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Method of infinite ascent applied on $-(2^p \cdot A^6) + B^3 = C^2$

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Abstract. In this paper, the author shows a technique of generating an infinite number of coprime integral solutions for (A, B, C) of the Diophantine equation $-(2^p \cdot A^6) + B^3 = C^2$ for any positive integral values of p when $p \equiv 1 \pmod 6$ or $p \equiv 2 \pmod 6$. For doing this, we will be using a published result of this author in The Mathematics Student, a periodical of the Indian Mathematical Society.

1 Introduction

Many people, viz., Lebesgue [14], Ljunggren [15], Nagell [19], [20], Chao [8], Cohn [10], Mignotte & de Weger [18], Bugeaud, Mignotte & Siksek [7] have investigated on the solution of the Diophantine equation $x^2 + C = y^n$ with $x \geq 1$, $y \geq 1$, $n \geq 3$ and C is any integer, positive or negative for different values of $|C| \leq 100$. Le [13], Luca [16]; Arif & Muriefah [1] have considered a different form of the equation $x^2 + C = y^n$, when C is no longer a fixed integer but the power of one or two fixed primes.

For other related results concerning equation $x^2 + C = y^n$ see [2], [3], [4], [5], [9], [11], [17], [21], [22], [23], [24]. For a survey relating equation $x^2 + C = y^n$ see [6]. Allowing C to be the product of some power of 2 and an integral sixth power, Theorem 3 and Theorem 4 give the main results of this paper. From a paper of Jena [12], we reproduce two useful Theorems relating to the Diophantine equation

$$mA^6 + nB^3 = C^2 \tag{1}$$

for any pair of integers $(m\,, n)$ and the integral variables (A, B, C). Basing on these two theorems we obtain the main results of this paper.

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Theorem 1 (Jena [12]). *For any integer m, p and q,*

$$m(2pq)^6 + (mp^6 - q^2)(9mp^6 - q^2)^3 = (27m^2p^{12} - 18mp^6q^2 - q^4)^2. \tag{2}$$

Proof. The proof is got by expanding the terms of both the LHS and RHS of (2) and noting their equality. □

Theorem 2 (Jena [12]). *If (A_t, B_t, C_t) is a solution of the Diophantine equation $mA^6 + nB^3 = C^2$ with m, n, A, B and C as integers then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also a solution of the same equation such that*

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \left\{(2A_tC_t), -B_t(9mA_t^6 - C_t^2), (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)\right\} \tag{3}$$

and if mA_t, nB_t and C_t are pairwise coprime where nB_t is an odd integer and 3 is not a factor of C_t then mA_{t+1}, nB_{t+1} and C_{t+1} are also pairwise coprime where nB_{t+1} is an odd integer and 3 is not a factor of C_{t+1}; in addition to this, mA_{t+1} will be always an even integer and C_{t+1} always an odd integer.

Proof. We can get details of the proof in paper [12]. □

Now, let us proceed to the next section to note the principal results of this paper.

2 Results

In this paper, we prove that for any positive integer p, when $p \equiv 1 \pmod 6$ or $p \equiv 2 \pmod 6$ the Diophantine equation $-(2^p \cdot A^6) + B^3 = C^2$ has infinitely many coprime integral solutions for (A, B, C). This is equivalent to proving the statements of Theorem 3 and Theorem 4.

Theorem 3. *For any positive integer $q \geq 1$, the Diophantine equation*

$$-(2^{6q-5} \cdot A^6) + B^3 = C^2 \tag{4}$$

has infinitely many coprime integral solutions for (A, B, C).

Proof. We will prove Theorem 3 in three steps. Firstly, we have to establish that equation (4) has infinitely many coprime integral solutions for (A, B, C) when $q = 1$. Secondly, we will see how to use these coprime solutions of first step to find the initial coprime solutions for (A, B, C) of equation (4) for other values of $q > 1$. Next, we will show that the conditions of generating infinite number of coprime integral solutions, as proposed by Theorem 2, are applicable to (4) for each value of q.

Step I. Putting $q = 1$ in (4) we get

$$-(2^1 \cdot A^6) + B^3 = C^2. \tag{5}$$

We will denote the i^{th} solution for (A, B, C) of equation (4) when $q = j$ as $(A_i, B_i, C_i)_{q=j}$, where i and j take positive integral values. Now, we know that

$$-2 \cdot 1^6 + 3^3 = 5^2. \tag{6}$$

Using the result of (6), we get the starting solution for (A, B, C) of equation (4) as

$$(A_1, B_1, C_1)_{q=1} = (1, 3, 5). \tag{7}$$

Comparing (5) with (1) we get $m = -2$ and $n = 1$. The conditions of generating an infinite number of coprime integral solutions as proposed by Theorem 2 are applicable for equation (5), because the three terms mA_1, nB_1 and C_1 take values -2, 3 and 5 respectively, and are pairwise coprime; nB_1 is an odd integer and 3 is not a factor of C_1. Thus, Theorem 2 can be used repeatedly to generate an infinite number of coprime integral solutions for (A, B, C). Using (3) we have

$$\begin{aligned}(A_2, B_2, C_2)_{q=1} &= \big\{(2A_1C_1), -B_1(9mA_1^6 - C_1^2),\\ &\qquad (27m^2A_1^{12} - 18mA_1^6C_1^2 - C_1^4)\big\}\\ &= \big\{(2 \cdot 1 \cdot 5), -3 \cdot (9 \cdot (-2) \cdot 1^6 - 5^2),\\ &\qquad (27 \cdot (-2)^2 \cdot 1^{12} - 18 \cdot (-2) \cdot 1^6 \cdot 5^2 - 5^4)\big\}\\ &= (2^1 \cdot 5, 129, 383).\end{aligned} \tag{8}$$

Using equation (3), we calculate the k^{th} solution of (5) as

$$(A_k, B_k, C_k) = (2^{k-1} \cdot A'_k, B_k, C_k)$$

where the integer $k > 1$, $A_k = 2^{k-1}A'_k$ and all three terms A'_k, B_k and C_k are odd. By repeated use of equation (3) one can find any number of coprime integral solutions for (A, B, C) of equation (5).

Step II. The first solution for (A, B, C) of equation (5) is $(1, 3, 5)$. Using these values for (A, B, C) in (5) we have

$$\begin{aligned}-2 \cdot 1^6 + 3^3 &= 5^2.\\ \text{Or} \quad -2 \cdot 2^0 \cdot 1^6 + 3^3 &= 5^2.\end{aligned} \tag{9}$$

The second solution for (A, B, C) of equation (5) is $(2^1 \cdot 5, 129, 383)$. Using these values for (A, B, C) in (5) we get

$$\begin{aligned}-2 \cdot 2^6 \cdot 5^6 + 129^3 &= 383^2.\\ \text{Or} \quad -2^7 \cdot 5^6 + 129^3 &= 383^2.\end{aligned} \tag{10}$$

The k^{th} solution for (A, B, C) of equation (5) is $(2^{k-1} \cdot A'_k, B_k, C_k)$. Using these values for (A, B, C) in (5) we obtain

$$-(2^{6k-5} \cdot A'^6_k) + B_k^3 = C_k^2. \tag{11}$$

When $q = 1$, from (9) we get the starting solution for (A, B, C) of equation (4) as $(2^0 \cdot 1, 3, 5)$.
When $q = 2$, from (10) we get the starting solution for (A, B, C) of equation (4) as $(5, 129, 383)$.

When $q = k$, from (11) we get the starting solution for (A, B, C) of equation (4) as (A'_k, B_k, C_k).

Step III. In Step I, we have already proved the validity of the statement of Theorem 3 for $q = 1$. Putting $q = 2$ in (4) we get

$$-(2^7 \cdot A^6) + B^3 = C^2. \tag{12}$$

Now, for each integral value of $q > 1$, there is a starting solution for (A, B, C) for equation (4) as we showed in Step II. Since the values of B and C in these starting solutions are the same values which are generated by the subsequent solutions of equation (4), they should be coprime; B and C are odd integers; and 3 is not a factor of C. Hence, for any integer $q > 1$, the statement of Theorem 3 is valid, because the conditions of generating infinite number of coprime integral solutions as proposed by Theorem 2 are satisfied.

Thus, combining these three steps, we complete the proof of Theorem 3. □

Theorem 4. *For any positive integer $q \geq 1$, the Diophantine equation*

$$-(2^{6q-4} \cdot A^6) + B^3 = C^2 \tag{13}$$

has infinitely many coprime integral solutions for (A, B, C).

Proof. Since $-(2^2 \cdot 1^6) + 5^3 = 11^2$, we get the first coprime solution for (A, B, C) of the Diophantine equation (13) when $q = 1$ as

$$(A_1, B_1, C_1)_{q=1} = (1, 5, 11). \tag{14}$$

Using Theorem 2 we obtain

$$(A_2, B_2, C_2)_{q=1} = (2^1 \cdot 11, 785, -5497) = (2^1 \cdot 11, 785, 5497). \tag{15}$$

We can use (15) to get the first coprime solution for (A, B, C) of the Diophantine equation (13) when $q = 2$ as

$$(A_1, B_1, C_1)_{q=2} = (11, 785, 5497).$$

Steps similar to the proof of Theorem 3 should be followed in establishing the statement of Theorem 4. □

3 Conclusion

The proof of Theorem 3 and Theorem 4 establishes the infinitude characteristics of the Diophantine equation

$$-(2^p \cdot A^6) + B^3 = C^2$$

for any positive integral values of p when $p \equiv 1 \pmod 6$ or, $p \equiv 2 \pmod 6$. But, what about the status of this equation when $p \equiv 0$, 3, 4, or 5 (mod 6)? Well, we don't have the answer, because an initial starting coprime solution for (A, B, C) in each of these cases is not available with us. It needs further investigation.

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