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Kybernetika, Vol. 49 (2013), No. 6, 948--961

Persistent URL: http://dml.cz/dmlcz/143581

KYBERNETIKA — VOLUME *49* (2013), NUMBER 6, PAGES 948–961

LEFT AND RIGHT SEMI-UNINORMS ON A COMPLETE LATTICE

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Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval, and left (right) semi-uninorms are non-commutative and non-associative extensions of uninorms. In this paper, we firstly introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples. Then, we lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. Finally, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

Keywords: fuzzy connective, uninorm, left (right) semi-uninorm, upper (lower) approximation

Classification: 03B52, 03E72

1. INTRODUCTION

Uninorms, introduced by Yager and Rybalov [30], and studied by Fodor et al. [9], are special aggregation operators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling [10, 22, 27, 28, 29]. Uninorms are interesting because their structure is a special combination of t-norms and t-conorms [9]. It is well known that a uninorm U can be conjunctive or disjunctive whenever $U(0,1) = 0$ or 1, respectively. This fact allows to use uninorms in defining fuzzy implications and coimplications [3, 19, 20].

There are real-life situations when truth functions can not be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. [17, 18] introduced the concepts of left and right uninorms on $[0,1]$, Wang and Fang [25, 26] studied the residual operators and the residual coimplicators of left (right) uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [15] introduced the concept of semi-uninorms on a complete lattice. In this paper, motivated by these generalizations, we will generalize the concepts of both left (right) uninorms and semi-uninorms, introduce a new concept, called the left (right) semi-uninorm, illustrate these notions by means of some examples and lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a given binary operation on a complete lattice.

This paper is organized as follows. In section 2, we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these concepts by means of some examples. In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In section 4, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

The knowledge about lattices required in this paper can be found in [5].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. LEFT AND RIGHT SEMI-UNINORMS

Noting that the commutativity and associativity are not desired for aggregation operators in a lot of cases. In this section, based on [15, 17, 25, 26], we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples.

Definition 2.1. A binary operation U on L is called a left (right) semi-uninorm if it satisfies the following two conditions:

(U1) there exists a left (right) neutral element, i. e., an element $e_L \in L$ ($e_R \in L$) satisfying $U(e_L, x) = x$ ($U(x, e_R) = x$) for all $x \in L$,

(U2) U is non-decreasing in each variable.

For any left (right) semi-uninorm U on L, U is said to be left-conjunctive (right-conjunctive) if $U(0, 1) = 0$ ($U(1, 0) = 0$). U is said to be conjunctive if both $U(0, 1) = 0$ and $U(1, 0) = 0$ since it satisfies the classical boundary conditions of AND. If $U(1, 0) = 1$ ($U(0, 1) = 1$), then we call U left-disjunctive (right-disjunctive). We call U disjunctive if both $U(1, 0) = 1$ and $U(0, 1) = 1$ by a similar reason.

If a left (right) semi-uninorm U is associative, then U is the left (right) uninorm (see [25, 26]).

If a left (right) semi-uninorm U with left (right) neutral element e_L (e_R) has a right (left) neutral element e_R (e_L), then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, U is the semi-uninorm (see [15]). In particular, if the neutral element $e = 1$, then the semi-uninorm U becomes a t-seminorm (see [21]) or a semi-copula (see [4, 8]); if the neutral element $e = 0$, then the semi-uninorm U becomes a t-semiconorm (see [7]).

Clearly, $U(0, 0) = 0$ and $U(1, 1) = 1$ hold for any left (right) semi-uninorm U on L. Moreover, the left (right) neutral elements need not to be unique. In fact, the projection operator given by $U(x, y) = x$ for all $x, y \in L$ is such that any element in L is a right neutral element. But, left (right) neutral elements are all idempotent (see [2]) because $U(e_L, e_L) = e_L$ ($U(e_R, e_R) = e_R$) for any left (right) neutral element e_L (e_R) of U.

Definition 2.2. (Wang and Fang [26]) A binary operation U on L is called left (right) infinitely $\vee$-distributive if

$$U\big(\bigvee_{j\in J} x_j, y\big) = \bigvee_{j\in J} U(x_j, y) \ \Big(U\big(x, \bigvee_{j\in J} y_j\big) = \bigvee_{j\in J} U(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L;$$

left (right) infinitely $\wedge$-distributive if

$$U\big(\bigwedge_{j\in J} x_j, y\big) = \bigwedge_{j\in J} U(x_j, y) \ \Big(U\big(x, \bigwedge_{j\in J} y_j\big) = \bigwedge_{j\in J} U(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L.$$

If a binary operation U is left infinitely $\vee$-distributive ($\wedge$-distributive) and also right infinitely $\vee$-distributive ($\wedge$-distributive), then U is said to be infinitely $\vee$-distributive ($\wedge$-distributive).

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1 (see [6]), we have that

$$U(0, y) = U\big(\bigvee_{j\in\emptyset} x_j, y\big) = \bigvee_{j\in\emptyset} U(x_j, y) = 0 \ \Big(U(x, 0) = U\big(x, \bigvee_{j\in\emptyset} y_j\big) = \bigvee_{j\in\emptyset} U(x, y_j) = 0\Big)$$

for any $x, y \in L$ when U is left (right) infinitely $\vee$-distributive and

$$U(1, y) = U\big(\bigwedge_{j\in\emptyset} x_j, y\big) = \bigwedge_{j\in\emptyset} U(x_j, y) = 1 \ \Big(U(x, 1) = U\big(x, \bigwedge_{j\in\emptyset} y_j\big) = \bigwedge_{j\in\emptyset} U(x, y_j) = 1\Big)$$

for any $x, y \in L$ when U is left (right) infinitely $\wedge$-distributive.

When $L = [0, 1]$, a binary function f on $[0, 1]^2$ is infinitely sup-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are left-continuous and increasing and $f(x, 0) = f(0, y) = 0$ for any $x, y \in [0, 1]$; and f is infinitely inf-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are right-continuous and increasing and $f(x, 1) = f(1, y) = 1$ for any $x, y \in [0, 1]$ (see [11]).

For the sake of convenience, we introduce the following symbols:

$\mathcal{U}_s^{e_L}(L)$: the set of all left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_s^{e_R}(L)$: the set of all right semi-uninorms with right neutral element e_R on L;

$\mathcal{U}_{s\vee}^{e_L}(L)$: the set of all right infinitely $\vee$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{\vee s}^{e_R}(L)$: the set of all left infinitely $\vee$-distributive right semi-uninorms with right neutral element e_R on L;

$\mathcal{U}_{s\wedge}^{e_L}(L)$: the set of all right infinitely $\wedge$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{\wedge s}^{e_R}(L)$: the set of all left infinitely $\wedge$-distributive right semi-uninorms with right neutral element e_R on L.

Now, we illustrate the notions of left (right) semi-uninorms by means of some examples.

Example 2.3. Let $L = \{0, a, b, c, d, 1\}$ be a lattice, where $0 < a < b < d < 1,\ 0 < a < c < d < 1$, $b \wedge c = a$ and $b \vee c = d$. Define two binary operations U_1, U_2 on L as follows:

U_1	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	c	c	1
b	0	a	b	c	d	1
c	0	a	c	d	d	1
d	0	a	d	d	d	1
1	0	1	1	1	1	1

U_2	0	a	b	c	d	1
0	0	0	0	0	0	1
a	0	0	a	0	c	1
b	0	a	b	c	d	1
c	0	0	c	0	c	1
d	0	d	d	d	d	1
1	1	1	1	1	1	1

Obviously, U_1 and U_2 are neither commutative nor associative. It is easy to verify that U_1 is a conjunctive infinitely $\vee$-distributive semi-uninorm with the neutral element b and U_2 is a disjunctive infinitely $\wedge$-distributive semi-uninorm with the neutral element b.

Example 2.4. Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a < b < 1,\ 0 < a < c < 1$, $b \wedge c = a$ and $b \vee c = 1$. Define a binary operation U on L as follows:

U	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	c	1
b	0	a	b	c	1
c	0	a	b	c	1
1	0	1	1	1	1

Clearly, U is a conjunctive left semi-uninorm with two left neutral elements b and c. But, U has no right neutral element. It is easy to see that U is neither commutative nor associative. Moreover, U is neither left infinitely $\vee$-distributive ($\wedge$-distributive) nor right infinitely $\vee$-distributive ($\wedge$-distributive).

Example 2.5. Let $e_L \in L$,

$$U_{sW}^{e_L}(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ 0 & \text{otherwise}, \end{cases} \qquad U_{sM}^{e_L}(x, y) = \begin{cases} y & \text{if } x \leq e_L, \\ 1 & \text{otherwise}, \end{cases}$$

$$U_{sW}^{e_L\,*}(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x \geq e_L,\ y \neq 1, \\ 0 & \text{otherwise}, \end{cases} \quad U_{sM}^{e_L\,*}(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ y & \text{if } x \leq e_L,\ y \neq 0, \\ 1 & \text{otherwise}, \end{cases}$$

where x and y are elements of L. Then $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_s^{e_L}(L)$; $U_{sW}^{e_L}$ and $U_{sM}^{e_L\,*}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{s\vee}^{e_L}(L)$; $U_{sW}^{e_L\,*}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{s\wedge}^{e_L}(L)$.

Example 2.6. Let $e_R \in L$,

$$U_{sW}^{e_R}(x,y) = \begin{cases} x & \text{if } y \geq e_R, \\ 0 & \text{otherwise,} \end{cases} \qquad U_{sM}^{e_R}(x,y) = \begin{cases} x & \text{if } y \leq e_R, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sW}^{e_R\,*}(x,y) = \begin{cases} 1 & \text{if } x = 1, \\ x & \text{if } y \geq e_R,\ x \neq 1, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_R\,*}(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x & \text{if } y \leq e_R,\ x \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. Then $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_s^{e_R}(L)$; $U_{sW}^{e_R}$ and $U_{sM}^{e_R\,*}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{\vee s}^{e_R}(L)$; $U_{sW}^{e_R\,*}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{\wedge s}^{e_R}(L)$.

3. THE UPPER AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A BINARY OPERATION

Constructing logic operators is an interesting work. Recently, Jenei and Montagna [12, 13, 14] introduced several new types of constructions of left-continuous t-norms and Wang [24] laid bare the formulas for calculating the smallest pseudo-t-norm that is stronger than a binary operation. In this section, we continue the work in [24] and give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

For any nonempty subfamily $\{T_j \mid j \in J\}$ of $L^{L\times L}$, the least upper bound $\vee_{j\in J} T_j$ and the greatest lower bound $\wedge_{j\in J} T_j$ of T_j's, respectively, define by

$$\Big(\bigvee_{j\in J} T_j\Big)(x,y) = \bigvee_{j\in J} T_j(x,y) \quad\text{and}\quad \Big(\bigwedge_{j\in J} T_j\Big)(x,y) = \bigwedge_{j\in J} T_j(x,y) \quad \forall x,y \in L.$$

It is easy to verify that $(L^{L\times L}, \leq, \vee, \wedge)$ is a complete lattice. Moreover, we have the following two theorems.

Theorem 3.1.

1. $\mathcal{U}_s^{e_L}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ as its minimal and maximal elements, respectively.
2. $\mathcal{U}_s^{e_R}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ as its minimal and maximal elements, respectively.

Theorem 3.2.

1. $\mathcal{U}_{s\wedge}^{e_L}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_L\,*}$ and $U_{sM}^{e_L}$ as its minimal and maximal elements, respectively.

2. $\mathcal{U}^{e_R}_{\wedge s}(L)$ is a complete sublattice of $L^{L\times L}$ with ${U^{e_R}_{sW}}^*$ and $U^{e_R}_{sM}$ as its minimal and maximal elements, respectively.

3. $\mathcal{U}^{e_L}_{s\vee}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_L}_{sW}$ and ${U^{e_L}_{sM}}^*$ as its minimal and maximal elements, respectively.

4. $\mathcal{U}^{e_R}_{\vee s}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_R}_{sW}$ and ${U^{e_R}_{sM}}^*$ as its minimal and maximal elements, respectively.

P r o o f. We only prove that statement (1) holds.

Suppose that $U_j \in \mathcal{U}^{e_L}_{s\wedge}(L)$ $(j \in J)$ and $J \neq \emptyset$. Then it follows from Theorem 3.1 that $\wedge_{j\in J} U_j \in \mathcal{U}^{e_L}_{s}(L)$. Moreover, we have

$$\begin{aligned}&\big(\bigwedge_{j\in J} U_j\big)\big(x, \bigwedge_{k\in K} y_k\big) = \bigwedge_{j\in J} U_j\big(x, \bigwedge_{k\in K} y_k\big) = \bigwedge_{j\in J}\bigwedge_{k\in K} U_j(x, y_k)\\ =\ & \bigwedge_{k\in K}\bigwedge_{j\in J} U_j(x, y_k) = \bigwedge_{k\in K}\big(\bigwedge_{j\in J} U_j(x, y_k)\big) = \bigwedge_{k\in K}\Big(\big(\bigwedge_{j\in J} U_j\big)(x, y_k)\Big),\end{aligned}$$

where K is any index set, and x and y_k $(k \in K)$ are any elements of L. Hence, $\wedge_{j\in J} U_j \in \mathcal{U}^{e_L}_{s\wedge}(L)$. Noting that fact $U^{e_L}_{sM} \in \{U \in \mathcal{U}^{e_L}_{s\wedge}(L) \mid U_j \leq U\ \forall j \in J\}$, let $U^* = \wedge\{U \in \mathcal{U}^{e_L}_{s\wedge}(L) \mid U_j \leq U\ \forall j \in J\}$, then $U^* \in \mathcal{U}^{e_L}_{s\wedge}(L)$ and $U^* = \vee_{j\in J} U_j$. Thus, $\mathcal{U}^{e_L}_{s\wedge}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_L}_{sM}$ and ${U^{e_L}_{sW}}^*$ as its maximal and minimal elements, respectively. □

For a binary operation A on L, if there exists $U \in \mathcal{U}^{e_L}_{s}(L)$ such that $A \leq U$, then it follows from Theorem 3.1 that $\bigwedge\{U \mid A \leq U, U \in \mathcal{U}^{e_L}_{s}(L)\}$ is the smallest left semi-uninorm that is stronger than A on L, we call it the upper approximation left semi-uninorm of A and written as $[A)^{e_L}_{s}$; if there exists $U \in \mathcal{U}^{e_L}_{s}(L)$ such that $U \leq A$, then $\bigvee\{U \mid U \leq A, U \in \mathcal{U}^{e_L}_{s}(L)\}$ is the largest left semi-uninorm that is weaker than A on L, we call it the lower approximation left semi-uninorm of A and written as $(A]^{e_L}_{s}$.

Similarly, we introduce the following symbols:

$[A)^{e_R}_{s}$: the upper approximation right semi-uninorm of A;

$(A]^{e_R}_{s}$: the lower approximation right semi-uninorm of A;

$(A]^{e_L}_{s\wedge}$: the right infinitely $\wedge$-distributive lower approximation left semi-uninorm of A;

$(A]^{e_R}_{\wedge s}$: the left infinitely $\wedge$-distributive lower approximation right semi-uninorm of A;

$[A)^{e_L}_{s\vee}$: the right infinitely $\vee$-distributive upper approximation left semi-uninorm of A;

$[A)^{e_R}_{\vee s}$: the left infinitely $\vee$-distributive upper approximation right semi-uninorm of A.

Now we consider how to construct the upper and lower approximation left (right) semi-uninorms of a binary operation.

Definition 3.3. Let $A \in L^{L\times L}$. Define the upper approximation A_u and the lower approximation A_l of A as follows:

$$A_u(x,y) = \bigvee\{A(u,v) \mid u \le x, v \le y\},\ A_l(x,y) = \bigwedge\{A(u,v) \mid u \ge x, v \ge y\}\ \ \forall\, x,y \in L.$$

Theorem 3.4. Let $A, B \in L^{L\times L}$. Then the following statements hold:

1. $A_l \le A \le A_u$.
2. $(A \vee B)_u = A_u \vee B_u$ and $(A \wedge B)_l = A_l \wedge B_l$.
3. A_u and A_l are non-decreasing in its each variable.
4. If A is non-decreasing in its each variable, then $A_u = A_l = A$.

Proof. Clearly, statements (1) and (2) hold.

3. We only prove that A_l is non-decreasing in its first variable.
If $x_1 \le x_2$, then

$$\{A(u,v) \mid u \ge x_1, v \ge y\} \supseteq \{A(u,v) \mid u \ge x_2, v \ge y\}.$$

Thus $A_l(x_1,y) \le A_l(x_2,y)$ for any $y \in L$ by Definition 3.3, i. e., A_l is non-decreasing in its first variable.

4. If A is non-decreasing in its each variable, then

$$A_l(x,y) = \bigwedge\{A(u,v) \mid u \ge x, v \ge y\} \ge \bigwedge\{A(x,y) \mid u \ge x, v \ge y\} = A(x,y)\ \forall x,y \in L$$

and hence $A_l \ge A$. Thus, it follows from statement (1) that $A_l = A$.
Similarly, we can show that $A_u = A$. □

As usual, the upper or lower approximation of a binary operation is neither a left semi-uninorm nor a right semi-uninorm.

Example 3.5. Let

$$A(x,y) = \begin{cases} \frac{1}{4}y & \text{if } x \le \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $A \le U_{sM}^{(\frac{1}{2})_L}$ and $A_u = A$. Clearly, A_u is not a left semi-uninorm. Let

$$U(x,y) = \begin{cases} \frac{1}{4}y & \text{if } x < \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that U is the upper approximation left semi-uninorm with left neutral element $\frac{1}{2}$ of A.

The following two theorems give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

Theorem 3.6. Let $A \in L^{L\times L}$ and $e_L \in L$.

1. If $A \leq U_{sM}^{e_L}$, then $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$.
2. If $U_{sW}^{e_L} \leq A$, then $(A]_s^{e_L} = U_{sM}^{e_L} \wedge A_l$.
3. If $A \leq {U_{sM}^{e_L}}^*$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$.
4. If ${U_{sW}^{e_L}}^* \leq A$ and A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, then $(A]_{s\wedge}^{e_L} = U_{sM}^{e_L} \wedge A$.

P r o o f. We only prove the statements (1) and (3) hold.

1. Let $U = U_{sW}^{e_L} \vee A_u$. Clearly, $U \geq A$ and $U_{sW}^{e_L} \leq U \leq U_{sM}^{e_L}$. Thus, $U(e_L, x) = x$ for all $x \in L$. By Theorem 3.4(3) and the monotonicity of $U_{sW}^{e_L}$, we see that U is non-decreasing in its each variable. So, $U \in \mathcal{U}_s^{e_L}(L)$. If $A \leq U_1$ and $U_1 \in \mathcal{U}_s^{e_L}(L)$, then $U_1 = (U_1)_u \geq A_u$ and $U_1 \geq U_{sW}^{e_L} \vee A_u = U$. Therefore, $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$.

3. Let $U^* = U_{sW}^{e_L} \vee A$. If A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then A is non-decreasing in its each variable and so $A_u = A$. Noting that $U_{sW}^{e_L}$ and A are all right infinitely $\vee$-distributive, we can see that U^* is also right infinitely $\vee$-distributive. By the proof of statement (1), we have that $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$. □

In Theorem 3.6(3), $A(x, 0) = 0$ for any $x \in L$ when A is right infinitely $\vee$-distributive. Thus, $A \leq {U_{sM}^{e_L}}^*$ can be replaced by $A \leq U_{sM}^{e_L}$.

Similarly, ${U_{sW}^{e_L}}^* \leq A$ can be replaced by $U_{sW}^{e_L} \leq A$ in Theorem 3.6(4).

Analogous to Theorem 3.6, we have the following theorem.

Theorem 3.7. Let $A \in L^{L\times L}$ and $e_R \in L$.

1. If $A \leq U_{sM}^{e_R}$, then $[A)_s^{e_R} = U_{sW}^{e_R} \vee A_u$.
2. If $U_{sW}^{e_R} \leq A$, then $(A]_s^{e_R} = U_{sM}^{e_R} \wedge A_l$.
3. If $A \leq U_{sM}^{e_R}$ and A is non-decreasing in its second variable and left infinitely $\vee$-distributive, then $[A)_{\vee s}^{e_R} = U_{sW}^{e_R} \vee A$.
4. If $U_{sW}^{e_R} \leq A$ and A is non-decreasing in its second variable and left infinitely $\wedge$-distributive, then $(A]_{\wedge s}^{e_R} = U_{sM}^{e_R} \wedge A$.

The following example shows that analogous to the above theorems may not hold for calculating the right (left) infinitely $\wedge$-distributive upper approximation left (right) semi-uninorm and the right (left) infinitely $\vee$-distributive lower approximation left (right) semi-uninorm of a binary operation.

Example 3.8. Let $L = \{0, a, b, 1\}$ be a lattice, where $0 < a < 1$, $0 < b < 1$, $a \vee b = 1$ and $a \wedge b = 0$. Define two binary operations A and B on L as follows:

A	0	a	b	1
0	0	0	0	0
a	a	1	a	1
b	0	0	0	0
1	a	1	a	1

B	0	a	b	1
0	0	b	0	b
a	1	1	1	1
b	0	b	0	b
1	1	1	1	1

Clearly, $A \leq U_{sM}^{0_L}$, $U_{sW}^{1_L} \leq B$, A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, and B is non-decreasing in its first variable and right infinitely $\vee$-distributive. Let $U_1 = {U_{sW}^{0_L}}^* \vee A$ and $U_2 = {U_{sM}^{1_L}}^* \wedge B$. Then

U_1	0	a	b	1
0	0	a	b	1
a	a	1	1	1
b	0	a	b	1
1	a	1	1	1

U_2	0	a	b	1
0	0	0	0	b
a	0	a	b	1
b	0	0	0	b
1	0	a	b	1

It is easy to see that U_1 is not right infinitely $\wedge$-distributive and U_2 is not right infinitely $\vee$-distributive. This shows that U_1 is not the right infinitely $\wedge$-distributive upper approximation left semi-uninorm of A and U_2 is not the right infinitely $\vee$-distributive lower approximation left semi-uninorm of B.

4. THE RELATIONS BETWEEN THE UPPER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A GIVEN BINARY OPERATION AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF ITS DUAL OPERATION

In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In this section, we investigate the relations between the upper approximation left (right) semi-uninorm of a given binary operation and the lower approximation left (right) semi-uninorm of its dual operation.

We firstly review some basic concepts and properties which will be used in this section.

Definition 4.1. (Ma and Wu [16]) A mapping $N : L \to L$ is called a negation if

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $x \leq y,\ x, y \in L \Rightarrow N(y) \leq N(x)$.

A negation N is called strong if it is an involution, i. e., $N(N(x)) = x$ for any $x \in L$.

Theorem 4.2. (Wang and Yu [23]) Let $x_j \in L$ $(j \in J)$. If N is a strong negation on L, then

$$N(\bigvee_{j\in J} x_j) = \bigwedge_{j\in J} N(x_j),\ N(\bigwedge_{j\in J} x_j) = \bigvee_{j\in J} N(x_j).$$

Definition 4.3. (De Baets [1]) Consider a strong negation N on L. The N-dual operation of a binary operation A on L is the binary operation A_N on L defined by

$$A_N(x,y) = N^{-1}\big(A(N(x),N(y))\big) \ \ \forall x,y \in L.$$

Note that $(A_N)_{N^{-1}} = (A_N)_N = A$ for any binary operation A on L.

The following theorem about N-dual is easily verified.

Theorem 4.4. Let A, B be two binary operations and N a strong negation on L. Then the following statements hold:

1. $(A \wedge B)_N = A_N \vee B_N$ and $(A \vee B)_N = A_N \wedge B_N$.
2. If A is left (right) infinitely $\vee$-distributive, then A_N is left (right) infinitely $\wedge$-distributive.
3. If A is left (right) infinitely $\wedge$-distributive, then A_N is left (right) infinitely $\vee$-distributive.
4. If A is increasing (decreasing) in its ith variable, then A_N is increasing (decreasing) in its ith variable ($i = 1, 2$).
5. The N-dual operation of a left (right) semi-uninorm with a left (right) neutral element e_L (e_R) is a left (right) semi-uninorm with a left (right) neutral element $N(e_L)$ ($N(e_R)$).
6. $(U_{sW}^{e_L})_N = U_{sM}^{N(e_L)}$, $(U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$, $(U_{sW}^{e_R})_N = U_{sM}^{N(e_R)}$ and $(U_{sM}^{e_R})_N = U_{sW}^{N(e_R)}$.

Theorem 4.5. If A is a binary operation and N a strong negation on L, then $(A_N)_u = (A_l)_N$ and $(A_N)_l = (A_u)_N$.

Proof. By Definition 4.3 and Theorem 4.2, we can see that

$$\begin{aligned}
(A_N)_u(x,y) &= \bigvee\{A_N(u,v) \mid u \le x, v \le y\} \\
&= \bigvee\{N^{-1}(A(N(u),N(v))) \mid u \le x, v \le y\} \\
&= N^{-1}\big(\bigwedge\{A(N(u),N(v)) \mid u \le x, v \le y\}\big) \\
&= N^{-1}\big(\bigwedge\{A(u',v') \mid u' \ge N(x), v' \ge N(y)\}\big) \\
&= N^{-1}\big(A_l(N(x),N(y))\big) = (A_l)_N(x,y) \ \ \forall x,y \in L.
\end{aligned}$$

Moreover, we have that $(A_u)_N = \big(((A_N)_N)_u\big)_N = \big(((A_N)_l)_N\big)_N = (A_N)_l$. $\square$

Below, we investigate the relations between the upper approximation left (right) semi-uninorms of a given binary operation and lower approximation left (right) semi-uninorms of its dual operation.

Theorem 4.6. Let A, N and e_L be a binary operation, strong negation and fixed element on L, respectively. Then the following statements hold:

1. If $A \le U_{sM}^{e_L}$, then $[A)_s^{e_L} = ((A_N]_s^{N(e_L)})_N$.
2. If $U_{sW}^{e_L} \le A$, then $(A]_s^{e_L} = ([A_N)_s^{N(e_L)})_N$.
3. If $A \le U_{sM}^{e_L}$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $[A)_{s\vee}^{e_L} = ((A_N]_{s\wedge}^{N(e_L)})_N$.
4. If $U_{sW}^{e_L} \le A$ and A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, then $(A]_{s\wedge}^{e_L} = ([A_N)_{s\vee}^{N(e_L)})_N$.

P r o o f. We only prove the statements (1) and (3) hold.

1. If $A \le U_{sM}^{e_L}$, then $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$ by Theorem 3.6 and $A_N \ge (U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$ by Theorem 4.4. Thus, $(A_N]_s^{N(e_L)} = U_{sM}^{N(e_L)} \wedge (A_N)_l$ by Theorem 3.6. Moreover, by virtue of Theorems 3.6, 4.4 and 4.5, we see that

$$\begin{aligned} & \big((A_N]_s^{N(e_L)}\big)_N = \big(U_{sM}^{N(e_L)} \wedge (A_N)_l\big)_N = \big(U_{sM}^{N(e_L)} \wedge (A_u)_N\big)_N \\ = \ & (U_{sM}^{N(e_L)})_N \vee \big((A_u)_N\big)_N = U_{sW}^{e_L} \vee A_u = [A)_s^{e_L}. \end{aligned}$$

3. If $A \le U_{sM}^{e_L}$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $A_u = A$ by Theorem 3.4(4), $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$ by Theorem 3.6, $A_N \ge (U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$ and A_N is is non-decreasing in its first variable and right infinitely $\wedge$-distributive by Theorem 4.4. Thus, $(A_N]_{s\wedge}^{N(e_L)} = U_{sM}^{N(e_L)} \wedge A_N$ by Theorem 3.6. Moreover, we see that $[A)_{s\vee}^{e_L} = ((A_N]_{s\wedge}^{N(e_L)})_N$ by the proof of statement (1). □

Analogous to Theorem 4.6, we have the following theorem.

Theorem 4.7. Let A, N and e_R be a binary operation, strong negation and fixed element on L, respectively. Then the following statements hold:

1. If $A \le U_{sM}^{e_R}$, then $[A)_s^{e_R} = ((A_N]_s^{N(e_R)})_N$.
2. If $U_{sW}^{e_R} \le A$, then $(A]_s^{e_R} = ([A_N)_s^{N(e_R)})_N$.
3. If $A \le U_{sM}^{e_R}$ and A is non-decreasing in its second variable and left infinitely $\vee$-distributive, then $[A)_{\vee s}^{e_R} = ((A_N]_{\wedge s}^{N(e_R)})_N$.
4. If $U_{sW}^{e_R} \le A$ and A is non-decreasing in its second variable and left infinitely $\wedge$-distributive, then $(A]_{\wedge s}^{e_R} = ([A_N)_{\vee s}^{N(e_R)})_N$.

5. CONCLUSIONS AND FUTURE WORKS

Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval. Noting that the associative binary operators are often used to generate n-ary aggregation operators and the commutativity is not desired for these aggregation operators in a lot of cases, Mas et al. [17, 18] introduced the concepts of left and right uninorms on $[0, 1]$ by eliminating the commutativity from the axioms of uninorm, Wang and Fang [25, 26] studied the residual operations and the residual coimplications of left (right) uninorms on a complete lattice, and Liu [15] discussed the concept of semi-uninorms on a complete lattice by removing the associativity and commutativity from the axioms of uninorms. In this paper, motivated by these generalizations, we introduce the concepts of left and right semi-uninorms on a complete lattice, lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation, and discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

In a forthcoming paper, we will investigate the relationships among left (right) semi-uninorms, implications and coimplications on a complete lattice.

ACKNOWLEDGEMENT

This paper is an expanded version of the report presented at the Conference on Quantitative Logic and Soft Computing, Xi'an, May 12 – 15, 2012.

This work is supported by the National Natural Science Foundation of China (61379064), Jiangsu Provincial Natural Science Foundation of China (BK2012672) and Science Foundation of Yancheng Teachers University (13YSYJB0108).

(Received June 20, 2012)

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