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COMPLETE CONVERGENCE IN MEAN FOR DOUBLE ARRAYS OF RANDOM VARIABLES WITH VALUES IN BANACH SPACES

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Abstract. The rate of moment convergence of sample sums was investigated by Chow (1988) (in case of real-valued random variables). In 2006, Rosalsky et al. introduced and investigated this concept for case random variable with Banach-valued (called complete convergence in mean of order p). In this paper, we give some new results of complete convergence in mean of order p and its applications to strong laws of large numbers for double arrays of random variables taking values in Banach spaces.

Keywords: complete convergence in mean; double array of random variables with values in Banach space; martingale difference double array; strong law of large numbers; p-uniformly smooth space

MSC 2010: 60B11, 60B12, 60F15, 60F25

1. Introduction

Let $\mathbb{E}$ be a real separable Banach space with norm $\|\cdot\|$ and $\{X_n, n \geqslant 1\}$ a sequence of random variables taking values in $\mathbb{E}$ ($\mathbb{E}$-valued r.v.'s for short). Recall that X_n is said to converge completely to 0 in mean of order p if

$$\sum_{n=1}^{\infty} E\|X_n\|^p < \infty.$$

This mode of convergence was investigated for the first time by Chow [2] for the sequence of real-valued random variables and by Rosalsky et al. [6] for the sequence

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of random variables taking values in a Banach space. In this paper, we introduce and study the complete convergence in mean of order p to 0 of double arrays of $\mathbb{E}$-random variables. In Section 3 some properties of the complete convergence in mean of order p are given and a new characterization of a p-uniformly smooth Banach space $\mathbb{E}$ in terms of the complete convergence in mean of order p of double arrays of $\mathbb{E}$-valued r.v.'s is obtained. These results are used in Section 4 to obtain some strong laws of large numbers for martingale difference double arrays of random variables taking values in Banach spaces.

2. Preliminaries and some useful lemmas

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same in each appearance. The set of all non-negative integers will be denoted by $\mathbb{N}$ and the set of all positive integers by $\mathbb{N}^*$. For (k, l) and $(m, n) \in \mathbb{N}^2$, the notation $(k, l) \preceq (m, n)$ (or $(m, n) \succeq (k, l)$) means that $k \leqslant m$ and $l \leqslant n$.

Definition 2.1. Let $\mathbb{E}$ be a real separable Banach space with norm $\|\cdot\|$ and let $\{S_{mn}; (m, n) \succeq (1, 1)\}$ be an array of $\mathbb{E}$-valued r.v.'s.

(1) S_{mn} is said to *converge completely to* 0 and we write $S_{mn} \xrightarrow{c} 0$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P(\|S_{mn}\| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

(2) S_{mn} is said to *converge to* 0 *in mean of order* p (or in $\mathcal{L}_p$ for short) as $m \vee n \to \infty$ and we write $S_{mn} \xrightarrow{\mathcal{L}_p} 0$ as $m \vee n \to \infty$ if

$$E\|S_{mn}\|^p \to 0 \quad \text{as } m \vee n \to \infty.$$

S_{mn} is said to converge completely to 0 in mean of order p and we write $S_{mn} \xrightarrow{c,\mathcal{L}_p} 0$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\|S_{mn}\|^p < \infty.$$

(3) S_{mn} is said to converge almost surely to 0 as $m \vee n \to \infty$ and we write $S_{mn} \to 0$ a.s. as $m \vee n \to \infty$ if

$$P\Big(\lim_{m \vee n \to \infty} \|S_{mn}\| = 0\Big) = 1.$$

It is clear that $S_{mn} \xrightarrow{c,\mathcal{L}_p} 0$ implies $S_{mn} \xrightarrow{\mathcal{L}_p} 0$ as $m \vee n \to \infty$. By the Markov inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\{\|S_{mn}\| > \varepsilon\} < \infty \quad \text{for all } \varepsilon > 0$$

we also see that $S_{mn} \xrightarrow{c,\mathcal{L}_p} 0$ implies $S_{mn} \xrightarrow{c} 0$ and $S_{mn} \xrightarrow{\text{a.s.}} 0$.

For an $\mathbb{E}$-valued r.v. X and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(X \mid \mathcal{G})$ is defined and enjoys the usual properties (see [7]).

A real separable Banach space $\mathbb{E}$ is said to be *p-uniformly smooth* ($1 \leqslant p \leqslant 2$) if there exists a finite positive constant C such that for any L^p integrable $\mathbb{E}$-valued martingale difference sequence $\{X_n, n \geqslant 1\}$,

$$E\Big\|\sum_{i=1}^{n} X_i\Big\|^p \leqslant C\sum_{i=1}^{n} E\|X_i\|^p.$$

Clearly every real separable Banach space is 1-uniformly smooth and every Hilbert space is 2-uniformly smooth. If a real separable Banach space is p-uniformly smooth for some $1 < p \leqslant 2$ then it is r-uniformly smooth for all $r \in [1, p)$. For more details, the reader may refer to Pisier [5].

Let $\{X_{mn}, (m,n) \succeq (1,1)\}$ be a double array of $\mathbb{E}$-valued r.v.'s, let $\mathcal{F}_{ij}$ be the σ-field generated by the family of $\mathbb{E}$-random variables $\{X_{kl}\,;\ k < i \text{ or } l < j\}$ and $\mathcal{F}_{11} = \{\emptyset;\ \Omega\}$.

The array of $\mathbb{E}$-valued r.v.'s $\{X_{mn}, (m,n) \succeq (1,1)\}$ is said to be an $\mathbb{E}$-valued *martingale difference double array* if $E(X_{mn} \mid \mathcal{F}_{mn}) = 0$ for all $(m,n) \succeq (1,1)$.

The following lemmas are necessary for proving the main results in the paper.

Lemma 2.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leqslant p \leqslant 2$ and let $\{X_{mn}\,;\ (m,n) \succeq (1,1)\}$ be a double array of $\mathbb{E}$-valued r.v.'s satisfying $E(X_{ij} \mid \mathcal{F}_{ij})$ which is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j) \preceq (m,n)$. Then*

$$E\max_{\substack{1\leqslant k\leqslant m\\ 1\leqslant l\leqslant n}}\Big\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij} - E(X_{ij} \mid \mathcal{F}_{ij}))\Big\|^p \leqslant C\sum_{i=1}^{m}\sum_{j=1}^{n} E\|X_{ij}\|^p,$$

where the constant C is independent of m and n.

P r o o f. The proof is completely similar to that of Lemma 2 of Dung et al. [3] after replacing $S_{kl} = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} V_{ij}$ by $S_{kl} = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l}(X_{ij} - E(X_{ij} \mid \mathcal{F}_{ij}))$. $\square$

The following lemma is a version of Lemma 3 of Adler and Rosalsky [1] for arrays of positive constants.

Lemma 2.2. *Let $p > 0$ and let $\{b_{mn};\ (m,n) \succeq (1,1)\}$ be an array of positive constants with $b^p_{ij}/ij \leqslant b^p_{mn}/mn$ for all $(i,j) \preceq (m,n)$ and $\lim\limits_{m\vee n\to\infty} b^p_{mn}/mn = \infty$. Then*

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{1}{b^p_{ij}} = \mathcal{O}\Big(\frac{mn}{b^p_{mn}}\Big) \quad \textit{as } m\vee n\to\infty$$

if and only if

$$\liminf_{m\vee n\to\infty}\frac{b^p_{rm,sn}}{b^p_{mn}} > rs \quad \textit{for some integers } r,s\geqslant 2.$$

Proof. Set $c_{mn} = \frac{b^p_{mn}}{mn}$, $(m,n) \preceq (1,1)$ then $c_{ij} \leqslant c_{mn}$ for all $(i,j) \preceq (m,n)$ and $\lim\limits_{m\vee n\to\infty} c_{mn} = \infty$. It is required to show that

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{1}{ijc_{ij}} = \mathcal{O}\Big(\frac{1}{c_{mn}}\Big) \quad \text{as } m\vee n\to\infty \tag{2.1}$$

if and only if

$$\liminf_{m\vee n\to\infty}\frac{c_{rm,sn}}{c_{mn}} > 1 \quad \text{for some integers } r,s\geqslant 2. \tag{2.2}$$

If (2.2) holds, then exits $\delta > 1$ and $n_o \in \mathbb{N}$ such that $c_{rm,sn} \geqslant \delta c_{mn}$ for all $m\vee n \geqslant n_o$, so

$$\begin{aligned}\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{1}{ijc_{ij}} &\leqslant \sum_{k,l=0}^{\infty}\sum_{i=mr^k}^{mr^{k+1}-1}\sum_{j=ns^l}^{ns^{l+1}-1}\frac{1}{klc_{kl}} \leqslant \sum_{k,l=0}^{\infty}\frac{(r-1)(s-1)}{c_{mr^k,ns^l}}\\ &\leqslant (r-1)(s-1)\frac{1}{c_{mn}}\Big(\sum_{k=1}^{\infty}\frac{1}{\delta^k}\Big)^2.\end{aligned}$$

Then, we have (2.1).

Conversely, assume that (2.2) does not hold. Then $\liminf\limits_{m\vee n\to\infty} c_{rm,sn}/c_{mn} = 1$ for any $r,s\geqslant 2$, then $c_{rm,sn} < 2c_{mn}$ for any $r,s\geqslant 2$ and an infinite numbers pair of values of (m,n) and so,

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{1}{ijc_{ij}} > \sum_{i=m}^{mr}\sum_{j=n}^{ns}\frac{1}{ijc_{ij}} \geqslant \frac{(\log r)(\log s)}{c_{rm,sn}} > \frac{(\log r)(\log s)}{2c_{m,n}}.$$

Since r, s is arbitrary, (2.1) does not hold as well. □

3. The complete convergence in mean

From now on, $\mathbb{E}$ be a real separable Banach space and for each double array of $\mathbb{E}$-valued r.v.'s $\{X_{mn};\ (m,n)\succeq(1,1)\}$; we always denote $\mathcal{F}_{ij}$ is σ-field generated by the family of $\mathbb{E}$-random variables $\{X_{kl};\ k<i \text{ or } l<j\}$, $\mathcal{F}_{11}=\{\emptyset;\ \Omega\}$,

$$S_{kl}=\sum_{i=1}^{k}\sum_{j=1}^{l}X_{ij} \text{ and } S^*_{kl}=\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-\mathbb{E}(X_{ij}\mid\mathcal{F}_{ij}));$$

$\{b_{mn};\ (m,n)\succeq(1,1)\}$ be a sequence of positive constants satisfying $b_{ij}\leqslant b_{mn}$ for all $(i,j)\preceq(m,n)$ and $\lim\limits_{m\vee n\to\infty} b_{mn}=\infty$.

Firstly, we show a condition under which the complete convergence in mean order p implies the convergence a.s. and the convergence in $\mathcal{L}_p$.

Theorem 3.1. *Let $\{X_{mn};\ (m,n)\succeq(1,1)\}$ be a double array of $\mathbb{E}$-valued r.v.'s. Suppose that*

$$M=\sup_{m,n}\frac{b_{2^{m+1}2^{n+1}}}{b_{2^m2^n}}<\infty. \tag{3.1}$$

If

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}\|}{(mn)^{1/p}b_{mn}}\xrightarrow{c,\mathcal{L}_p}0 \quad \text{for some } 1\leqslant p\leqslant 2, \tag{3.2}$$

then

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}\|}{b_{mn}}\to 0 \text{ a.s. and in } \mathcal{L}_p \text{ as } m\vee n\to\infty. \tag{3.3}$$

P r o o f. Set $A_{mn}=\{(k,l),(2^n,2^m)\preceq(k,l)\prec(2^{m+1},2^{n+1})\}$. We see that

$$\begin{aligned}
&\sum_{(m,n)\succeq(0,0)}E\Big(\frac{\max_{(k,l)\preceq(2^m,2^n)}\|S_{kl}\|}{b_{2^m2^n}}\Big)^p \\
&\quad\leqslant\sum_{(m,n)\succeq(0,0)}E\Big(\frac{M\max_{(k,l)\preceq(2^m,2^n)}\|S_{kl}\|}{b_{2^{m+1}2^{n+1}}}\Big)^p \\
&\quad\leqslant M^p\sum_{(m,n)\succeq(0,0)}\min_{(k.l)\in A_{mn}}E\Big(\frac{\max_{(i,j)\preceq(k,l)}\|S_{ij}\|}{b_{kl}}\Big)^p \\
&\quad\leqslant M^p\sum_{(m,n)\succeq(0,0)}\sum_{(k,l)\in A_{mn}}\frac{1}{2^m2^n}E\Big(\frac{\max_{(i,j)\preceq(k,l)}\|S_{ij}\|}{b_{kl}}\Big)^p
\end{aligned}\tag{3.4}$$

$$\leqslant M^p \sum_{(m,n)\succeq(0,0)} \sum_{(k,l)\in A_{mn}} \frac{4}{kl} E\Big(\frac{\max_{(i,j)\preceq(k,l)} \|S_{ij}\|}{b_{kl}}\Big)^p$$
$$\leqslant 4M^p \sum_{(m,n)\succeq(1,1)} \frac{1}{mn} E\Big(\frac{\max_{(k,l)\preceq(m,n)} \|S_{kl}\|^p}{b_{mn}^p}\Big)$$
$$\leqslant 4M^p \sum_{(m,n)\succeq(1,1)} E\Big(\frac{\max_{(k,l)\preceq(m,n)} \|S_{kl}\|}{(mn)^{1/p} b_{mn}}\Big)^p < \infty.$$

This implies that

$$E\Big(\frac{\max_{(k,l)\preceq(2^m,2^n)} \|S_{kl}\|}{b_{2^m 2^n}}\Big)^p \to 0 \quad \text{as } m\vee n\to\infty. \tag{3.5}$$

Now for $(k,l)\in A_{nm}$ we have

$$E\Big(\frac{\max_{(i,j)\preceq(k,l)} \|S_{ij}\|}{b_{kl}}\Big)^p \leqslant E\Big(\frac{\max_{(k,l)\preceq(2^{m+1},2^{n+1})} \|S_{kl}\|}{b_{kl}}\Big)^p \tag{3.6}$$
$$\leqslant E\Big(\frac{\max_{(k,l)\preceq(2^{m+1},2^{n+1})} \|S_{kl}\|}{b_{2^m 2^n}}\Big)^p \leqslant M^p E\Big(\frac{\max_{(k,l)\preceq(2^{m+1},2^{n+1})} \|S_{kl}\|}{b_{2^{m+1} 2^{n+1}}}\Big)^p.$$

From (3.5) and (3.6) we conclude that $\Big(\sup\limits_{(k,l)\preceq(m,n)} \Big\|\sum\limits_{j=1}^{k}\sum\limits_{i=1}^{l} X_{ij}\Big\|\Big)/b_{mn} \xrightarrow{\mathcal{L}_p} 0$ as $m\vee n\to\infty$.

By (3.4) and the Markov inequality, for all $\varepsilon>0$ we have

$$\sum_{(m,n)\succeq(0,0)} P\Big(\max_{(k,l)\preceq(2^m,2^n)} \|S_{kl}\| \geqslant \varepsilon b_{2^m 2^n}\Big)$$
$$\leqslant \frac{4M^p}{\varepsilon^p} \sum_{(m,n)\succeq(1,1)} E\Big(\frac{\max_{(k,l)\preceq(m,n)} \|S_{kl}\|}{(mn)^{1/p} b_{mn}}\Big)^p < \infty.$$

This implies by the Borel-Cantelli lemma that

$$\frac{\max_{(k,l)\preceq(2^m,2^n)} \|S_{kl}\|}{b_{2^m 2^n}} \xrightarrow{\text{a.s.}} 0 \quad \text{as } m\vee n\to\infty.$$

By the same argument as in (3.6), we have

$$\frac{\sup_{(k,l)\preceq(m,n)} \big\|\sum_{j=1}^{k}\sum_{i=1}^{l} X_{ij}\big\|}{b_{mn}} \xrightarrow{\text{a.s.}} 0 \quad \text{as } m\vee n\to\infty.$$

The proof of the theorem is completed. □

The following theorem shows that the rate of the convergence of strong laws of large numbers may be obtained as a consequence of the complete convergence in mean.

Theorem 3.2. *Let $\alpha, \beta \in \mathbb{R}$ and let $\{X_{mn};\ (m,n) \succeq (1,1)\}$ be a double array of $\mathbb{E}$-valued r.v.'s. If*

$$\frac{1}{(m^\alpha n^\beta)^{1/p} b_{mn}} \max_{(k,l)\preceq(m,n)} \|S_{kl}\| \xrightarrow{c,\mathcal{L}_p} 0 \quad \text{for some } 1 \leqslant p \leqslant 2,$$

then

$$\sum_{(m,n)\succeq(1,1)} m^{-\alpha} n^{-\beta} P\Big(b_{mn}^{-1} \max_{(k,l)\preceq(m,n)} \|S_{kl}\| > \varepsilon\Big) < \infty \quad \text{for every } \varepsilon > 0. \tag{3.7}$$

In the case of $\alpha < 1$, $\beta < 1$ and $\{b_{mn};\ (m,n) \succeq (1,1)\}$ satisfying (3.1), (3.7) *implies*

$$P\Big(\sup_{(k,l)\succeq(m,n)} \frac{\|S_{kl}\|}{b_{kl}} > \varepsilon\Big) = o\Big(\frac{1}{m^{1-\alpha} n^{1-\beta}}\Big) \quad \text{as } m \vee n \to \infty \text{ for every } \varepsilon > 0.$$

P r o o f. By Markov inequality, for all $\varepsilon > 0$

$$\begin{aligned}&\sum_{(m,n)\succeq(1,1)} m^{-\alpha} n^{-\beta} P\Big(b_{mn}^{-1} \max_{(k,l)\preceq(m,n)} \|S_{kl}\| \geqslant \varepsilon\Big)\\ &\qquad\qquad \leqslant \frac{1}{\varepsilon^p} \sum_{(m,n)\succeq(1,1)} m^{-\alpha} n^{-\beta} E\Big(\frac{\max_{(k,l)\preceq(m,n)} \|S_{kl}\|}{b_{mn}}\Big)^p < \infty.\end{aligned}$$

Then, we have (3.7).

Let $\alpha < 1$, $\beta < 1$. Fix $\varepsilon > 0$, and set $A_{mn} = \{(k,l), (2^{n-1}, 2^{m-1}) \prec (k,l) \preceq (2^m, 2^n)\}$. We see that

$$\begin{aligned}&\sum_{(m,n)\succeq(1,1)} m^{-\alpha} n^{-\beta} P\Big(\sup_{(k,l)\succeq(m,n)} b_{kl}^{-1} \|S_{kl}\| > \varepsilon\Big)\\ &\quad = \sum_{(i,j)\succeq(1,1)} \sum_{m=2^{i-1}}^{2^i-1} \sum_{n=2^{j-1}}^{2^j-1} m^{-\alpha} n^{-\beta} P\Big(\sup_{(k,l)\succeq(m,n)} b_{kl}^{-1} \|S_{kl}\| > \varepsilon\Big)\\ &\quad \leqslant C \sum_{(i,j)\succeq(1,1)} \sum_{m=2^{i-1}}^{2^i-1} \sum_{n=2^{j-1}}^{2^j-1} 2^{-i\alpha} 2^{-j\beta} P\Big(\sup_{(k,l)\succeq(2^{i-1},2^{j-1})} b_{kl}^{-1} \|S_{kl}\| > \varepsilon\Big)\\ &\quad \leqslant C \sum_{(i,j)\succeq(1,1)} 2^{i(1-\alpha)} 2^{j(1-\beta)} P\Big(\sup_{(u,v)\succeq(i,j)} \max_{(k,l)\in A_{uv}} b_{kl}^{-1} \|S_{kl}\| > \varepsilon\Big)\\ &\quad \leqslant C \sum_{(i,j)\succeq(1,1)} 2^{i(1-\alpha)} 2^{j(1-\beta)} \sum_{(u,v)\succeq(i,j)} P\Big(b_{2^{u-1}2^{v-1}}^{-1} \max_{(k,l)\preceq(2^u,2^v)} \|S_{kl}\| > \varepsilon\Big)\end{aligned}$$

$$\leqslant C \sum_{(u,v)\succeq(1,1)} P\Big(b^{-1}_{2^{u-1}2^{v-1}} \max_{(k,l)\preceq(2^u,2^v)} \|S_{kl}\| > \varepsilon\Big) \sum_{(i,j)\preceq(u,v)} 2^{i(1-\alpha)}2^{j(1-\beta)}$$
$$\leqslant C \sum_{(u,v)\succeq(1,1)} 2^{u(1-\alpha)}2^{v(1-\beta)} P\Big(b^{-1}_{2^u2^v} \max_{(k,l)\preceq(2^u,2^v)} \|S_{kl}\| > \frac{\varepsilon}{M}\Big)$$
$$\leqslant C \sum_{(m,n)\succeq(1,1)} m^{-\alpha}n^{-\beta} P\Big(b^{-1}_{mn} \max_{(k,l)\preceq(m,n)} \|S_{kl}\| > \frac{\varepsilon}{M}\Big) < \infty \quad \text{(by (3.7))}.$$

Since $\big\{P\big(\sup_{(k,l)\succeq(m,n)} b^{-1}_{kl}\|S_{kl}\| > \varepsilon\big), (m,n) \in \mathbb{N}^{*2}\big\}$ are non-increasing in (m,n) for order relationship $\preceq$ in $\mathbb{N}^{*2}$, it follows that

$$P\Big(\sup_{(k,l)\succeq(m,n)} b^{-1}_{kl}\|S_{kl}\| > \varepsilon\Big) = o\Big(\frac{1}{m^{1-\alpha}n^{1-\beta}}\Big) \quad \text{as } m\vee n\to\infty \text{ for all } \varepsilon>0.$$

□

Now we establish sufficient conditions for complete convergence in mean of order p.

Theorem 3.3. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1\leqslant p\leqslant 2$. Let $\{X_{mn};\ (m,n)\succeq(1,1)\}$ be a double array of $\mathbb{E}$-valued r.v.'s such that $E(X_{ij}|\mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j)\preceq(m,n)$. Suppose that*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b^{-p}_{mn} < \infty. \tag{3.8}$$

If

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varphi(m,n)E\|X_{mn}\|^p < \infty, \tag{3.9}$$

where $\varphi(m,n) = \sum\limits_{i=m}^{\infty}\sum\limits_{j=n}^{\infty} b^{-p}_{ij}$, then

$$\frac{1}{b_{mn}} \max_{(k,l)\preceq(m,n)} \|S^*_{kl}\| \xrightarrow{c,\mathcal{L}_p} 0. \tag{3.10}$$

Proof. We have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\frac{\max_{(k,l)\preceq(m,n)}\|S^*_{kl}\|^p}{b^p_{mn}} \leqslant C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sum_{i=1}^m\sum_{j=1}^n E\|X_{ij}\|^p}{b^p_{mn}} \quad \text{(by Lemma 2.1)}$$
$$\leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E\|X_{ij}\|^p\Big(\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{b^p_{mn}}\Big)$$
$$\leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varphi(i,j)E\|X_{ij}\|^p < \infty \text{ (by (3.9))}.$$

□

A characterization of p-uniformly smooth Banach spaces in terms of the complete convergence in mean of order p is presented in the following theorem.

Theorem 3.4. *Let $1 \leqslant p \leqslant 2$, let $\mathbb{E}$ be a real separable Banach space. Then the following statements are equivalent:*

(i) *$\mathbb{E}$ is of p-uniformly smooth.*

(ii) *For every double array of random variables $\{X_{mn};\ (m,n) \succeq (1,1)\}$ with values in $\mathbb{E}$ such that $E(X_{ij} \mid \mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j) \preceq (m,n)$, and every double array of positive constants $\{b_{mn};\ (m,n) \succeq (1,1)\}$ with $b_{ij} \leqslant b_{mn}$ for all $(i,j) \preceq (m,n)$ and satisfying*

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{1}{b_{ij}^{p}} = \mathcal{O}\Big(\frac{mn}{b_{mn}^{p}}\Big), \tag{3.11}$$

the condition

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\frac{E\|X_{mn}\|^{p}}{b_{mn}^{p}} < \infty \tag{3.12}$$

implies

$$\frac{1}{b_{mn}}\max_{(k,l)\preceq(m,n)}\|S_{kl}^{*}\| \xrightarrow{c,\mathcal{L}_p} 0. \tag{3.13}$$

(iii) *For every double array of random variables $\{X_{mn};\ (m,n) \succeq (1,1)\}$ with values in $\mathbb{E}$ such that $E(X_{ij} \mid \mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j) \preceq (m,n)$, the condition*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|X_{mn}\|^{p}}{(nm)^{p}} < \infty \tag{3.14}$$

implies

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}^{*}\|}{(mn)^{(p+1)/p}} \xrightarrow{c,\mathcal{L}_p} 0. \tag{3.15}$$

P r o o f. (i)$\to$(ii), because by (3.11) and (3.12) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\varphi(m,n)E\|X_{mn}\|^{p} < \infty,$$

which implies by Theorem 3.3 that (3.13) holds.

(ii)$\to$(iii): we choose $b_{mn} = (mn)^{(p+1)/p}$, then

$$\liminf_{m\vee n\to\infty} \frac{b^p_{km,ln}}{b^p_{mn}} = (kl)^{p+1} > kl \quad (k \geqslant 2,\ l \geqslant 2)$$

and, by Lemma 2.2, (3.11) holds and by (3.14), (3.12) holds. Thus by (ii), we have the conclusion (3.15).

(iii)$\to$(i): let $\{X_n, \mathcal{G}_n, n \geqslant 1\}$ be an arbitrary martingale differences sequence such that

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|^p}{n^p} < \infty.$$

For $n \geqslant 1$, set $X_{mn} = X_n$ if $m = 1$, and $X_{mn} = 0$ if $m \geqslant 2$. Then $\{X_{mn};\ (m,n) \succeq (1,1)\}$ is an array of random variables with

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|X_{mn}\|^p}{(mn)^p} = \sum_{n=1}^{\infty} \frac{E\|X_n\|^p}{n^p} < \infty.$$

By (iii) and noting that $\mathcal{F}_{1n} = \sigma\{X_i;\ i < n\} \subseteq \mathcal{G}_{n-1}$ for all $n > 1$, hence $E(X_{mn} \mid \mathcal{F}_{mn}) = 0$ for all $(m,n) \succeq (1,1)$, we have

$$\frac{\sum_{i=1}^n X_i}{(mn)^{(p+1)/p}} \xrightarrow{c,\mathcal{L}_p} 0,$$

and by Theorem 3.1 (with $b_{mn} = mn$) then $\Big(\sum\limits_{i=1}^{n} X_i\Big)/mn \xrightarrow{\text{a.s.}} 0$ as $m \vee n \to \infty$. Taking $m = 1$ and letting $n \to \infty$, we obtain that $1/n \sum\limits_{i=1}^{n} X_i \to 0$ a.s.

Then by Theorem 2.2 in [4], $\mathbb{E}$ is p-uniformly smooth. $\square$

For $b_{mn} = m^{\alpha+1/p} n^{\beta+1/p}$ $(\alpha, \beta > 0)$, from (ii) of Theorem 3.4 we get the following corollary.

Corollary 3.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leqslant p \leqslant 2$. Let $\alpha, \beta > 0$ and let $\{X_{mn};\ (m,n) \succeq (1,1)\}$ be an array of $\mathbb{E}$-valued r.v.'s such that $E(X_{ij} \mid \mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j) \preceq (m,n)$. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|X_{mn}\|^p}{n^{\alpha p} m^{\beta p}} < \infty,$$

then

$$\frac{\sup_{(k,l)\preceq(m,n)} \|S^*_{kl}\|}{m^{\alpha+1/p} n^{\beta+1/p}} \xrightarrow{c,\mathcal{L}_p} 0.$$

4. Applications to the strong law of large numbers

By applying the theorems about complete convergence in mean in Section 3 we establish some results on strong laws of large numbers for double arrays of martingale differences with values in p-uniformly smooth Banach spaces.

Theorem 4.1. *Let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1 \leqslant p \leqslant 2$ and let $\{X_{mn}, (m,n) \succeq (1,1)\}$ be an $\mathbb{E}$-valued martingale differences double array. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|X_{mn}\|^p}{n^{\alpha p}m^{\beta p}} < \infty,$$

then

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}\|}{m^{\alpha}n^{\beta}} \to 0 \text{ a.s.} \quad \text{and} \quad \text{in } \mathcal{L}_p \text{ as } m\vee n\to\infty.$$

Proof. By Corollary 3.1, we have

$$\frac{\sup_{(k,l)\preceq(m,n)}\|S_{kl}\|}{m^{\alpha+1/p}n^{\beta+1/p}} \xrightarrow{c,\mathcal{L}_p} 0.$$

Applying Theorem 3.1 with $b_{mn} = m^{\alpha}n^{\beta}$, we have

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}\|}{m^{\alpha}n^{\beta}} \to 0 \text{ a.s.} \quad \text{and} \quad \text{in } \mathcal{L}_p \text{ as } m\vee n\to\infty.$$

$\square$

The following theorem is a Marcinkiewicz-Zygmund type law of large numbers for double arrays of martingale differences.

Theorem 4.2. *Let $1 \leqslant r \leqslant s < q < p \leqslant 2$, let $\mathbb{E}$ be a p-uniformly smooth Banach space. Suppose that $\{X_{mn}, (m,n) \succeq (1,1)\}$ is an $\mathbb{E}$-valued martingale differences double array which is stochastically dominated by an $\mathbb{E}$-random variable X in the sense that for some $0 < C < \infty$,*

$$P\{\|X_{mn}\| \geqslant x\} \leqslant CP\{\|X\| \geqslant x\}$$

for all $(m,n) \succeq (1,1)$ and $x > 0$.

If $E(X_{ij}I(\|X_{ij}\| \leqslant i^{1/q}j^{1/r}) \mid \mathcal{F}_{ij})$ is measurable with respect to $\mathcal{F}_{mn}$ for all $(i,j) \preceq (m,n)$ and $E\|X\|^q < \infty$ then

$$\frac{\max_{(k,l)\preceq(m,n)}\|S_{kl}\|}{m^{1/q}n^{1/r}} \to 0 \text{ a.s.} \quad \text{and} \quad \text{in } \mathcal{L}_s \text{ as } m\vee n\to\infty. \tag{4.1}$$

P r o o f. For each $(m,n) \succeq (1,1)$ set

$$Y_{mn} = X_{mn} I(\|X_{mn}\| \leqslant m^{1/q} n^{1/r}),\ Z_{mn} = X_{mn} I(\|X_{mn}\| > m^{1/q} n^{1/r}),$$
$$U_{mn} = Y_{mn} - E(Y_{mn} \mid \mathcal{F}_{mn}),\ V_{mn} = Z_{mn} - E(Z_{mn} \mid \mathcal{F}_{mn}).$$

It is clear that $X_{mn} = U_{mn} + V_{mn}$.

First,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|Y_{mn}\|^p}{(m^{1/q}n^{1/r})^p} &\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^{1/q}n^{1/r})^p} \int_0^{m^{1/q}n^{1/r}} p x^{p-1} P\{\|X_{mn}\| > x\}\,\mathrm{d}x \\
&\leqslant C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^{1/q}n^{1/r})^p} \int_0^{m^{1/q}n^{1/r}} p x^{p-1} P\{\|X\| > x\}\,\mathrm{d}x \\
&= C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_0^1 P\{\|X\| > t^{1/p} m^{1/q} n^{1/r}\}\,\mathrm{d}t \\
&= C \int_0^1 \Bigg(\sum_{n=1}^{\infty} \Bigg(\sum_{m=1}^{\infty} P\Big\{ \frac{\|X\|}{t^{1/p} n^{1/r}} > m^{1/q} \Big\} \Bigg) \Bigg)\,\mathrm{d}t \\
&= C E(\|X\|^q) \int_0^1 \bigg(\frac{1}{t^{q/p}} \sum_{n=1}^{\infty} \frac{1}{n^{q/r}} \bigg)\,\mathrm{d}t < \infty.
\end{aligned}$$

By applying Corollary 3.1, it follows that

$$\frac{\sup_{(k,l)\preceq(m,n)} \sum_{i=1}^{k}\sum_{j=1}^{l} U_{ij}}{m^{1/q+1/p} n^{1/r+1/p}} \xrightarrow{c,\mathcal{L}_p} 0,$$

and by Theorem 3.1, we get

$$\frac{\sup_{(k,l)\preceq(m,n)} \sum_{i=1}^{k}\sum_{j=1}^{l} U_{ij}}{m^{1/q} n^{1/r}} \to 0 \text{ a.s.} \quad \text{and} \quad \text{in } \mathcal{L}_p \text{ as } m \vee n \to \infty.$$

Then

$$\frac{\sup_{(k,l)\preceq(m,n)} \sum_{i=1}^{k}\sum_{j=1}^{l} U_{ij}}{m^{1/q} n^{1/r}} \to 0 \text{ a.s.} \quad \text{and} \quad \text{in } \mathcal{L}_s \text{ as } m \vee n \to \infty. \tag{4.2}$$

Next,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|Z_{mn}\|^s}{(m^{1/q}n^{1/r})^s} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{s/q} n^{s/r}} \int_0^{\infty} s x^{s-1} P\{\|Z_{mn}\| > x\}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s/q}n^{s/r}}\int_0^{m^{1/q}n^{1/r}} sx^{s-1}P\{\|X_{mn}\| > m^{1/q}n^{1/r}\}\,\mathrm{d}x\\
&\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s/q}n^{s/r}}\int_{m^{1/q}n^{1/r}}^{\infty} sx^{s-1}P\{\|X_{mn}\| > x\}\,\mathrm{d}x\\
&\leqslant C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s/q}n^{s/r}}\int_0^{m^{1/q}n^{1/r}} x^{s-1}P\{\|X\| > m^{1/q}n^{1/r}\}\,\mathrm{d}x\\
&\quad + C\sum\nolimits_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s/q}n^{s/r}}\int_{m^{1/q}n^{1/r}}^{\infty} x^{s-1}P\{\|X\| > x\}\,\mathrm{d}x\\
&= C\biggl(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}P\Bigl\{\frac{\|X\|}{n^{1/r}} > m^{1/q}\Bigr\}\\
&\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_1^{\infty} t^{s-1}P\{\|X\| > tm^{1/q}n^{1/r}\}\,\mathrm{d}t\biggr)\\
&\leqslant C\biggl(\sum_{n=1}^{\infty}\frac{E\|X\|^q}{n^{q/r}} + \int_1^{\infty} t^{s-1}\biggl(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}P\Bigl\{\frac{\|X\|}{n^{1/r}t} > m^{1/q}\Bigr\}\biggr)\,\mathrm{d}t\biggr)\\
&\leqslant C\biggl(\sum_{n=1}^{\infty}\frac{E\|X\|^q}{n^{q/r}} + \int_1^{\infty} t^{s-1}\biggl(\sum_{n=1}^{\infty}\frac{E\|X\|^q}{n^{q/r}t^q}\biggr)\,\mathrm{d}t\biggr)\\
&\leqslant CE\|X\|^q\sum_{n=1}^{\infty}\frac{1}{n^{q/r}}\biggl(\int_1^{\infty}\frac{1}{t^{q-s+1}}\,\mathrm{d}t + 1\biggr) < \infty.
\end{aligned}
$$

By applying Corollary 3.1, it follows that

$$
\frac{\sup_{(k,l)\preceq(m,n)}\sum_{i=1}^{k}\sum_{j=1}^{l}V_{ij}}{m^{1/q+1/s}n^{1/r+1/s}} \xrightarrow{c,\mathcal{L}_s} 0
$$

and by Theorem 3.1 we have

$$
\frac{\sup_{(k,l)\preceq(m,n)}\sum_{i=1}^{k}\sum_{j=1}^{l}V_{ij}}{m^{1/q}n^{1/r}} \to 0 \quad \text{a.s. and in } \mathcal{L}_s \text{ as } m\vee n\to\infty. \tag{4.3}
$$

By (4.2), (4.3) and since the inequality $E\|X+Y\|^s \leqslant 2^{s-1}(E\|X\|^s + E\|Y\|^s)$ holds for $1\leqslant s\leqslant 2$ we have (4.1). The proof is completed. □

Finally, we establish the rate of convergence in the strong law of large numbers.

Theorem 4.3. *Let $0<r<p$, $0<s<p$, let $\mathbb{E}$ be a p-uniformly smooth Banach space for some $1\leqslant p\leqslant 2$ and $\{X_{mn};\ (m,n)\succeq(1,1)\}$ an $\mathbb{E}$-valued martingale differences double array. If*

$$
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|X_{mn}\|^p}{n^{p-r}m^{p-s}} < \infty,
$$

then

$$(4.4)\qquad P\Big(\sup_{(k,l)\succeq(m,n)}\frac{\|S_{kl}\|}{kl}>\varepsilon\Big)=o\Big(\frac{1}{m^r n^s}\Big)\ \text{as } m\vee n\to\infty \text{ for every } \varepsilon>0.$$

Proof. By (ii) in Theorem 3.4 and Lemma 2.2 (with $\{b_{mn}=m^{1+(1-r)/p}\times n^{1+(1-s)/p};\ (m,n)\succeq(1,1)\}$), we have

$$\frac{1}{m^{1+(1-r)/p}n^{1+(1-s)/p}}\max_{(k,l)\preceq(m,n)}\|S_{kl}\|\xrightarrow{c,\mathcal{L}_p}0,$$

and by Theorem 3.2 (with $\alpha=1-r,\beta=1-s$), we have (4.4). □

References

[1] *A. Adler, A. Rosalsky*: Some general strong laws for weighted sums of stochastically dominated random variables. Stochastic Anal. Appl. *5* (1987), 1–16.

[2] *Y. S. Chow*: On the rate of moment convergence of sample sums and extremes. Bull. Inst. Math., Acad. Sin. *16* (1988), 177–201.

[3] *L. V. Dung, T. Ngamkham, N. D. Tien, A. I. Volodin*: Marcinkiewicz-Zygmund type law of large numbers for double arrays of random elements in Banach spaces. Lobachevskii J. Math. *30* (2009), 337–346.

[4] *J. Hoffmann-Jørgensen, G. Pisier*: The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. *4* (1976), 587–599.

[5] *G. Pisier*: Martingales with values in uniformly convex spaces. Isr. J. Math. *20* (1975), 326–350.

[6] *A. Rosalsky, L. V. Thanh, A. I. Volodin*: On complete convergence in mean of normed sums of independent random elements in Banach spaces. Stochastic Anal. Appl. *24* (2006), 23–35.

[7] *F. S. Scalora*: Abstract martingale convergence theorems. Pac. J. Math. *11* (1961), 347–374.

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