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ON THE f- AND h-TRIANGLE OF THE BARYCENTRIC SUBDIVISION OF A SIMPLICIAL COMPLEX

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Abstract. For a simplicial complex Δ we study the behavior of its f- and h-triangle under the action of barycentric subdivision. In particular we describe the f- and h-triangle of its barycentric subdivision $\mathrm{sd}(\Delta)$. The same has been done for f- and h-vector of $\mathrm{sd}(\Delta)$ by F. Brenti, V. Welker (2008). As a consequence we show that if the entries of the h-triangle of Δ are nonnegative, then the entries of the h-triangle of $\mathrm{sd}(\Delta)$ are also nonnegative. We conclude with a few properties of the h-triangle of $\mathrm{sd}(\Delta)$.

Keywords: symmetric group; simplicial complex; f- and h-vector (triangle); barycentric subdivision of a simplicial complex

MSC 2010: 05A05, 05E40, 05E45

1. Introduction

Let Δ be a simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$, that is, a subset $\Delta \subseteq 2^{[n]}$ of the powerset $2^{[n]}$ such that $A \subseteq B \in \Delta$ implies $A \in \Delta$. For an $A \in \Delta$, set $\dim A = \#A - 1$ and $\dim \Delta = \max\limits_{A \in \Delta} \dim A$. Elements of Δ are called faces and inclusionwise maxima faces are called facets. If a simplicial complex is generated by a single facet of dimension $(d-1)$, then it is called $(d-1)$-simplex. For a $(d-1)$-dimensional simplicial complex Δ the f-vector is defined to be $f^{\Delta} = (f_{-1}^{\Delta}, f_0^{\Delta}, f_1^{\Delta}, f_2^{\Delta}, \ldots, f_{d-1}^{\Delta})$, where f_i^{Δ} is the number of i-dimensional faces of Δ. The polynomial $f^{\Delta}(t) = \sum\limits_{i=0}^{d} f_{i-1}^{\Delta} t^{d-i}$ is called the f-polynomial. The f-polynomial relates to commutative algebra in the following way:

Let $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n variables over the field K. Recall that a monomial ideal $I \subset S$ is an ideal which is generated by the monomials in

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S. By I_Δ we denote the monomial ideal generated by $(x_{i_1}\dots x_{i_r};\ \{i_1,\dots,i_r\}\notin\Delta)$. The ring $K[\Delta]=S/I_\Delta$ is called the Stanley-Reisner ring of Δ. There is a one-to-one correspondence between the square-free monomial ideals in n variables and the simplicial complexes over the vertex set of cardinality n. This creates a relation between commutative algebra and combinatorics. Moreover, if we define the h-vector $h^\Delta=(h_1^\Delta,\dots,h_d^\Delta)$ by $h_k^\Delta=\sum\limits_{i=0}^{k}(-1)^{k-i}\binom{d-i}{k-i}f_{i-1}^\Delta$, then the Hilbert series

$$\mathrm{Hilb}(K[\Delta],t)=\sum_{i\geqslant 0}\dim_K(K[\Delta])_i t^i$$

of $K[\Delta]$ is given by $h_0^\Delta+\dots+h_d^\Delta t^d/(1-t)^d$. Here we denote by $(K[\Delta])_i$ the K-vector space generated by the images of the monomials of degree i in the ring $K[\Delta]$. For details, we refer the reader to [3] and [4].

A simplicial complex is said to be pure if all its facets have equal dimension. A pure simplicial complex Δ is shellable if the facets of Δ can be given a linear order $F_1,\dots,F_n$ such that $\langle F_i\rangle\cap\langle F_1,\dots,F_{i-1}\rangle$ is generated by a nonempty set of maximal proper faces of F_i for $i=1,\dots,n$, where $\langle\dots\rangle$ denotes the simplicial complex generated by the face within the brackets. Shellability is a well-known concept in combinatorics with several useful consequences of algebraic and topological nature. The h-vector of Δ can be directly read off from the shelling. To extend this concept for a non-pure simplicial complex the idea of the f- and h-triangle of a simplicial complex was introduced in [1]. A formal definition will follow in Section 2.

In this research we study the behavior of the f- and h-triangle of a simplicial complex under the operations motivated from geometry, namely the barycentric subdivision. In particular we answer the following questions:

Given a simplicial complex Δ, describe the f- and h-triangle of its barycentric subdivision. This has been done for the f- and h-vector in [2].

2. Main results

Let $A\in\Delta$ be a face of Δ. The degree of A, denoted by $\delta(A)$, is defined as follows:

$$\delta(A)=\max\{|F|:\ A\subseteq F\in\Delta\}.$$

Björner and Wachs [1] introduce the f- and h-triangles in the following way:

Definition 2.1. For a $(d-1)$-complex Δ, let

(1) $f_{i,j}^\Delta$ denote number of faces of degree i and cardinality j,

(2) $h^{\Delta}_{i,j} = \sum\limits_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f^{\Delta}_{i,k}$,

(3) the triangular integer arrays $\mathfrak{f}^{\Delta} = (f^{\Delta}_{i,j})_{0 \leqslant j \leqslant i \leqslant d}$ and $\mathfrak{h}^{\Delta} = (h^{\Delta}_{i,j})_{0 \leqslant j \leqslant i \leqslant d}$ be called the f-triangle and h-triangle of Δ, respectively.

For example, $\mathfrak{f}^{\Delta}$ has following representation:

$$\begin{array}{llll} f_{0,0} & & & \\ f_{1,0} & f_{1,1} & & \\ \vdots & & \ddots & \\ f_{d,0} & f_{d,1} & \dots & f_{d,d} \end{array}$$

Note that the indexing of the f-triangle is by the cardinality and that of the f-vector is by the dimension of faces of Δ.

To give an idea about the barycentric subdivision of a $(d-1)$-dimensional simplicial complex Δ, let $(f_0, \dots, f_{d-1})$ be the f-vector of Δ and $\chi = \sum\limits_{i=0}^{d-1} f_i$. We take each face of $\Delta_{d-1} \setminus \emptyset$ as a vertex and label the set of vertices by $v_1, \dots, v_{\chi}$. Now a j-dimensional face of the barycentric subdivision of Δ is a chain of vertices $v_{i_1}, \dots, v_{i_j}$ such that $v_{i_1} \subset \dots \subset v_{i_j}$. The collection of $\emptyset$, all vertices and all such chains forms a simplicial complex called the *barycentric subdivision* of Δ and is denoted by $\mathrm{sd}(\Delta)$. It is well known that Δ and $\mathrm{sd}(\Delta)$ are homeomorphic, that is, both define the cellulations and triangulations of the same space.

The f-triangle of $\mathrm{sd}(\Delta)$ is described as follows:

Lemma 2.2. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then,*

$$f^{\mathrm{sd}(\Delta)}_{i,j} = \sum_{k=0}^{i} j!\, S(k,j) f^{\Delta}_{i,k},$$

for $0 \leqslant j \leqslant i \leqslant d$, where $S(k,j)$ is the Stirling number of the second kind.

Proof. An (i,j)-face of the barycentric subdivision $\mathrm{sd}(\Delta)$ of Δ is given by a subset $\{F_1, \dots, F_j\}$ of j faces of $\Delta \setminus \{\emptyset\}$ such that

$$F_1 \subset F_2 \subset \dots \subset F_j,$$

with $\delta(F_j) = i$. For a face $F \neq \emptyset$ of Δ of cardinality greater or equel to j, we can identify a chain $F_1 \subset \dots \subset F_j = F$ in the barycentric subdivision with the ordered set partition $F_1 \mid F_2 \setminus F_1 \mid \dots \mid F_j \setminus F_{j-1}$ of $F = F_j$.

If F has degree i, then this gives a bijection between the faces of cardinality j and degree i of the barycentric subdivision with top element F and the ordered set of partitions of F into j nonempty blocks. An ordered partition of a set with j elements into j nonempty blocks is counted by the formula $j!S(k,j)$, where k denotes the cardinality of $F = F_j$.

We have $f^{\triangle}_{i,k}$ such faces, so we multiply it with $j!S(k,j)$ to get the result for our case. Now by summing over all faces, that is, from $k = 0$ to i, we have the required formula. $\square$

The following example will demonstrate the above lemma:

Example 2.3. Let Δ be the simplicial complex given in Figure 1 (a) and its barycentric subdivision $\mathrm{sd}(\Delta)$ in Figure 1 (b). By Lemma 2.2, the f-triangle of Δ and $\mathrm{sd}(\Delta)$ is obtained as follows:

$$\begin{array}{llll} 0 & & & \\ 0 & 0 & & \\ 0 & 1 & 1 & \\ 1 & 3 & 3 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{llll} 0 & & & \\ 0 & 0 & & \\ 0 & 2 & 2 & \\ 1 & 7 & 12 & 6 \end{array}$$

In [2], Brenti and Welker define the number $A(d,i,j)$ in the following way: let $\sigma \in S_d$ be a permutation of the symmetric group S_d and let $D(\sigma)$ be the set of descents of σ, i.e., $D(\sigma) = \{i \in [d-1] \colon \sigma(i) > \sigma(i+1)\}$. Set $\mathrm{des}(\sigma) = \#D(\sigma)$. For $1 \leqslant d$, $1 \leqslant j \leqslant d$ and $0 \leqslant i \leqslant d-1$, $A(d,i,j)$ denotes the number of permutations $\sigma \in S_d$ such that $\sigma(1) = j$ and $\mathrm{des}(\sigma) = i$.

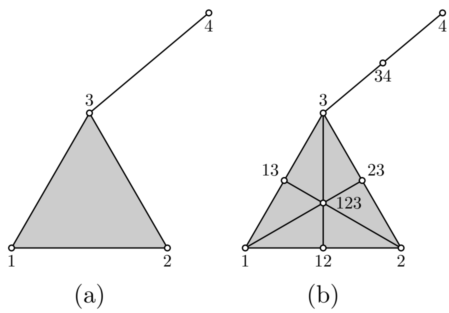

Figure 1.

We modify the number $A(d,i,j)$ in the following way: we denote by $B(d,i,j)$ the number of permutations $\sigma \in S_d$ such that $\mathrm{des}(\sigma) = i$ and $\sigma(d) = j$. The h-triangle of the barycentric subdivision is then given by:

Theorem 2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then*

$$h_{i,j}^{\mathrm{sd}(\Delta)} = \sum_{r=0}^{i} B(i+1, j, i+1-r) h_{i,r}^{\Delta},$$

for $0 \leqslant j \leqslant i \leqslant d$.

Proof. By applying the definition of the h-triangle of the barycentric subdivision, we have

$$h_{i,j}^{\mathrm{sd}(\Delta)} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}^{\mathrm{sd}(\Delta)}.$$

By substituting the value of $f_{i,k}$ from Lemma 2.2, we get

$$h_{i,j}^{\mathrm{sd}(\Delta)} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} \sum_{t=0}^{i} k! S(t,k) f_{i,t}^{\mathrm{sd}(\Delta)}$$

$$h_{i,j}^{\mathrm{sd}(\Delta)} = \sum_{k=0}^{j} \sum_{t=0}^{i} (-1)^{j-k} \binom{i-k}{j-k} k! S(t,k) f_{i,t}^{\mathrm{sd}(\Delta)}.$$

Now applying the reverse relation of $f_{i,t}^{\mathrm{sd}(\Delta)}$, we get

$$\begin{aligned} (2.1) \qquad h_{i,j}^{\mathrm{sd}(\Delta)} &= \sum_{k=0}^{j} \sum_{t=0}^{i} (-1)^{j-k} \binom{i-k}{j-k} k!\, S(t,k) \sum_{r=0}^{t} \binom{i-r}{i-t} h_{i,r}^{\Delta} \\ &= \sum_{r=0}^{i} \left(\sum_{t=0}^{i} \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} \binom{i-r}{i-t} k!\, S(t,k) \right) h_{i,r}^{\Delta}. \end{aligned}$$

By [2], we have

$$\begin{aligned} &\sum_{t=0}^{i} \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} \binom{i-r}{i-t} k! S(t,k) \\ &\quad = \sum_{\{T \subseteq [i], \#T = j\}} \#\{\sigma \in P_i\,;\ D(\sigma) = j,\ \sigma(i) = i+1-r\} \\ &\quad = \#\{\sigma \in P_{i+1}\,;\ \mathrm{des}(\sigma) = j,\ \sigma(i+1) = i+1-r\}, \end{aligned}$$

which describes the number $B(i+1, j, i+1-r)$, hence Equation 2.1 implies:

$$h_{i,j}^{\mathrm{sd}(\Delta)} = \sum_{r=0}^{i} B(i+1, j, i+1-r) h_{i,r}^{\Delta}.$$

□

It is easy to see that Theorem 2.4 verifies the following elementary properties of $h^{\mathrm{sd}(\Delta)}$ given in (Lemma 3.3, [1]):

Corollary 2.5.

(i) $h_{d,0}^{\mathrm{sd}(\Delta)} = 1$ *and* $h_{s,0}^{\mathrm{sd}} = 0$ *for* $0 \leqslant s < d$.

(ii) $\sum\limits_{j=0}^{s} h_{s,j}^{\mathrm{sd}(\Delta)}$ *equals the number of* $(s-1)$*-dimensional facets of* $\mathrm{sd}(\Delta)$.

Proof. (i) $h_{d,0}^{\mathrm{sd}(\Delta)} = \sum\limits_{r=0}^{d} B(d+1, 0, i+1-r) h_{d,r}^{\Delta} = h_{d,0}^{\Delta} = 1$, $h_{d,i} = 0$ for $i > 0$. Analogously $h_{s,0}^{\mathrm{sd}} = \sum\limits_{r=0}^{s} B(s+1, 0, i+1-r) h_{d,r}^{\Delta} = 0$.

(ii) It follows from Theorem 2.4. □

We conclude with the following important result:

Corollary 2.6. *If* $h_{i,j}^{\Delta} \geqslant 0$, *then the following holds:*

(i) $h_{i,j}^{\mathrm{sd}(\Delta)} \geqslant 0$,

(ii) $h_{i,j}^{\mathrm{sd}(\Delta)} \geqslant h_{i,j}^{h_{i,j}}$,

for all $0 \leqslant j \leqslant i \leqslant d$.

Proof. (i) By definition, the number $B(d, i, j)$ is nonnegative, so by Theorem 2.4 the result holds.

(ii) By hypothesis and again by Theorem 2.4, $h_{i,j}^{\mathrm{sd}(\Delta)} \geqslant B(i+1, j, i+1-j) h_{i,j}^{\Delta}$. Thus if $B(i+1, j, i+1-j) \geqslant 1$, then we are done, i.e., there is at least one element $\sigma \in S_{i+1}$ such that $\sigma(i+1) = i+1-j$ with $\mathrm{des}(\sigma) = j$. But $\sigma(l) = i+2-l$ for $1 \leqslant l \leqslant j$ and $\sigma(l) = l-j$ for $j+1 \leqslant l \leqslant i+1$ is the required element. □

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