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TIME ASYMPTOTIC DESCRIPTION OF AN ABSTRACT CAUCHY PROBLEM SOLUTION AND APPLICATION TO TRANSPORT EQUATION

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Abstract. In this paper, we study the time asymptotic behavior of the solution to an abstract Cauchy problem on Banach spaces without restriction on the initial data. The abstract results are then applied to the study of the time asymptotic behavior of solutions of an one-dimensional transport equation with boundary conditions in L_1-space arising in growing cell populations and originally introduced by M. Rotenberg, J. Theoret. Biol. 103 (1983), 181–199.

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1. Introduction

Let T be the generator of a strongly continuous semigroup $(U(t))_{t\geqslant 0}$ on a Banach space X. Let $w(U)$ denote the type of the semigroup $(U(t))_{t\geqslant 0}$ defined by:

$$w(U) = \inf\{w > 0 \text{ such that } \exists M_w \text{ satisfying } \|U(t)\| \leqslant M_w \mathrm{e}^{wt}\ \forall t \geqslant 0\}.$$

Let $\mathcal{L}(X)$ denote the set of all bounded linear operators in X. If $K \in \mathcal{L}(X)$, by the classical perturbation theory (see, for instance, [13, Proposition 1.4]), $A := T + K$ generates a strongly continuous semigroup $(V(t))_{t\geqslant 0}$ given by the Dyson-Phillips expansion:

$$V(t) = \sum_{j\geqslant 0} U_j(t), \tag{1.1}$$

where

$$U_0(t) = U(t) \quad \text{and} \quad U_j(t) = \int_0^t U(s) K U_{j-1}(t-s)\,\mathrm{d}s \quad \forall j \geqslant 1. \tag{1.2}$$

The series (1.1) converges in $\mathcal{L}(X)$ uniformly in bounded time. The remainder term of order n is given by

$$R_n(t) = \sum_{j\geqslant n} U_j(t) = \int_{s_1+\ldots+s_n\leqslant t,\ s_i\geqslant 0} U(s_1)K\ldots U(s_n)KV\Big(t-\sum_{i=1}^{n} s_i\Big)\,\mathrm{d}s_1\ldots\mathrm{d}s_n.$$

So, the Cauchy problem

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = A\psi(t) := (T+K)\psi(t), \quad \psi(0)=\psi_0, \quad \psi_0\in\mathcal{D}(T), \tag{1.3}$$

has a unique classical solution given by $\psi(t)=V(t)\psi_0$. In general, this result follows from the Hille-Yosida theorem. Unfortunately, the Hille-Yosida theorem is not constructive, so the knowledge of the spectrum of A or $(V(t))_{t\geqslant 0}$ plays a central role in getting more information on the solution of problem (1.3), in particular, its behavior for large times.

In [12], M. Mokhtar-Kharroubi has shown that under the following conditions:

$$(\mathcal{A}_1)\qquad \begin{cases} \text{There exists an integer } m \text{ and } w>w(U) \text{ such that}\\ \text{(i) } [(\lambda-T)^{-1}K]^m \text{ is compact for all } \lambda \text{ with } \operatorname{Re}\lambda > w(U),\\ \text{(ii) } \lim\limits_{|\operatorname{Im}\lambda|\to\infty} \|[(\lambda-T)^{-1}K]^m\| = 0 \text{ uniformly on } \{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda\geqslant w\}, \end{cases}$$

one gets the two following results:

$(\mathcal{R}_1)$: $\sigma(A)\cap\{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda>w(U)\}$ consists at most of discrete eigenvalues with finite algebraic multiplicities and $\sigma(A)\cap\{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda\geqslant w>w(U)\}=\{\lambda_i,\ i=1,\ldots,n\}$ is finite.

$(\mathcal{R}_2)$: For any initial data $\psi_0\in\mathcal{D}(A^2)$, the solution of the Cauchy problem (1.3) satisfies

$$\Big\|\psi(t)-\sum_{i=1}^{n} \mathrm{e}^{\lambda_i t}\mathrm{e}^{tD_i}P_i\psi_0\Big\| = o(\mathrm{e}^{\beta^* t}) \quad \text{for all } \beta^*\in\,]\beta_1,\beta_2[, \tag{1.4}$$

where $\beta_1=\sup\{\operatorname{Re}\lambda,\lambda\in\sigma(A) \text{ and } \operatorname{Re}\lambda<w\}$, $\beta_2=\min\{\operatorname{Re}\lambda_i;\ 1\leqslant i\leqslant n\}$, P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with the eigenvalue λ_i, $i=1,2,\ldots,n$.

His analysis was clarified and refined later by B. Abdelmoumen, A. Jeribi, and M. Mnif [1] who showed that the result $(\mathcal{R}_2)$ is obtained even if $\psi_0\in\mathcal{D}(A)$. In fact, they have replaced the assumption $(\mathcal{A}_1)$ by a stronger assumption:

$$(\mathcal{A}_2)\qquad \begin{cases} \text{(i) There exists } m\in\mathbb{N} \text{ such that } [(\lambda-T)^{-1}K]^m \text{ is compact} \\ \quad\text{for all } \lambda \text{ with } \operatorname{Re}\lambda > w(U). \\ \text{(ii) There exists a real } r_0>0, \text{ and for } w>w(U), \text{ there exists } C(w) \\ \quad\text{such that } |\operatorname{Im}\lambda|^{r_0}\|[(\lambda-T)^{-1}K]^m\| \text{ is bounded on } \{\lambda\in\mathbb{C}; \\ \quad\operatorname{Re}\lambda\geqslant w,\ |\operatorname{Im}\lambda|\geqslant C(w)\}. \\ \text{(iii) There exists } c\in\mathbb{R} \text{ such that } \|(\lambda-A)^{-1}\| \text{ is bounded} \\ \quad\text{on } \{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda\geqslant c\}. \end{cases}$$

They have proved that under conditions $(\mathcal{A}_2)$(i)–(ii) the result $(\mathcal{R}_1)$ holds true. Moreover, they have proved that if the assumption $(\mathcal{A}_2)$ is fulfilled, then the requirement $\psi_0\in\mathcal{D}(A^2)$ can be eliminated and the result $(\mathcal{R}_2)$ holds true for any initial data ψ_0 in $\mathcal{D}(A)$.

The purpose of the first part of this paper is to give a description of the large time behavior of solutions to the abstract Cauchy problem (1.3) on Banach spaces without restriction on the initial data. More precisely, we will prove that if we change the assumption $(\mathcal{A}_2)$ to the assumption:

$$(\mathcal{H}_1)\qquad \begin{cases} \text{(i) There exists } m\in\mathbb{N} \text{ such that } [(\lambda-T)^{-1}K]^m \text{ is compact} \\ \quad\text{for all } \lambda \text{ with } \operatorname{Re}\lambda > w(U). \\ \text{(ii) There exists a real } r_0>0, \text{ and for } w>w(U), \text{ there exists } C(w) \\ \quad\text{such that } |\operatorname{Im}\lambda|^{r_0}\|(\lambda-T)^{-1}B_\lambda^m K(\lambda-A)^{-1}\| \text{ is bounded on} \\ \quad\{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda\geqslant w,\ |\operatorname{Im}\lambda|\geqslant C(w)\}, \text{ where } B_\lambda:=K(\lambda-T)^{-1}, \end{cases}$$

then we get the same results as those obtained in [1]. In fact, by using the assumption $(\mathcal{H}_1)$, which is weaker than $(\mathcal{A}_2)$ and $(\mathcal{A}_1)$, we get the result $(\mathcal{R}_1)$ of M. Mokhtar-Kharroubi [12] and we show that the condition $\psi_0\in\mathcal{D}(A^2)$ may be weakened, that is, (1.4) holds true for all ψ_0 belonging to $\mathcal{D}(A)$. Moreover, we give a description of the large time behavior of solutions to the associated Cauchy problem (1.3) (see Theorem 2.1).

In the second part of this paper, we apply the abstract result to the study of the time asymptotic behavior of solutions of the following initial boundary value problem originally introduced by M. Rotenberg [14]:

$$(1.5)\qquad \begin{cases} \dfrac{\partial\psi}{\partial t}(\mu,v,t) = -v\dfrac{\partial\psi}{\partial\mu}(\mu,v,t) - \left[\displaystyle\int_a^b r(\mu,v,v')\,\mathrm{d}v'\right]\psi(\mu,v,t) \\ \qquad\qquad + \displaystyle\int_a^b r(\mu,v,v')\psi(\mu,v',t)\,\mathrm{d}v' \\ \qquad = T_K\psi(\mu,v,t)+B\psi(\mu,v,t) := A_K\psi(\mu,v,t), \\ \psi(\mu,v,0)=\psi_0(u,v). \end{cases}$$

Our paper is organized as follows. In Section 2, we give a description of the large time behavior of solutions to the associated Cauchy problem (1.3). In Section 3, we apply our results directly to discuss the time asymptotic behavior of the solution of a time-dependent linear transport equation with boundary conditions arising in growing cell populations.

2. Time asymptotic description of the solution to the abstract Cauchy problem (1.3)

In this section, we present Theorem 2.1 which gives, on the Banach space X, a description of the large time behavior of solutions to the associated Cauchy problem (1.3). For $w > w(U)$, consider $\mathcal{R}_w := \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geqslant w\}$. We begin by the following proposition:

Proposition 2.1. *Let Q be a complex polynomial satisfying $Q(0) = 0$ and $Q(1) \neq 0$. Assume that $(\lambda - T)^{-1}Q(B_\lambda)$ is compact in $\mathcal{R}_w$, where $B_\lambda := K(\lambda - T)^{-1}$. Then, $\sigma(A) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > w(U)\}$ consists at most of a countable set of isolated points λ_k. Each λ_k is an eigenvalue of finite multiplicity and is a pole for the resolvent $(\lambda - A)^{-1}$.*

P r o o f. The proof of this proposition is inspired and adapted from [16, Theorem II]. For some $n \in \mathbb{N}^*$, we suppose that the polynomial Q is written as:

$$Q(X) = a_1 X + a_2 X^2 + a_3 X^3 + ... + a_n X^n,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{C}$. The function $\lambda \to Q(B_\lambda)$ is regular analytic in the half-plane $\operatorname{Re}\lambda > w(U)$ and its values $Q(B_\lambda)$ are by assumption compact operators. Therefore, for any $\sigma > \operatorname{Re}\lambda$ we have

$$\|Q(B_\lambda)\| \leqslant \frac{a_1\|K\|}{\sigma - w(U)} + \frac{a_2\|K\|^2}{(\sigma - w(U))^2} + \ldots + \frac{a_n\|K\|^n}{(\sigma - w(U))^n}.$$

So, $Q(B_\lambda) \to 0$ if $\operatorname{Re}\lambda \to \infty$. Therefore, $\mu = \sum_{i=1}^{n} a_i \neq 0$ is not an eigenvalue for $Q(B_\lambda)$. Hence Smul'yan's theorem in [15] applies. Then, except for a discrete set of values $\lambda_k \in \mathcal{R}_w$ the operator $\mu I - Q(B_\lambda)$ has a bounded everywhere defined inverse, while $(\mu I - Q(B_\lambda))^{-1}$ has a pole at each of the points λ_k.

On the other hand, we have

$$\mu I - Q(B_\lambda) = a_1(I - B_\lambda) + a_2(I - B_\lambda^2) + \ldots + a_n(I - B_\lambda^n).$$

Then,

$$\begin{aligned}(\mu I - Q(B_\lambda))^{-1} &= [a_1(I - B_\lambda) + a_2(I - B_\lambda^2) + \ldots + a_n(I - B_\lambda^n)]^{-1} \\ &= [(I - B_\lambda)(a_1 I + a_2(I + B_\lambda) + \ldots + a_n(I + B_\lambda + \ldots + B_\lambda^{n-1}))]^{-1}.\end{aligned}$$

So,

$$[a_1 I + a_2(I + B_\lambda) + \ldots + a_n(I + B_\lambda + \ldots + B_\lambda^{n-1})](\mu I - Q(B_\lambda))^{-1} = (I - B_\lambda)^{-1}.$$

Let λ be such that $(I - B_\lambda)^{-1}$ exists. Put

$$R_\lambda = (\lambda - T)^{-1}(I - B_\lambda)^{-1}.$$

Write $A_\lambda = \lambda - A = \lambda - T - K$. We have

$$A_\lambda R_\lambda = (\lambda - A)(\lambda - T - K)^{-1} = I.$$

In the same way, we have $R_\lambda A_\lambda = I$. Hence A_λ^{-1} exists for such λ as a bounded everywhere defined operator and is equal to R_λ. Consequently, the resolvent $(\lambda - A)^{-1} = R_\lambda$ is an analytic function of λ in the half-plane $\operatorname{Re}\lambda > w(U)$ with the exception of a discrete set of values λ_k where R_λ has a pole.

Any pole λ_k of R_λ is an eigenvalue of A. A corresponding eigenfunction φ satisfies the equation $B_{\lambda_k}\varphi = \varphi$. The equation $(\mu I - Q(B_{\lambda_k}))\varphi = 0$ implies $(Q(B_{\lambda_k})/\mu)\varphi = \varphi$. The operator $Q(B_{\lambda_k})$ being compact, the space of solutions of this equation is finite dimensional. This implies that the space of eigenfunctions of A corresponding to the eigenvalue λ_k is finite dimensional, too. □

We deduce from Proposition 2.1 that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}, \ldots$ of A lying in the half plane $\operatorname{Re}\lambda > w(U)$ can be ordered in such a way that the real part decreases [10, page 109], i.e., $\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \ldots > \operatorname{Re}\lambda_{n+1} > \ldots > w(U)$ and $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > w(U)\} \setminus \{\lambda_n, n = 1, 2, \ldots\} \subset \varrho(A)$, where $\varrho(A)$ is the resolvent set of A.

The main result of this section is the following theorem:

Theorem 2.1. *Assume the hypothesis $(\mathcal{H}_1)$ is true and the conditions of Proposition 2.1 are satisfied. Then, for any $\varepsilon > 0$, there exists $M > 0$ such that*

$$\|V(t)(I - P)\| \leqslant M \mathrm{e}^{(\operatorname{Re}\lambda_{n+1} + \varepsilon)t} \quad \forall t > 0,$$

where $P = P_1 + \ldots + P_n$ is the spectral projection of the compact set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

P r o o f. Let $\varepsilon > 0$ and set $\beta_{n,\varepsilon} = \operatorname{Re}\lambda_{n+1} + \varepsilon$. For every λ with $\operatorname{Re}\lambda > \beta_{n,\varepsilon} - \varepsilon/2$, define $f(\lambda) := (\lambda - T)^{-1} B_\lambda^m K(\lambda - A)^{-1}(I-P)\psi$, where $B_\lambda = K(\lambda - T)^{-1}$. It follows from the hypothesis $(\mathcal{H}_1)$ that there exists $\eta > 0$ such that

$$\|f(\lambda)\| \leqslant \frac{\eta}{|\operatorname{Im}\lambda|^{r_0}} \tag{2.1}$$

uniformly on $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \beta_{n,\varepsilon} - \varepsilon/2\}$.

According to [7, Theorem 6.6.1], the function

$$g(t) = \frac{1}{2\mathrm{i}\pi} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{e}^{\lambda t} f(\lambda)\,\mathrm{d}\lambda, \quad \gamma > \max(0, \beta_{n,\varepsilon}), \quad t \geqslant 0, \tag{2.2}$$

is continuous and

$$\int_0^\infty \mathrm{e}^{-\lambda t} g(t)\,\mathrm{d}t = f(\lambda). \tag{2.3}$$

On the other hand, set

$$W(t) = V(t)(I-P) - \sum_{k=0}^m U_k(t).$$

It is easy to see that $t \mapsto W(t)$ is strongly continuous for $t \geqslant 0$. For every $\psi \in X$, we have

$$W(t)(I-P)\psi = V(t)(I-P)\psi - \sum_{k=0}^m U_k(t)(I-P)\psi. \tag{2.4}$$

From [7], [13], for any λ such that $\operatorname{Re}\lambda > \omega(U)$, one can write

$$\int_0^\infty \mathrm{e}^{-\lambda t} U_k(t)\psi\,\mathrm{d}t = (\lambda - T)^{-1} B_\lambda^k \psi, \quad \psi \in X,\ k \in \mathbb{N} \tag{2.5}$$

and

$$\|U_k(t)\| \leqslant \mathrm{e}^{(\omega(U)+\varepsilon)t} \widetilde{M}^{k+1} \|K\|^k \frac{t^k}{k!}, \quad k \in \mathbb{N}, \tag{2.6}$$

where $\widetilde{M} \geqslant 1$ such that $\|U(t)\| \leqslant \widetilde{M}\mathrm{e}^{(\omega(U)+\varepsilon)t}$ for all $t \geqslant 0$. Hence, the use of Eqs. (2.4) and (2.5) leads to

$$\int_0^\infty \mathrm{e}^{-\lambda t} W(t)(I-P)\psi\,\mathrm{d}t = (\lambda - A)^{-1}(I-P)\psi - \sum_{k=0}^m (\lambda - T)^{-1} B_\lambda^k (I-P)\psi.$$

The fact that

$$(\lambda - A)^{-1} = \sum_{k=0}^{\infty} (\lambda - T)^{-1} B_\lambda^k$$

yields

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t} W(t)(I-P)\psi \,\mathrm{d}t &= \sum_{k=m+1}^{+\infty} (\lambda - T)^{-1} B_\lambda^k (I-P)\psi \\ &= (\lambda - T)^{-1} B_\lambda^m K(\lambda - A)^{-1}(I-P)\psi.\end{aligned}$$

Hence,

$$f(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} W(t)(I-P)\psi \,\mathrm{d}t. \tag{2.7}$$

By virtue of the uniqueness of the Laplace integral, Eqs. (2.3) and (2.7) imply

$$W(t)(I-P)\psi = g(t).$$

Since $\lambda \to f(\lambda)$ is analytic in the region $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \beta_{n,\varepsilon} - \varepsilon/2\}$, the integral path on the right-hand side of Eq. (2.3) can be shifted to $\operatorname{Re}\lambda = \beta_{n,\varepsilon}$, i.e.,

$$\begin{aligned} g(t) = \frac{1}{2\mathrm{i}\pi} \lim_{y\to\infty} \Bigg[&\int_{\beta_{n,\varepsilon}-\mathrm{i}y}^{\beta_{n,\varepsilon}+\mathrm{i}y} \mathrm{e}^{t\lambda} f(\lambda)\,\mathrm{d}\lambda + \int_{\beta_{n,\varepsilon}}^{\gamma} \mathrm{e}^{t(x+\mathrm{i}y)} f(x+\mathrm{i}y)\,\mathrm{d}x \\ &+ \int_{\gamma}^{\beta_{n,\varepsilon}} \mathrm{e}^{t(x-\mathrm{i}y)} f(x-\mathrm{i}y)\,\mathrm{d}x \Bigg].\end{aligned}$$

From Eq. (2.1) and using the Lebesgue dominated convergence theorem, the second term and the third term of the above equation tend to zero, so

$$g(t) = \frac{1}{2\mathrm{i}\pi} \int_{\beta_{n,\varepsilon}-\mathrm{i}\infty}^{\beta_{n,\varepsilon}+\mathrm{i}\infty} \mathrm{e}^{t\lambda} f(\lambda)\,\mathrm{d}\lambda. \tag{2.8}$$

We have

$$\|g(t)\| \leqslant \frac{1}{2\pi} \mathrm{e}^{t\beta_{n,\varepsilon}} \int_{-\infty}^{\infty} \|f(\beta_{n,\varepsilon} + \mathrm{i}y)\|\,\mathrm{d}y.$$

We deduce from Eqs. (2.1) and (2.8) that

$$\|g(t)\| \leqslant C\mathrm{e}^{\beta_{n,\varepsilon} t}, \tag{2.9}$$

where $C = (1/2\pi)\eta/|\operatorname{Im}\lambda_{n+1}|^{r_0}$.

Finally, from Eqs. (2.4), (2.6) and (2.9), we get

$$
\begin{aligned}
\|V(t)(I-P)\| &\leqslant \|W(t)(I-P)\| + \sum_{k=0}^{m} \|U_k(t)(I-P)\| \\
&\leqslant C\mathrm{e}^{t\beta_{n,\varepsilon}} + \sum_{k=0}^{m} \mathrm{e}^{(\omega(U)+\varepsilon)t} \widetilde{M}^{k+1} \|K\|^k \frac{t^k}{k!} \\
&\leqslant M\mathrm{e}^{\beta_{n,\varepsilon} t},
\end{aligned}
$$

where

$$
M = \sup_{t \geqslant 0} \left(C + \mathrm{e}^{(\omega(U)-\operatorname{Re}\lambda_{n+1})t} \sum_{k=0}^{m} \widetilde{M}^{k+1} \|K\|^k \frac{t^k}{k!} \right).
$$

This completes the proof. □

3. Application to transport equation

The goal of this section is to apply our result (Theorem 2.1) to the following initial boundary value problem originally introduced by M. Rotenberg [14]:

$$
\left\{
\begin{aligned}
\frac{\partial \psi}{\partial t}(\mu, v, t) = &\, -v \frac{\partial \psi}{\partial \mu}(\mu, v, t) - \left[\int_a^b r(\mu, v, v')\,\mathrm{d}v' \right] \psi(\mu, v, t) \\
&+ \int_a^b r(\mu, v, v') \psi(\mu, v', t)\,\mathrm{d}v' \\
&= T_K \psi(\mu, v, t) + B\psi(\mu, v, t) := A_K \psi(\mu, v, t), \\
\psi(\mu, v, 0) = &\, \psi_0(u, v),
\end{aligned}
\right.
\tag{3.1}
$$

with the following boundary operator:

$$
v\psi(0, v, t) = p \int_a^b \kappa(v, v') v' \psi(1, v', t)\,\mathrm{d}v',
$$

where $\mu \in [0,1]$, $v, v' \in [a,b]$ with $0 < a < b < \infty$, $p \geqslant 0$ denotes the medium number of daughter cells which are descended from mother cells and $\kappa(\cdot,\cdot)$ is the kernel of correlation which satisfies the normalization condition:

$$
\int_a^b \kappa(v, v')\,\mathrm{d}v = 1.
$$

Eq. (3.1) describes the growth and the density of the cell population as a function of the degree of maturity μ, the maturation velocity v and time t. The degree of maturity μ is defined so that $\mu = 0$ at birth and $\mu = 1$ at death.

The function $r(\cdot,\cdot,\cdot)$ denotes the transition rate at which cells change their velocities from v to v'. We denote by $\sigma(\cdot,\cdot)$ the total transition cross section in $L^\infty([0,1]\times[a,b])$, defined by

$$\sigma(\mu,v)=\int_a^b r(\mu,v,v')\,\mathrm{d}v'.$$

We begin by introducing the different notations and preliminaries which we shall need in the sequel. Let us first precise the functional setting of the problem:

Let

$$X:=L_1([0,1]\times[a,b];\,\mathrm{d}\mu\,\mathrm{d}v).$$

We denote by X^0 and X^1 the following boundary spaces:

$$\begin{aligned} X^0 &:= L_1(\{0\}\times[a,b];v\,\mathrm{d}v),\\ X^1 &:= L_1(\{1\}\times[a,b];v\,\mathrm{d}v), \end{aligned}$$

endowed with their natural norms.

Let $\mathcal{W}$ be the space defined by

$$\mathcal{W}=\Big\{\psi\in X;\ v\frac{\partial\psi}{\partial\mu}\in X\Big\}.$$

It is well known (see [6]) that any ψ in $\mathcal{W}$ has traces on the spatial boundary $\{0\}$ and $\{1\}$ which belong to the spaces X^0 and X^1, respectively.

Let K be the following boundary operator:

$$\begin{cases} K\colon\ X^1\to X^0,\\ \qquad \psi\mapsto \dfrac{p}{v}\displaystyle\int_a^b \kappa(v,v')\psi(1,v')v'\,\mathrm{d}v'. \end{cases}$$

Consider the transport operator $A_K:=T_K+B$, where T_K is the free streaming operator defined by:

$$\begin{cases} T_K\colon\ \mathcal{D}(T_K)\subseteq X\to X,\\ \qquad\qquad \psi\mapsto T_K\psi(u,v)=-v\dfrac{\partial\psi}{\partial\mu}(u,v)-\sigma(u,v)\psi(u,v),\\ \mathcal{D}(T_K)=\{\psi\in\mathcal{W};\ \psi^0=K\psi^1\}, \end{cases}$$

where $\psi^0:=\psi(0,v)$, $\psi^1:=\psi(1,v), v\in[a,b]$, and the collision operator B (the integral part of A_K) is a bounded partially integral operator on X defined by:

$$\begin{cases} B\colon\ X\to X,\\ \qquad \psi\mapsto \displaystyle\int_a^b r(\mu,v,v')\psi(\mu,v',t)\,\mathrm{d}v'. \end{cases}\tag{3.2}$$

For more information on this model, see, for example, [3] and [14].

Now, we shall give the expression of the resolvent $(\lambda - T_K)^{-1}$. To do so, we need to determine the solution of the operator equation $(\lambda - T_K)\psi = g$, where $\lambda \in \mathbb{C}$, ψ must belong to $\mathcal{D}(T_K)$ and g is a given function in X.

Let $\underline{\sigma} = \operatorname{ess\,inf} \sigma(\cdot,\cdot)$. For $\operatorname{Re}\lambda > -\underline{\sigma}$, a simple calculation leads to

$$\begin{aligned}\psi(u,v) = \psi(0,v)\exp\left(\frac{-1}{v}\int_0^{\mu}(\lambda+\sigma(\mu',v))\,\mathrm{d}\mu'\right)\\ + \frac{1}{v}\int_0^{\mu}\exp\left(\frac{-1}{v}\int_{\mu'}^{\mu}(\lambda+\sigma(\tau,v))\,\mathrm{d}\tau\right)g(\mu',v)\,\mathrm{d}\mu',\end{aligned}$$

and therefore,

$$\begin{aligned}\psi^1 = \psi(0,v)\exp\left(\frac{-1}{v}\int_0^{1}(\lambda+\sigma(\mu',v))\,\mathrm{d}\mu'\right)\\ + \frac{1}{v}\int_0^{1}\exp\left(\frac{-1}{v}\int_{\mu'}^{1}(\lambda+\sigma(\tau,v))\,\mathrm{d}\tau\right)g(\mu',v)\,\mathrm{d}\mu'.\end{aligned}$$

Observe that the operator B acts only on the maturation velocity v', so μ may be viewed merely as a parameter in $[0,1]$. Hence, we may consider B as a function

$$\begin{aligned}B\colon\ [0,1] &\to \mathcal{L}(L_1([a,b],\,\mathrm{d}v)),\\ \mu &\to B(\mu).\end{aligned}$$

For our subsequent analysis, we introduce the following operators:

$$\begin{cases} P_\lambda\colon\ X^0 \to X^1,\\ \qquad u \mapsto (P_\lambda u)(0,v) := u(0,v)\exp\left(\dfrac{-1}{v}\displaystyle\int_0^1(\lambda+\sigma(\mu',v))\,\mathrm{d}\mu'\right),\end{cases}$$

and

$$\begin{cases} Q_\lambda\colon\ X^0 \to X,\\ \qquad u \mapsto (Q_\lambda u)(0,v) := u(0,v)\exp\left(\dfrac{-1}{v}\displaystyle\int_0^\mu(\lambda+\sigma(\mu',v))\,\mathrm{d}\mu'\right).\end{cases}$$

The operators P_λ and Q_λ are bounded and satisfy the following estimates:

$$\|P_\lambda\| \leqslant \mathrm{e}^{(-1/b)(\operatorname{Re}\lambda+\underline{\sigma})} \tag{3.3}$$

and

$$\|Q_\lambda\| \leqslant \frac{1}{\operatorname{Re}\lambda+\underline{\sigma}}. \tag{3.4}$$

Let Π_λ and R_λ denote the following operators:

$$\begin{cases} \Pi_\lambda \colon \ X \to X^1, \\ \qquad g \mapsto (\Pi_\lambda g)(\mu, v) := \dfrac{1}{v} \displaystyle\int_0^1 \exp\left(\frac{-1}{v} \int_{\mu'}^1 (\lambda + \sigma(\tau, v))\,\mathrm{d}\tau \right) g(\mu', v)\,\mathrm{d}\mu', \end{cases}$$

and

$$\begin{cases} R_\lambda \colon \ X \to X, \\ \qquad g \mapsto (R_\lambda g)(\mu, v) := \dfrac{1}{v} \displaystyle\int_0^\mu \exp\left(\frac{-1}{v} \int_{\mu'}^\mu (\lambda + \sigma(\tau, v))\,\mathrm{d}\tau \right) g(\mu', v)\,\mathrm{d}\mu'. \end{cases}$$

By using [9], a straightforward calculation using Hölder's inequality shows that Π_λ and R_λ are bounded and satisfy:

$$\|\Pi_\lambda\| \leqslant 1 \tag{3.5}$$

and

$$\|R_\lambda\| \leqslant \frac{1}{\operatorname{Re}\lambda + \underline{\sigma}}. \tag{3.6}$$

In the sequel, we shall use the following definition:

Definition 3.1. The collision operator B defined in (3.2) is said to be regular operator if $\{r(\mu, \cdot, v'), (\mu, v') \in [0,1] \times [a,b]\}$ is a relatively weak compact subset of $L_1([a,b], \mathrm{d}v)$.

Theorem 3.1. *We assume that the collision operator B is non-negative, regular and the boundary operator K is positive. Let $\overline{\lambda} := \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(T_K)\}$ be the leading eigenvalue of the operator T_K. Then, for any $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \overline{\lambda}$, the operator $(\lambda - T_K)^{-1}B$ is weakly compact on X.*

Proof. Let λ_0 denote the real number defined by:

$$\lambda_0 := \begin{cases} -\underline{\sigma} & \text{if } \|K\| \leqslant 1, \\ -\underline{\sigma} + b\log(\|K\|) & \text{if } \|K\| > 1. \end{cases}$$

Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \lambda_0$, by virtue of the proof of Theorem 3.1 in [9], we can write

$$(\lambda - T_K)^{-1}B = Q_\lambda K(I - P_\lambda K)^{-1}\Pi_\lambda B + R_\lambda B.$$

So, by using relations (3.4), (3.5), and (3.6), we have

$$
\begin{aligned}
\|(\lambda - T_K)^{-1}B\| &\leqslant \|Q_\lambda\|\,\|K\|\frac{\|\Pi_\lambda\|}{1-\|P_\lambda K\|}\|B\| + \|R_\lambda\|\,\|B\| \\
&\leqslant \frac{1}{\operatorname{Re}\lambda + \underline{\sigma}}\left(\frac{\|K\|}{1-\|P_\lambda\|\|K\|}+1\right)\|B\|.
\end{aligned}
$$

Let $\varepsilon > 0$. For $\operatorname{Re}\lambda > \lambda_0 + \varepsilon$ we have in view of (3.3)

$$
\|(\lambda - T_K)^{-1}B\| \leqslant \frac{1}{\varepsilon}\left(\frac{\|K\|}{1-\mathrm{e}^{-\varepsilon/b}\|K\|}+1\right)\|B\|.
$$

So, $(\lambda - T_K)^{-1}B$ depends continuously on B, uniformly on $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \lambda_0 + \varepsilon\}$. According to [11, Theorem 2.4] and [2, Proposition 2.1 (i)], it suffices to prove the result when B is dominated by a rank-one operator in $\mathcal{L}(L_1([a,b],\,\mathrm{d}v))$. Moreover, by [2, Remark 2.2] and [2, Proposition 2.1 (ii)], we may assume that B itself is a rank-one collision operator in $\mathcal{L}(L_1([a,b],\,\mathrm{d}v))$. This asserts that B has kernel

$$
\kappa(u,v) = \kappa_1(u)\kappa_2(v); \quad \kappa_1(\cdot) \in L_1([a,b]), \quad \kappa_2(\cdot) \in L^\infty([a,b]).
$$

To conclude, it suffices to show that $Q_\lambda K(I - P_\lambda K)^{-1}\Pi_\lambda B$ and $R_\lambda B$ are weakly compact on X. We claim that $\Pi_\lambda B$ and $R_\lambda B$ are weakly compact on X. Consider $\psi \in X$

$$
\begin{aligned}
(\Pi_\lambda B\psi)(v) &= \frac{1}{v}\int_0^1 \exp\left(\frac{-1}{v}\int_{\mu'}^1 (\lambda + \sigma(\tau,v))\,\mathrm{d}\tau\right) B\psi(\mu',v)\,\mathrm{d}\mu' \\
&= \frac{1}{v}\int_0^1\!\!\int_{-1}^1 \exp\left(\frac{-1}{v}\int_{\mu'}^1 (\lambda + \sigma(\tau,v))\,\mathrm{d}\tau\right)\kappa_1(u)\kappa_2(v)\psi(\mu',v)\,\mathrm{d}\mu'\,\mathrm{d}v \\
&= J_\lambda U_\lambda \psi,
\end{aligned}
$$

where U_λ and J_λ denote the following bounded operators:

$$
\begin{cases}
U_\lambda\colon\ X \to L_1([0,1],\,\mathrm{d}\mu'), \\
\qquad\quad \psi \mapsto \displaystyle\int_{-1}^1 \kappa_2(v)\psi(\mu',v)\,\mathrm{d}v,
\end{cases}
$$

and

$$
\begin{cases}
J_\lambda\colon\ L_1([0,1],\,\mathrm{d}\mu') \to X^0, \\
\qquad\qquad \varphi \mapsto \displaystyle\frac{1}{v}\int_0^1 \exp\left(\frac{-1}{v}\int_{\mu'}^1 (\lambda + \sigma(\tau,v))\,\mathrm{d}\tau\right)\kappa_1(u)\varphi(\mu')\,\mathrm{d}\mu'.
\end{cases}
$$

It is now sufficient to show that J_λ is weakly compact. To do so, let $\mathcal{O}$ be a bounded set of $L_1([0,1], \mathrm{d}\mu')$ and let $\varphi \in \mathcal{O}$. We have

$$\int_E |J_\lambda \varphi(v)|\,|v|\,\mathrm{d}v \leqslant \|\varphi\| \int_E |\kappa_1(u)|\,\mathrm{d}u,$$

for all measurable subsets E of $[0,1]$. Next, applying [5, Corollary 11, page 294], we infer that the set $J_\lambda(\mathcal{O})$ is weakly compact, since $\lim\limits_{|E|\to 0} \int_E |\kappa_1(u)|\,\mathrm{d}u = 0$, where $|E|$ is the measure of E. This completes the proof. □

It is well known that the streaming operator T_K generates a strongly continuous positive semigroup $(\widetilde{U}(t))_{t\geqslant 0}$ on X (see, for example, [6]). Since the collision operator B is bounded and positive, the transport operator A_K generates also a strongly continuous positive semigroup $(\widetilde{V}(t))_{t\geqslant 0}$ on X given by the Dyson-Phillips expansion.

Theorem 3.2. *We assume that the collision operator B is non-negative, regular and the boundary operator K is positive. Then, the following assertions hold:*

(i) $\sigma(A_K) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \operatorname{Re}\overline{\lambda}\}$ *consists at most of a countable set of isolated points λ_k. Each λ_k is an eigenvalue of finite multiplicity and is a pole for the resolvent $(\lambda - A_K)^{-1}$.*

(ii) *For $w > 0$, let $\sigma(A_K) \cap \{\operatorname{Re}\lambda > \operatorname{Re}\overline{\lambda} + w\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, let $\beta_1 = \sup\{\operatorname{Re}\lambda, \lambda \in \sigma(A_K)$ and $\operatorname{Re}\lambda < \operatorname{Re}\overline{\lambda} + w\}$ and $\beta_2 = \min\{\operatorname{Re}\lambda_j;\ 1 \leqslant j \leqslant n\}$. Clearly $\beta_1 < \beta_2$. Let β^* be such that $\beta_1 < \beta^* < \beta_2$ and $\psi_0 \in \mathcal{D}(A_K)$. Then, the solution $\psi(t)$ of the Cauchy problem* (3.1) *is given by:*

$$\psi(t) = R(t) + \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \mathrm{e}^{tD_j} P_j \psi_0,$$

where

$$R(t) = \lim_{\gamma\to+\infty} \frac{1}{2\mathrm{i}\pi} \int_{\beta^*-\mathrm{i}\gamma}^{\beta^*+\mathrm{i}\gamma} \mathrm{e}^{\lambda t} (\lambda - A_K)^{-1} \psi_0 \,\mathrm{d}\lambda$$

and where P_j and D_j are, respectively, the spectral projection and the nilpotent operator associated with the eigenvalue λ_j.

Proof. (i) By Theorem 3.1, $(\lambda - T_K)^{-1}B$ is weakly compact. Since the space $X := L_1([0,1]\times[a,b];\, \mathrm{d}\mu\,\mathrm{d}v)$ has the Dunford-Pettis property (see [4]), the use of [8, Lemma 2.1] affirms that $[(\lambda - T_K)^{-1}B]^2$ is compact. Hence, if we consider $Q(X) = aX^2$ a complex polynomial with $a \neq 0$, then, for any $w > -\operatorname{Re}\overline{\lambda}$, $(\lambda - T_K)^{-1}Q(B_\lambda)$ is compact in $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geqslant w\}$. By virtue of Proposition 2.1, $\sigma(A_K) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \operatorname{Re}\overline{\lambda}\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.

(ii) The result follows immediately from Proposition 2.1. □

Theorem 3.3. *Assume that the hypotheses of Theorem 3.2 hold true. Then for any $\varepsilon > 0$, there exists a positive constant M such that*

$$\|\widetilde{V}(t)(I-P)\| \leqslant M\mathrm{e}^{(\operatorname{Re}\lambda_{n+1}+\varepsilon)t} \quad \forall t > 0.$$

Proof. Let $w > \operatorname{Re}\overline{\lambda}$. It follows from [2, Theorem 2.2 (ii)] that there exists $C(w)$ such that $|\operatorname{Im}\lambda|\,\|(\lambda - T_K)^{-1}B\|$ is bounded on $\Delta_w := \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geqslant w,\ |\operatorname{Im}\lambda| \geqslant C(w)\}$. On the other hand, by virtue of the assumptions on K and B, it follows from [12, Lemma 1.1] that $\|(\lambda - A_K)^{-1}\|$ is uniformly bounded on $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geqslant \operatorname{Re}\overline{\lambda} + w,\ |\operatorname{Im}\lambda| \geqslant C(w)\}$. Then, by the relation

$$|\operatorname{Im}\lambda|\,\|(\lambda - T_K)^{-1}B_\lambda B(\lambda - A_K)^{-1}\| \leqslant |\operatorname{Im}\lambda|\,\|(\lambda - T_K)^{-1}B\|^2\|(\lambda - A_K)^{-1}\|,$$

we deduce that $|\operatorname{Im}\lambda|\,\|(\lambda - T_K)^{-1}B_\lambda B(\lambda - A_K)^{-1}\|$ is bounded on Δ_w. The result follows immediately from Theorems 2.1 and 3.2 (i). $\square$

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