} \leq \exp\{C_b + \lambda x^2\},\ x \in \mathbb{R},$$

holds for some constant $C_b \geq 0$.
By (2.4) for $\{A_n(t), t \in [0,T]\}$ and (3.1) we get

$$\begin{aligned}
&\left|\frac{\partial}{\partial t} f(t,x)\right| = \left|\langle (A(t) - Hu_n(t)B^2)U_n(t,0)x_0, S_B^*(x)y\rangle_V\right|\\
&\leq \left|\langle (A(t) - Hu_n(t)B^2)U_n(t,0)(A(0) - Hu_n(0)B^2)^{-1}(A(0) - Hu_n(0)B^2)x_0, S_B^*(x)y\rangle_V\right|\\
&\leq C\|(A(0) - Hu_n(0)B^2)x_0\|_V M_B \exp\{\omega_B|x|\}\|y\|_V \leq C_f \mathrm{e}^{\lambda x^2}
\end{aligned}$$

and by (2.10) and (3.1)

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x^2} f(t,x)\right| &= \langle U_n(t,0)x_0, S_B^*(x)(B^*)^2 y\rangle_V \leq \|U_n(t,0)x_0\|_V \|S_B^*(x)(B^*)^2 y\|_V\\
&\leq K_U\|x_0\|_V M_B \exp\{\omega_B|x|\}\|(B^*)^2 y\|_V \leq C_f \mathrm{e}^{\lambda x^2}.
\end{aligned}$$

Now, Corollary 4.8 from [2] yields

$$\begin{aligned}\langle X_n(t), y\rangle_V &= f(t, B^H(t)) = f(0, B^H(0)) + \int_0^t \frac{\partial}{\partial r} f(r, B^H(r))\,\mathrm{d}r \\ &+ \int_0^t \frac{\partial}{\partial x} f(r, B^H(r))\,\mathrm{d}B^H(r) + \int_0^t H r^{2H-1} \frac{\partial^2}{\partial x^2} f(r, B^H(r))\,\mathrm{d}r \\ &= \langle x_0, y\rangle_V + \int_0^t \langle (A(r) - H u_n(r) B^2) U_n(r,0)x_0, S_B^*(B^H(r))y\rangle_V\,\mathrm{d}r \\ &+ \int_0^t \langle B S_B(B^H(r)) U_n(r,0)x_0, y\rangle_V\,\mathrm{d}B^H(r) \\ &+ \int_0^t \langle H r^{2H-1} B^2 S_B(B^H(r)) U_n(r,0)x_0, y\rangle_V\,\mathrm{d}r \quad \mathbb{P} - \text{a.s.}\end{aligned}$$

for all $t \in [0,T]$. Using the commutativity assumption (AB) we get

$$\begin{aligned}\langle X_n(t), y\rangle_V &= \langle x_0, y\rangle_V + \int_0^t \langle A(r) X_n(r), y\rangle_V\,\mathrm{d}r + \int_0^t \langle B X_n(r), y\rangle_V\,\mathrm{d}B^H(r) \\ &+ \int_0^t \langle H(r^{2H-1} - u_n(r)) B^2 X_n(r), y\rangle_V\,\mathrm{d}r \quad \mathbb{P} - \text{a.s.}\end{aligned}$$

for all $t \in [0,T]$ and $y \in \mathrm{Dom}((B^*)^2)$. Taking a countable subset of the domain $\mathrm{Dom}((B^*)^2)$ dense in V we obtain that the process $\{X_n(t), t \in [0,T]\}$ is D-valued and it is a strong solution to the equation (3.3).

Let $x_0 \in V$. To prove the second part take a sequence $\{x_k, k \in \mathbb{N}\}$ in D converging to x_0 in V and consider approximating processes $\{Y_k(t), t \in [0,T]\}, k \in \mathbb{N}$, of the process $\{X_n(t), t \in [0,T]\}$ defined as

$$Y_k(t) = S_B(B^H(t)) U_n(t,0) x_k.$$

By the previous part of the proof it is known that the process $\{Y_k(t), t \in [0,T]\}$ is a strong solution to the equation (3.3) with the initial value $Y_k(0) = x_k$ and for each $y \in D^*$

$$\begin{aligned}\langle Y_k(t), y\rangle_V &= \langle x_k, y\rangle_V + \int_0^t \langle Y_k(r), A^*(r) y\rangle_V\,\mathrm{d}r + \int_0^t \langle Y_k(r), B^* y\rangle_V\,\mathrm{d}B^H(r) \\ &+ \int_0^t \langle H(r^{2H-1} - u_n(r)) Y_k(r), (B^*)^2 y\rangle_V\,\mathrm{d}r \quad \mathbb{P} - \text{a.s.}\end{aligned} \tag{3.6}$$

for all $t \in [0,T]$.

Our aim is to pass to the limit in the equation (3.6) in the space $L^2(\Omega)$ for any fixed $t \in [0,T]$ and any fixed $y \in D^*$ and to use the closedness of the Skorokhod integral. By the Fernique theorem (see [7]) it is well-known that

$$\mathbb{E}\Big[\exp\{\zeta \sup\{|B^H(t)|; t \in [0,T]\}\}\Big] < +\infty \tag{3.7}$$

for any constant $\zeta > 0$.
Using (3.1), (3.7) and (2.10)

$$\begin{aligned}
&\mathbb{E}\left|\langle Y_k(t), y\rangle_V - \langle X_n(t), y\rangle_V\right|^2 = \mathbb{E}\left|\langle Y_k(t) - X_n(t), y\rangle_V\right|^2 \\
&\quad = \mathbb{E}\left|\langle S_B(B^H(t))U_n(t,0)(x_k - x_0), y\rangle_V\right|^2 \\
&\quad \leq M_B^2 \mathbb{E}\Big[\exp\{2\omega_B \sup\{|B^H(r)|; r \in [0,T]\}\}\Big] K_U^2 \|y\|_V^2 \|x_k - x_0\|_V^2 \xrightarrow[k\to+\infty]{} 0, \qquad (3.8) \\
&\mathbb{E}\left|\langle x_k, y\rangle_V - \langle x_0, y\rangle_V\right|^2 = \langle x_k - x_0, y\rangle_V^2 \leq \|y\|_V^2 \|x_k - x_0\|_V^2 \xrightarrow[k\to+\infty]{} 0,
\end{aligned}$$

by (3.4)

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \langle (Y_k(r) - X_n(r)), A^*(r)y\rangle_V \,\mathrm{d}r\right|^2 = \mathbb{E}\left|\int_0^t \langle (S_B(B^H(t))U_n(t,0)(x_k - x_0)), A^*(r)y\rangle_V \,\mathrm{d}r\right|^2 \\
&\quad \leq M_B^2 \mathbb{E}\Big[\exp\{2\omega_B \sup\{|B^H(r)|; r \in [0,T]\}\}\Big] K_U^2 T^2 \|x_k - x_0\|_V^2 (C_0^*)^2 \|A^*(0)y\|_V^2 \xrightarrow[k\to+\infty]{} 0,
\end{aligned}$$

and by (U3)

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \langle H(r^{2H-1} - u_n(r))(Y_k(r) - X_n(r)), (B^*)^2 y\rangle_V \,\mathrm{d}r\right|^2 \\
&\quad = \mathbb{E}\left|\int_0^t \langle H(r^{2H-1} - u_n(r))S_B(B^H(t))U_n(t,0)(x_k - x_0), (B^*)^2 y\rangle_V \,\mathrm{d}r\right|^2 \\
&\quad \leq M_B^2 \mathbb{E}\Big[\exp\{2\omega_B \sup\{|B^H(r)|; r \in [0,T]\}\}\Big] K_U^2 T^{4H} \|x_k - x_0\|_V^2 \|(B^*)^2 y\|_V^2 \xrightarrow[k\to+\infty]{} 0.
\end{aligned}$$

Therefore we can pass to the limit in the equation (3.6) in the space $L^2(\Omega)$ and there exists a random variable $Y_{(n,y)}(t)$ such that

$$\int_0^t \langle Y_k(r), B^* y\rangle_V \,\mathrm{d}B^H(r) \xrightarrow[n\to+\infty]{} Y_{(n,y)}(t) \quad \text{in} \quad L^2(\Omega).$$

Analogous to (3.8) we get

$$\int_0^t \mathbb{E}\left|\langle Y_k(r), B^* y\rangle_V - \langle X_n(r), B^* y\rangle_V\right|^2 \mathrm{d}r \xrightarrow[k\to+\infty]{} 0$$

and

$$\{\langle Y_k(r), B^* y\rangle_V, r \in [0,t]\},\ \{\langle X_n(r), B^* y\rangle_V, r \in [0,t]\} \in L^2(\Omega; L^2([0,t]))$$

for any $k \in \mathbb{N}$ and by the Itô formula we know that the process $\{\langle Y_k(r), B^* y\rangle_V, r \in \in [0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion. Hence by the closedness of the Skorokhod integral we have that the process $\{\langle X_n(r), B^* y\rangle_V, r \in [0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion and

$$Y_{(n,y)}(t) = \int_0^t \langle X_n(r), B^* y\rangle_V \,\mathrm{d}B^H(r) \quad \mathbb{P} - \text{a.s.}$$

(see [2], Remark 3.4.2) for any $t \in [0,T]$. Thus the process $\{X_n(t), t \in [0,T]\}$ is a weak solution to the equation (3.3). □

Now we can define the process $\{X(t), t \in [0,T]\}$ as

$$X(t) = S_B(B^H(t))U(t,0)x_0, \; t \in [0,T],$$

and show the relation between processes $\{X_n(t), t \in [0,T]\}$ and $\{X(t), t \in [0,T]\}$.

Lemma 3.3 *For any $y \in V$ and any $t \in [0,T]$ the random variables $\langle X_n(t), y\rangle_V$ converge to the random variable $\langle X(t), y\rangle_V$ in the space $L^2(\Omega)$.*

Proof. Using (3.1), (3.7) and (2.11) we get

$$\mathbb{E}\left|\langle X_n(t), y\rangle_V - \langle X(t), y\rangle_V\right|^2 = \mathbb{E}\left|\langle X_n(t) - X(t), y\rangle_V\right|^2$$
$$= \mathbb{E}\left|\langle S_B(B^H(t))(U_n(t,0)x_0 - U(t,0))x_0, y\rangle_V\right|^2 \leq \|U_n(\,.\,,0)x_0 - U(\,.\,,0)x_0\|^2_{\mathscr{C}([0,T];V)}$$
$$\times \|y\|_V^2 M_B^2 \mathbb{E}\Big[\exp\{2\omega_B \sup\{|B^H(t)|; t \in [0,T]\}\}\Big] \xrightarrow[n\to+\infty]{} 0.$$

□

Now we can prove that the process $\{X(t), t \in [0,T]\}$ is a weak solution to the equation (3.2).

Theorem 3.4 *Assume that $\{A(t), t \in [0,T]\}$ and B are linear operators on V satisfying* (A1), (A2), (A3) *and* (B1), (B2). *Moreover, assume that* $\mathrm{Dom}(B^2) \supset \supset \mathrm{Dom}((-A(0))^\alpha)$ *for some $\alpha \in (0,1)$,* (2.8), (AB) *and* (3.4) *hold. Then for each $x_0 \in V$ the process $\{X(t), t \in [0,T]\}$ is a weak solution to the equation*

$$\begin{aligned} \mathrm{d}X(t) &= A(t)X(t)\mathrm{d}t + BX(t)\mathrm{d}B^H(t), \\ X(0) &= x_0. \end{aligned} \tag{3.9}$$

Proof. The proof is similar to the last part of the proof of Proposition 3.2. We pass to the limit in the equation

$$\langle X_n(t), y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \langle X_n(r), A^*(r)y\rangle_V\,\mathrm{d}r + \int_0^t \langle X_n(r), B^*y\rangle_V\,\mathrm{d}B^H(r)$$
$$+ \int_0^t \langle H(r^{2H-1} - u_n(r))X_n(r), (B^*)^2 y\rangle_V\,\mathrm{d}r \tag{3.10}$$

in the space $L^2(\Omega)$ for any fixed $t \in [0,T]$ and any fixed $y \in D^*$.
By (3.4), (3.7), (3.1) and (2.11) we have

$$\mathbb{E}\left|\int_0^t \langle (X_n(r) - X(r)), A^*(r)y\rangle_V\,\mathrm{d}r\right|^2$$
$$= \mathbb{E}\left|\int_0^t \langle S_B(B^H(r))(U_n(r,0)x_0 - U(r,0)x_0)), A^*(r)y\rangle_V\right|^2 \mathrm{d}t \leq (C_0^*)^2 \|A^*(0)y\|_V^2 T^2$$
$$\times M_B^2 \mathbb{E}\Big[\exp\{2\omega_B \sup\{|B^H(r)|; r \in [0,T]\}\}\Big] \|U_n(\,.\,,0)x_0 - U(\,.\,,0)x_0\|^2_{\mathscr{C}([0,T];V)} \xrightarrow[n\to+\infty]{} 0.$$

Hence

$$\int_0^t \langle X_n(r), A^*(r)y\rangle_V\,\mathrm{d}r \xrightarrow[n\to+\infty]{} \int_0^t \langle X(r), A^*(r)y\rangle_V\,\mathrm{d}r \quad \text{in} \quad L^2(\Omega).$$

Further, by (3.1), (2.10), (3.7), and (U2) we obtain

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \langle H(r^{2H-1}-u_n(r))X_n(r),(B^*)^2y\rangle_V\,\mathrm{d}r\right|^2\\
&=\mathbb{E}\left|\int_0^t \langle H(r^{2H-1}-u_n(r))S_B(B^H(r))U_n(r,0)x_0,(B^*)^2y\rangle_V\,\mathrm{d}r\right|^2\\
&\leq H^2\|(B^*)^2y\|_V^2M_B^2\mathbb{E}\Big[\exp\{2\omega_B\sup\{|B^H(r)|;r\in[0,T]\}\}\Big]K_U^2\|x_0\|_V^2\\
&\times\left(\int_0^T(r^{2H-1}-u_n(r))\,\mathrm{d}r\right)^2\xrightarrow[n\to+\infty]{}0,
\end{aligned}$$

thus

$$\int_0^t \langle H(r^{2H-1}-u_n(r))X_n(r),(B^*)^2y\rangle_V\,\mathrm{d}r\xrightarrow[n\to+\infty]{}0\quad\text{in}\quad L^2(\Omega).$$

From the proof of the previous lemma also follows that the left-hand side of (3.10) converges to $\langle X(t),y\rangle_V$, therefore there exists a random variable $Y_y(t)$ such that

$$\int_0^t\langle X_n(r),B^*y\rangle_V\,\mathrm{d}B^H(r)\xrightarrow[n\to+\infty]{}Y_y(t)\quad\text{in}\quad L^2(\Omega). \tag{3.11}$$

By Proposition 3.2 we have that the process $\{\langle X_n(r),B^*y\rangle_V, r\in[0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion. Moreover, analogous to Lemma 3.3 we obtain

$$\{\langle X_n(r),B^*y\rangle_V,r\in[0,t]\},\ \{\langle X(r),B^*y\rangle_V,r\in[0,t]\}\in L^2(\Omega;L^2([0,t]))$$

and

$$\int_0^t\mathbb{E}\left|\langle X_n(r),B^*y\rangle_V-\langle X(r),B^*y\rangle_V\right|^2\mathrm{d}r\xrightarrow[k\to+\infty]{}0$$

for any $n\in\mathbb{N}$. Hence by the closedness of the Skorokhod integral we have that the process $\{\langle X(r),B^*y\rangle_V, r\in[0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion and

$$Y_y(t)=\int_0^t\langle X(r),B^*y\rangle_V\,\mathrm{d}B^H(r)\quad\mathbb{P}-\text{a.s.}$$

(see [2], Remark 3.4.2) for any $t\in[0,T]$. Thus the process $\{X(t),t\in[0,T]\}$ satisfies the equality

$$\langle X(t),y\rangle_V=\langle x_0,y\rangle_V+\int_0^t\langle X(r),A^*(r)y\rangle_V\,\mathrm{d}r+\int_0^t\langle X(r),B^*y\rangle_V\,\mathrm{d}B^H(r)\quad\mathbb{P}-\text{a.s.}$$

for any $t\in[0,T]$ and $y\in D^*$ and Theorem 3.4 follows. □

4. Examples

In this section we give two examples of a stochastic partial differential equation illustrating the results obtained in the previous section.

Example 4.1 Consider the following stochastic parabolic equation of the second order

$$\frac{\partial u}{\partial t}(t,x) = L(t,x)u + bu(t,x)\frac{\mathrm{d}B^H}{\mathrm{d}t}, \tag{4.1}$$
$$u(0,x) = x_0(x),\ x \in \mathscr{O}$$
$$u(t,x) = 0,\ (t,x) \in [0,T] \times \partial\mathscr{O},$$

where $\mathscr{O} \subset \mathbb{R}^d$ is a bounded domain with the boundary of class $\mathscr{C}^2$, $b \in \mathbb{R} \setminus \{0\}$ and

$$L(t,x)u = a_0(t,x)u(t,x) + \sum_{i=1}^{d} a_i(t,x)\frac{\partial u}{\partial x_i}(t,x) + \sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)$$

is a uniformly strongly elliptic operator on $\mathscr{O}$, i.e. there exists a constant $\vartheta > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(t,x)\zeta_i\zeta_j > \vartheta\|\zeta\|^2_{\mathbb{R}^d}$$

for all $(t,x) \in [0,T] \times \bar{\mathscr{O}}$ and $0 \neq \zeta = (\zeta_1,\ldots,\zeta_d) \in \mathbb{R}^d$.
The functions $a_0(t,\,.\,), a_i(t,\,.\,), a_{ij}(t,\,.\,) \in \mathscr{C}^\infty(\bar{\mathscr{O}})$ for any $i,j = 1,\ldots,d$ and $t \in [0,T]$.
Equation (4.1) can be rewritten in the form

$$\mathrm{d}X(t) = A(t)X(t)\mathrm{d}t + BX(t)\mathrm{d}B^H(t) \tag{4.2}$$
$$X(0) = x_0$$

for $t \in [0,T]$, where $V = L^2(\mathscr{O})$,

$$(A(t)u)(x) = L(t,x)u,$$

where $\mathrm{Dom}(A(t)) = D = H^2(\mathscr{O}) \cap H^1_0(\mathscr{O})$ and $B = bI \in \mathscr{L}(V)$.
Assume that

$$\sup_{x\in\mathscr{O}}\{|a_0(t,x) - a_0(s,x)|, |a_i(t,x) - a_i(s,x)|, |a_{ij}(t,x) - a_{ij}(s,x)|\} \leq M|t-s|^\gamma$$

for any $s,t \in [0,T]$, $i,j = 1,\ldots,d$, and some constants $M > 0, 0 < \gamma < 1$ then the assumptions (A1), (A2), (A3) are satisfied (cf. Theorem 3.8.3, [13]). The adjoint operator $A^*(t)$ has the same form as the operator $A(t)$ only with other coefficients. So the domain $\mathrm{Dom}(A^*(t)) = D^* = D = \mathrm{Dom}(A(t))$ is independent of t. Also conditions (B1), (B2), (2.8), (AB) and $\mathrm{Dom}(B^2) \supset \mathrm{Dom}((-A(0))^\alpha)$ for some $\alpha \in (0,1)$ are trivially satisfied. Moreover, we have to assume that (2.6) and (3.4). Then the assumption of Theorem 3.4 are satisfied thus there exists a weak solution to the equation (4.2).

Example 4.2 Consider the equation

$$\frac{\partial u}{\partial t}(t,x) = -\frac{\partial^4 u}{\partial x^4}(t,x) - \alpha u(t,x) + \frac{\partial u}{\partial x}(t,x)\frac{\mathrm{d}B^H}{\mathrm{d}t}, \tag{4.3}$$
$$u(0,x) = x_0(x),$$

in the weighted space $V = L^2_\rho(\mathbb{R})$ with the weight $\mathrm{e}^{-\rho|x|}$, $x \in \mathbb{R}$, and some fixed positive constant ρ, where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. The operator $A = -\frac{\partial^4}{\partial x^4} - \alpha I$ defined on the domain $D = \mathrm{Dom}(A) = W^{4,2}(\mathbb{R})$ generates a strongly continuous semigroup $\{S_A(t), t \in [0, T]\}$ on V which is exponentially stable for any fixed $\alpha > 0$ (see e.g. [10]). The operator $B = \frac{\partial}{\partial x}$ with the domain $\mathrm{Dom}(B) = W^{1,2}(\mathbb{R})$ generates a strongly continuous group $\{S_B(t), t \in \mathbb{R}\}$ on V which is a shift operator

$$(S_B(t)u)(x) = u(t + x),\ t, x \in \mathbb{R}.$$

Moreover, $D = D^* = \mathrm{Dom}(A^*)$, $\mathrm{Dom}(B^2) = \mathrm{Dom}((B^*)^2) = W^{2,2}(\mathbb{R})$ and $S_B(t)$ commute with A on D for each $t \in [0, T]$. The operators

$$\left\{A_n(t) = -\frac{\partial^4}{\partial x^4} - \alpha I - Hu_n(t)\frac{\partial^2}{\partial x^2},\ t \in [0, T]\right\}$$

are strongly elliptic and generate a strongly continuous evolution system $\{U_n(t, s), 0 \leq \leq s \leq t \leq T\}$.

It remains to show (2.8), i.e.

$$\|S_A(t)B^2\|_{\mathscr{L}(V)} \leq \frac{C_A}{t^\beta}$$

for some constants $C_A > 0,\ 0 < \beta < 2H$ and $0 \leq t \leq T$.

Recall (see e.g. [6]) that there exists the fundamental solution $G \in \mathscr{C}^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$ to the operator $\frac{\partial}{\partial t} - A$ with the property

$$\left|\frac{\partial^2}{\partial x^2}G(t, x, y)\right| \leq K_1 t^{-1/2} g(K_2 t, |x - y|),\ t \in (0, T],\ x, y \in \mathbb{R} \tag{4.4}$$

for some constants $K_1, K_2 > 0$, where

$$g(t, z) = t^{-1/4} \exp\left\{-\left(\frac{z^4}{t}\right)^{1/3}\right\},\ t > 0,\ z \in \mathbb{R}.$$

Moreover, for any $u \in L^2(\mathbb{R})$

$$(S_A(t)u)(x) = \int_{\mathbb{R}} G(t, x, y)u(y)\,\mathrm{d}y,\ t > 0,\ x \in \mathbb{R}. \tag{4.5}$$

Since the semigroup $\{S_A(t), t \geq 0\}$ is self-adjoint on $L^2(\mathbb{R})$ the equality

$$\langle S_A(t)u, v\rangle_{L^2(\mathbb{R})} = \langle u, S_A(t)v\rangle_{L^2(\mathbb{R})},\ u, v \in L^2(\mathbb{R}),$$

holds, so using (4.5) and Fubini Theorem we obtain

$$\langle S_A(t)u, v\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} G(t, x, y)u(y)\,\mathrm{d}y\, v(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} G(t, x, y)u(y)v(x)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\langle u, S_A(t)v\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u(y)\int_{\mathbb{R}} G(t, y, x)v(x)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} G(t, y, x)u(y)v(x)\,\mathrm{d}x\,\mathrm{d}y.$$

Thus $G(t,x,y)=G(t,y,x), t>0, x,y\in\mathbb{R}$.
Let $\vartheta_\rho\in\mathscr{C}^\infty(\mathbb{R})$ be a smooth approximation of the weight $\mathrm{e}^{-\rho|x|}, x\in\mathbb{R}$, such that $\vartheta_\rho(x)=\mathrm{e}^{-\rho|x|}, |x|\geq 1$. Then

$$(g(t,\,.\,)*\vartheta_\rho)(x)\leq K_3\vartheta_\rho(x),\ t\in[0,T],\ x\in\mathbb{R}, \tag{4.6}$$

for some constant $K_3>0$.
Take $u\in\mathscr{C}_0^\infty(\mathbb{R})$. Then using (4.5), symmetry of G, (4.4), Jensen inequality and (4.6)

$$\begin{aligned}
&\int_{\mathbb{R}}\left|(S_A(t)B^2u)(x)\right|^2\vartheta_\rho(x)\,\mathrm{d}x=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}G(t,x,y)\frac{\partial^2}{\partial y^2}u(y)\,\mathrm{d}y\right|^2\vartheta_\rho(x)\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}G(t,y,x)\frac{\partial^2}{\partial y^2}u(y)\,\mathrm{d}y\right|^2\vartheta_\rho(x)\,\mathrm{d}x=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\frac{\partial^2}{\partial y^2}G(t,y,x)u(y)\,\mathrm{d}y\right|^2\vartheta_\rho(x)\,\mathrm{d}x\\
&\quad\leq K_1^2\,t^{-1}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}g(K_2t,|x-y|)|u(y)|\,\mathrm{d}y\Big)^2\vartheta_\rho(x)\,\mathrm{d}x\\
&\quad\leq KK_1^2\,t^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}g(K_2t,|x-y|)|u(y)|^2\,\mathrm{d}y\,\vartheta_\rho(x)\,\mathrm{d}x\leq KK_1^2K_3\,t^{-1}\int_{\mathbb{R}}|u(y)|^2\vartheta_\rho(y)\,\mathrm{d}y\\
&\quad=KK_1^2K_3\,t^{-1}\|u\|^2_{L^2_\rho(\mathbb{R})}.
\end{aligned}$$

Since $\mathscr{C}_0^\infty(\mathbb{R})$ is dense in $L^2_\rho(R)$ we obtain that

$$\|S_A(t)B^2\|_{\mathscr{L}(V)}\leq\frac{(KK_1^2K_3)^{1/2}}{t^{1/2}},\ t>0.$$

Hence the condition $1/2<2H$ can be satisfied only for $H>1/4$. Therefore, under this hypothesis $H>1/4$ the equation (4.3) has a weak solution

$$\{X(t)=S_B(B^H(t))U(t,0)u_0, t\in[0,T]\}$$

for any initial value $u_0\in V$.

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