Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Lubomír Kubáček; Jana Vrbková
Variance of Plug-in Estimators in Multivariate Regression Models

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 52 (2013), No. 2, 85--97

Persistent URL: `http://dml.cz/dmlcz/143541`

Terms of use:

© Palacký University Olomouc, Faculty of Science, 2013

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **52**, 2 (2013) 85–97

Variance of Plug-in Estimators in Multivariate Regression Models*

Lubomír KUBÁČEK [1] Jana VRBKOVÁ [1,2]

[1] *Department of Mathematical Analysis and Applications of Mathematics*
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: lubomir.kubacekl@upol.cz

[2] *Department of Geoinformatics, Faculty of Science, Palacký University*
17. listopadu 50, 771 46 Olomouc, Czech Republic
e-mail: jana.vrbkova@upol.cz

(Received June 25, 2013)

Abstract

Variance components in regression models are usually unknown. They must be estimated and it leads to a construction of plug–in estimators of the parameters of the mean value of the observation matrix. Uncertainty of the estimators of the variance components enlarge the variances of the plug–in estimators. The aim of the paper is to find this enlargement.

Key words: variance components, plug-in estimator, multivariate models

2010 Mathematics Subject Classification: 62J05, 62H12

1 Introduction

A construction of the best linear unbiased estimator (BLUE) of model parameters need a knowledge of the covariance matrix. If variance components are under discussion and they must be estimated, then a plug–in estimator of the model parameters must be used. This enlarges the variance of the BLUE. The aim of the paper is to find this enlargement.

*Supported by the Operational Program Education for Competitiveness – European Social Fund, project CZ.1.07/2.3.00/20.0170 of the Ministry of Education, Youth and Sports of the Czech Republic.

2 Notation and preliminaries

Four basic structures of multivariate models are under consideration (in more detail see in [2]).

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim_{nm} \left[(\mathbf{I}_m \otimes \mathbf{X}_{n,k})\,\mathrm{vec}(\mathbf{B}_{k,m}), \sum_{i=1}^{p} \vartheta_i \mathbf{V}_{i,(m,m)} \otimes \mathbf{I}_n \right], \tag{2.1}$$

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim_{nm} \left[(\mathbf{I}_m \otimes \mathbf{X}_{n,k})\,\mathrm{vec}(\mathbf{B}_{k,m}), \mathbf{I}_m \otimes \sum_{i=1}^{p} \vartheta_i \mathbf{V}_{i,(n,n)} \right], \tag{2.2}$$

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim_{nr} \left[(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k})\,\mathrm{vec}(\mathbf{B}_{k,m}), \sum_{i=1}^{p} \vartheta_i \mathbf{V}_{i,(r,r)} \otimes \mathbf{I}_n \right], \tag{2.3}$$

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim_{nr} \left[(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k})\,\mathrm{vec}(\mathbf{B}_{k,m}), \mathbf{I}_r \otimes \sum_{i=1}^{p} \vartheta_i \mathbf{V}_{i,(n,n)} \right]. \tag{2.4}$$

Here $\underline{\mathbf{Y}}$ is either $n \times m$ or $n \times r$ random matrix (observation matrix), $\mathbf{I}_m$ is $m \times m$ identity matrix, $\mathbf{X}$ is a given $n \times k$ matrix, $\mathbf{Z}$ is a given $m \times r$ matrix, $\mathbf{V}_1, \dots \mathbf{V}_p$ are given either $r \times r$ or $n \times n$ symmetric and positive semidefinite matrices and $\vartheta_1, \dots, \vartheta_p$ are unknown variance components.

Because of simplicity all models are considered to satisfy the following conditions.

$$r(\mathbf{X}_{n,k}) = k < n,\ \mathbf{V}_1, \dots, \mathbf{V}_p \text{ are symmetric and positive semidefinite,}$$
$$\vartheta_i > 0,\ i = 1, \dots, p,$$
$$\boldsymbol{\vartheta} \in \underline{\vartheta} \quad \text{(open set in the } p\text{-dimensional Euclidean space } E^p\text{)},$$
$$\boldsymbol{\vartheta} \in \underline{\vartheta} \Rightarrow \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \text{ is positive definite matrix,} \quad r(\mathbf{Z}_{m,r}) = m < r.$$

Let $\boldsymbol{\vartheta}_0$ be an approximate value of the vector $\boldsymbol{\vartheta}$ and $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{1,0}, \dots, \vartheta_{p,0})'$. Then $\boldsymbol{\vartheta}_0$-LBLUEs (locally best linear unbiased estimator) of the matrix $\mathbf{B}$ are

$$\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}, \quad \mathrm{Var}_{\vartheta_0}\left[\mathrm{vec}(\widehat{\mathbf{B}})\right] = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) \otimes (\mathbf{X}'\mathbf{X})^{-1}$$

in the model (1),

$$\widehat{\mathbf{B}} = \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}},\ \mathrm{Var}_{\vartheta_0}\left[\mathrm{vec}(\widehat{\mathbf{B}})\right] = \mathbf{I}_m \otimes \left[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}$$

in the model (2),

$$\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1},$$
$$\mathrm{Var}_{\vartheta_0}\left[\mathrm{vec}(\widehat{\mathbf{B}})\right] = \left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1} \otimes (\mathbf{X}'\mathbf{X})^{-1}$$

in the model (3) and

$$\widehat{\mathbf{B}} = \left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^{-1},$$
$$\mathrm{Var}_{\vartheta_0}\left[\,\mathrm{vec}(\widehat{\mathbf{B}})\right] = (\mathbf{Z}\mathbf{Z}')^{-1}\otimes\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}$$

in the model (4) (in more detail see in [2]).

In fact the model (2) is m-tuple of univariate models. Nevertheless it can be analysed as the multivariate one.

In the following text the symbol

$$\mathbf{M}_X = \mathbf{I} - \mathbf{X}\mathbf{X}^+$$

(here "+" means the Moore–Penrose generalized inverse of a matrix; in more detail see in [3]) and

$$\left[\mathbf{M}_X\mathbf{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+ = \mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) - \mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)$$

will be used.

If the vector $\boldsymbol{\vartheta}$ of variance components is estimable, then the $\boldsymbol{\vartheta}_0$-MINQUEs (minimum norm quadratic unbiased estimator; in more detail see in [4], [2], [1]) of $\boldsymbol{\vartheta}$ are

$$\widehat{\boldsymbol{\vartheta}} = \frac{1}{n-k}\mathbf{S}^{-1}_{\Sigma^{-1}(\vartheta_0)}\widehat{\boldsymbol{\gamma}},\ \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1,\dots,\widehat{\gamma}_p)',$$
$$\widehat{\gamma}_i = \mathrm{Tr}\left[\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right],\ i = 1,\dots,p,$$
$$\left\{\mathbf{S}_{\Sigma^{-1}(\vartheta_0)}\right\}_{i,j} = \mathrm{Tr}\left[\mathbf{V}_i\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right],\ i,j = 1,\dots,p,$$
$$\mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = \frac{2}{n-k}\mathbf{S}^{-1}_{\Sigma^{-1}(\vartheta_0)}\quad \text{(in the case of normality of } \underline{\mathbf{Y}})$$

in the model (1),

$$\widehat{\boldsymbol{\vartheta}} = \frac{1}{m}\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\widehat{\boldsymbol{\gamma}},\ \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1,\dots,\widehat{\gamma}_p)'$$
$$\widehat{\gamma}_i = \mathrm{Tr}\left\{\underline{\mathbf{Y}}'\left[\mathbf{M}_X\mathbf{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+\mathbf{V}_i\left[\mathbf{M}_X\mathbf{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+\underline{\mathbf{Y}}\right\},\ i,j = 1,\dots,p,$$
$$\left\{\mathbf{S}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right\}_{i,j} = \mathrm{Tr}\left\{\mathbf{V}_i\left[\mathbf{M}_X\mathbf{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_x\right]^+\mathbf{V}_j\left[\mathbf{M}_X\mathbf{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+\right\},$$
$$i,j = 1,\dots,p,$$
$$\mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = \frac{2}{m}\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\quad \text{(in the case of normality of } \underline{\mathbf{Y}})$$

in the model (2),

$$\widehat{\boldsymbol{\vartheta}} = \left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+}\right]^{-1}\widehat{\boldsymbol{\gamma}},\ \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1,\dots,\widehat{\gamma}_p)',$$
$$\widehat{\gamma}_i = \mathrm{Tr}\Big(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) + \underline{\mathbf{Y}}'\mathbf{P}_X\underline{\mathbf{Y}}\Big\{\left[\mathbf{M}_{Z'}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{Z'}\right]^+$$
$$\times\mathbf{V}_i\left[\mathbf{M}_{Z'}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{Z'}\right]^+\Big\}\Big),\ i=1,\dots,p,$$
$$\mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = 2\left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+}\right]^{-1}\quad \text{(in the case of normality of } \underline{\mathbf{Y}}\text{)}$$

in the model (3) and in the model (4) it is valid that

$$\widehat{\boldsymbol{\vartheta}} = \left[(r-m)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + m\mathbf{S}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right]^{-1}\widehat{\boldsymbol{\gamma}},\ \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1,\dots,\widehat{\gamma}_p)',$$
$$\widehat{\gamma}_i = \mathrm{Tr}\left[\underline{\mathbf{Y}}\mathbf{M}_{Z'}\underline{\mathbf{Y}}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right] + \mathrm{Tr}\Big\{\underline{\mathbf{Y}}\mathbf{P}_{Z'}\underline{\mathbf{Y}}'\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+$$
$$\times\mathbf{V}_i\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+\Big\},\ i=1,\dots,p,$$
$$\mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = 2\left[(r-m)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + m\mathbf{S}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right]^{-1}\quad \text{(in the case of normality of } \underline{\mathbf{Y}}\text{).}$$

Here $\mathbf{P}_X = \mathbf{X}\mathbf{X}^+$.

3 Variance of plug-in estimators

In this section the normality of $\underline{\mathbf{Y}}$ is assumed.

3.1 Model (1)

In the model (1) it is valid that the BLUE of the unbiasedly estimable function $\mathrm{Tr}(\mathbf{HB})$ for any given $m \times k$ matrix $\mathbf{H}$ is

$$\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}}) = \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}\right],\quad \mathrm{Var}_{\vartheta}\left[\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})\right] = \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right],$$

thus the plug–in estimator need not be used and only $\mathrm{Var}_{\vartheta}\left[\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})\right]$ must be estimated. The estimator of the dispersion of $\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})$ is

$$\mathrm{Var}_{\vartheta}\widehat{\left[\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})\right]} = \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\sum_{i=1}^{p}\widehat{\vartheta}_i\mathbf{V}_i\right] = \mathbf{g}'\widehat{\boldsymbol{\vartheta}},$$

where $\mathbf{g} = (g_1,\dots,g_p)',\ g_i = \mathrm{Tr}\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\mathbf{V}_i\right],\ i=1,\dots,p.$

Thus

$$\mathrm{Var}_{\vartheta}\widehat{\left[\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})\right]} = \mathbf{g}'\widehat{\boldsymbol{\vartheta}},\ \mathrm{Var}_{\vartheta_0}\left\{\mathrm{Var}_{\vartheta}\widehat{\left[\mathrm{Tr}(\mathbf{H}\widehat{\mathbf{B}})\right]}\right\} = \frac{2}{n-k}\mathbf{g}'\mathbf{S}^{-1}_{\Sigma^{-1}(\vartheta_0)}\mathbf{g}.$$

3.2 Model (2)

Lemma 1 *In the model* (2) *it is valid that*

$$\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i} = -\operatorname{Tr}\left[\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{v}}\right],$$

where $\mathbf{C}(\boldsymbol{\vartheta}_0) = \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}, \quad \underline{\mathbf{v}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0).$

Thus

$$E_{\vartheta_0}\left(\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i}\right) = 0, \quad E_{\vartheta_0}\left(\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i}\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_j}\right)$$

$$= \operatorname{Tr}\left\{\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^{+}\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}'\right\},$$

$$i, j = 1, \ldots, p.$$

Proof

$$\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i} = \frac{\partial \operatorname{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}}\right\}}{\partial \vartheta_i}$$

$$= \operatorname{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\right.$$

$$\left.\times\,\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}}\right\} - \operatorname{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}}\right\}$$

$$= -\operatorname{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{v}}\right\}.$$

Since

$$\operatorname{Var}_{\vartheta_0}\left[\operatorname{vec}(\underline{\mathbf{v}})\right] = \mathbf{I}\otimes\left[\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) - \mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\right]$$

and

$$\left[\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right]\operatorname{Var}_{\vartheta_0}\left[\operatorname{vec}(\underline{\mathbf{v}})\right]\left[\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right] = \mathbf{I}\otimes\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^{+},$$

it is valid that

$$E_{\vartheta_0}\left(\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i}\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_j}\right)$$

$$= \operatorname{Tr}\left\{\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^{+}\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}'\right\}.$$

□

Let $\mathbf{Z}_H(\boldsymbol{\vartheta}_0)$ be $p \times p$ matrix with entries given as

$$\{\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\}_{i,j} =$$
$$= \operatorname{Tr}\left\{\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\big[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\big]^+\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}'\right\},$$
$$i, j = 1, \dots, p.$$

Corollary 1 *Let* $\widehat{\delta\boldsymbol{\vartheta}} = \widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0$. *Since*

$$\operatorname{Tr}\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \delta\widehat{\boldsymbol{\vartheta}})\big] \approx$$
$$\approx \operatorname{Tr}\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\big] - \sum_{i=1}^{p} \operatorname{Tr}\big[\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{v}}\big]\widehat{\delta\vartheta}_i,$$

and $\underline{\mathbf{v}}$ *and* $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ *are non-correlated, it is valid that*

$$E_{\vartheta_0}\left(\operatorname{Tr}\left\{\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right) \approx \operatorname{Tr}(\mathbf{H}\mathbf{B}),$$
$$\operatorname{Var}_{\vartheta_0}\left(\operatorname{Tr}\left\{\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right) \approx \operatorname{Tr}\big[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\big] + \widehat{\delta\boldsymbol{\vartheta}}'\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\widehat{\delta\boldsymbol{\vartheta}}.$$

Lemma 2

$$\operatorname{Var}_{\vartheta_0}\left\{\operatorname{Tr}\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\right\} = \operatorname{Var}_{\vartheta_0}\left[E_{\vartheta_0}\left(\operatorname{Tr}\left\{\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right)\right]$$
$$+ E_{\vartheta_0}\left[\operatorname{Var}_{\vartheta_0}\left(\operatorname{Tr}\left\{\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right)\right].$$

Proof It is well known. □

With respect to Corollary 1 and Lemma 2 the following statement can be obtained.

Theorem 1

$$\operatorname{Var}_{\vartheta_0}\left\{\operatorname{Tr}\big[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\big]\right\} \approx$$
$$\approx \operatorname{Tr}\left\{\mathbf{H}\big[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\big]^{-1}\mathbf{H}'\right\} + \delta\boldsymbol{\vartheta}'\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} + \frac{2}{m}\operatorname{Tr}\left(\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right).$$

Remark 1 If the vector $\boldsymbol{\vartheta}$ is estimated by an iteration, i.e.

$$\widehat{\boldsymbol{\vartheta}}^{(i+1)} = \frac{1}{m}\mathbf{S}^{-1}_{[M_X\Sigma(\widehat{\vartheta}^{(i)})M_X]^+}\widehat{\boldsymbol{\gamma}}^{(i)},$$
$$\widehat{\gamma}_j^{(i)} = \operatorname{Tr}\left\{\underline{\mathbf{Y}}'\big[\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}^{(i)})\mathbf{M}_X\big]^+\mathbf{V}_j\big[\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}^{(i)})\mathbf{M}_X\big]^+\underline{\mathbf{Y}}\right\}, \quad j = 1, \dots, p,$$

then the estimator

$$\widehat{\operatorname{Var}_{\vartheta}\left\{\operatorname{Tr}\big[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\big]\right\}} \approx \operatorname{Tr}\left\{\mathbf{H}\big[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})\mathbf{X}\big]^{-1}\mathbf{H}'\right\} + \frac{2}{m}\left(\mathbf{Z}_H(\widehat{\boldsymbol{\vartheta}})\mathbf{S}^{-1}_{[M_X\Sigma(\widehat{\vartheta})M_X]^+}\right)$$

can be used.

3.3 Model (3)

Lemma 3 *In the model* (3) *it is valid that*

$$\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i} = -\operatorname{Tr}\left\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{v}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right\},$$
$$i = 1, \dots, p,$$

where $\underline{\mathbf{v}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\mathbf{Z}$.

Thus

$$E_{\vartheta_0}\left(\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i}\right) = 0, \quad E_{\vartheta_0}\left(\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_i}\frac{\partial \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial \vartheta_j}\right)$$
$$= \operatorname{Tr}\left\{\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right.$$
$$\left.\times\,\mathbf{V}_j\left[\mathbf{M}_{Z'}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{Z'}\right]^{+}\right\}, \quad i,j = 1, \dots, p.$$

Proof It is an analogy of the proof of Lemma 1. The equality

$$\operatorname{Var}_{\vartheta_0}\left[\operatorname{vec}(\underline{\mathbf{v}})\right] = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\otimes\mathbf{M}_X + \left\{\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) - \mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\mathbf{Z}\right\}\otimes\mathbf{P}_X$$

was utilized. Here $\mathbf{P}_X = \mathbf{X}\mathbf{X}^{+}$. □

Let $\mathbf{T}_H(\boldsymbol{\vartheta}_0)$ be the $p \times p$ matrix with the entries

$$\{\mathbf{T}_H(\boldsymbol{\vartheta}_0)\}_{i,j} =$$
$$= \operatorname{Tr}\left\{\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right.$$
$$\left.\times\,\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\left[\mathbf{M}_{Z'}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_{Z'}\right]^{+}\right\}, \quad i,j = 1, \dots, p.$$

Corollary 2 *Since*

$$\operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\right] \approx \operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right] - \sum_{i=1}^{p}\operatorname{Tr}\left\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{v}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\right.$$
$$\left.\times\,\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right\}\widehat{\delta\vartheta}_i$$

and $\underline{\mathbf{v}}$ *and* $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ *are non-correlated, it is valid that*

$$E_{\vartheta_0}\left(\operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right) \approx \operatorname{Tr}(\mathbf{H}\mathbf{B}),$$
$$\operatorname{Var}_{\vartheta_0}\left(\operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right) \approx$$
$$\approx \operatorname{Tr}\left\{\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right\} + \widehat{\delta\boldsymbol{\vartheta}}'\mathbf{T}_H(\boldsymbol{\vartheta}_0)\widehat{\delta\boldsymbol{\vartheta}}.$$

Since

$$\operatorname{Var}_{\vartheta_0}\left[E_{\vartheta_0}\left(\operatorname{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right)\right] \approx 0,$$

and

$$E_{\vartheta_0}\left[\mathrm{Var}_{\vartheta_0}\left(\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right)\right]\approx \mathrm{Tr}\left\{\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right\}$$
$$+\,\delta\boldsymbol{\vartheta}'\mathbf{T}_H(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}+2\,\mathrm{Tr}\left\{\mathbf{T}_H(\boldsymbol{\vartheta}_0)\left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)}+k\mathbf{S}^{-1}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+}\right]^{-1}\right\},$$

the following statement is valid.

Theorem 2 *In the model* (3)

$$\mathrm{Var}_{\vartheta_0}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\right\}\approx \mathrm{Tr}\left\{\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}'\right]^{-1}\right\}$$
$$+\,\delta\boldsymbol{\vartheta}'\mathbf{T}_H(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta}+2\,\mathrm{Tr}\left\{\mathbf{T}_H(\boldsymbol{\vartheta}_0)\left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)}+k\mathbf{S}^{-1}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+}\right]^{-1}\right\}.$$

Remark 2 If the vector $\boldsymbol{\vartheta}$ is estimated by iteration, then

$$\mathrm{Var}_{\vartheta}\left\{\widehat{\mathrm{Tr}\left[\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]}\right\}\approx \mathrm{Tr}\left\{\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})\mathbf{Z}'\right]^{-1}\right\}$$
$$+\,2\,\mathrm{Tr}\left\{\mathbf{T}_H(\widehat{\boldsymbol{\vartheta}})\left[(n-k)\mathbf{S}_{\Sigma^{-1}(\widehat{\vartheta})}+k\mathbf{S}_{[M_{Z'}\Sigma(\widehat{\vartheta})M_{Z'}]^+}\right]^{-1}\right\}.$$

3.4 Model (4)

Lemma 4 *In the model* (4) *it is valid that*

$$\frac{\partial\,\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial\vartheta_i}=-\,\mathrm{Tr}\left\{\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{v}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right\},$$
$$i=1,\dots,p,$$

where $\underline{\mathbf{v}}=\underline{\mathbf{Y}}-\mathbf{X}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\mathbf{Z}$. *Thus*

$$E_{\vartheta_0}\left(\frac{\partial\,\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial\vartheta_i}\right)=0,\ \ i=1,\dots,p,$$

$$\mathrm{cov}_{\vartheta_0}\left(\frac{\partial\,\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial\vartheta_i},\frac{\partial\,\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]}{\partial\vartheta_j}\right)$$
$$=\mathrm{Tr}\left\{\mathbf{V}_i\left[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X\right]^+\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{H}'(\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{H}\right.$$
$$\left.\times\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right\}=\{\mathbf{U}_H(\boldsymbol{\vartheta}_0)\}_{i,j},\quad i,j=1,\dots,p.$$

Proof The equality

$$\mathrm{Var}_{\vartheta_0}\left[\mathrm{vec}(\underline{\mathbf{v}})\right]=\mathbf{M}_{Z'}\otimes\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)+\mathbf{P}_{Z'}\otimes\left\{\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)-\mathbf{X}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\right\}$$

must be used. Further procedures are analogous as in the proof of Lemma 1. □

Since

$$\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right] \approx \mathrm{Tr}\left\{\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\right]$$
$$-\sum_{i=1}^{p}\mathrm{Tr}\left\{\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{v}}\mathbf{Z}'(\mathbf{ZZ}')^{-1}\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\right\}\widehat{\delta\vartheta}_i,$$

and $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ and $\underline{\mathbf{v}}$ are non-correlated, it is valid that

$$E_{\vartheta_0}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\} \approx \mathrm{Tr}(\mathbf{HB}),$$
$$\mathrm{Var}_{\vartheta_0}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\} \approx$$
$$\approx \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{H}'(\mathbf{ZZ}')^{-1}\right\} + \widehat{\delta\boldsymbol{\vartheta}}'\mathbf{U}_H(\boldsymbol{\vartheta}_0)\widehat{\delta\boldsymbol{\vartheta}}.$$

Thus the following statement is valid.

Theorem 3

$$\mathrm{Var}_{\vartheta_0}\left(E_{\vartheta_0}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right) \approx 0,$$
$$E_{\vartheta_0}\left(\mathrm{Var}_{\vartheta_0}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})\right]\Big|\widehat{\delta\boldsymbol{\vartheta}}\right\}\right) \approx \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{H}'(\mathbf{ZZ}')^{-1}\right\}$$
$$+\,\delta\boldsymbol{\vartheta}'\mathbf{U}_H(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} + 2\,\mathrm{Tr}\left\{\mathbf{U}_H(\boldsymbol{\vartheta}_0)\left[(r-m)\mathbf{S}^{-1}_{\Sigma^{-1}(\vartheta_0)} + m\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right]^{-1}\right\}.$$

Remark 3 If the estimator of $\boldsymbol{\vartheta}$ is determined by the iteration, the estimator of $\mathrm{Var}_\vartheta\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\}$ can be given as

$$\widehat{\mathrm{Var}_\vartheta\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\}} \approx \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})\mathbf{X}\right]^{-1}\mathbf{H}'(\mathbf{ZZ}')^{-1}\right\}$$
$$+\,2\,\mathrm{Tr}\left\{\mathbf{U}_H(\widehat{\boldsymbol{\vartheta}})\left[(r-m)\mathbf{S}^{-1}_{\Sigma^{-1}(\widehat{\vartheta})} + m\mathbf{S}^{-1}_{[M_X\Sigma(\widehat{\vartheta})M_X]^+}\right]^{-1}\right\}.$$

4 Numerical example

Let

$$\mathrm{vec}(\underline{\mathbf{Y}}_{6,3}) \sim [(\mathbf{I}\otimes\mathbf{X})\mathrm{vec}(\mathbf{B}), \mathbf{I}\otimes\boldsymbol{\Sigma}]$$

where

$$\mathbf{X}_{6,2} = \begin{pmatrix} 1, & -3 \\ 1, & -2 \\ 1, & -1 \\ 1, & 1 \\ 1, & 2 \\ 1, & 3 \end{pmatrix}, \quad \mathbf{B}_{2,3} = \begin{pmatrix} 0, & 0.5, & 1 \\ 0.5, & 1, & 1.5 \end{pmatrix}$$

$$\mathbf{\Sigma} = \vartheta_1 \begin{pmatrix} \mathbf{I}_{3,3}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}_{3,3} \end{pmatrix} + \vartheta_2 \begin{pmatrix} \mathbf{0}_{3,3}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}_{3,3} \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} 1, & 1 \\ 1, & 1 \\ 1, & 1 \end{pmatrix}, \quad \vartheta_1 = (0.1)^2, \quad \vartheta_2 = (0.3)^2, \quad \boldsymbol{\vartheta}_0 = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}.$$

Then

$$\sqrt{\mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{X}\right]^{-1}\mathbf{H}'\right\}} = 0.1946272,$$

$$\sqrt{\frac{1}{9999}\sum_{i=1}^{10000}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right] - \frac{1}{10000}\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\}^2} = 0.2145846,$$

$$\sqrt{\mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{H}'\right\} + \frac{2}{3}\mathrm{Tr}\left[\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right]} = 0.2037746.$$

Since

$$\frac{0.2037746}{0.2145846} = 0.9496236,$$

the approximate standard deviation of the plug-in estimator attains in the mean 95 % of the actual value.

Let in the same case $\vartheta_1 = 1^2$ and $\vartheta_2 = 3^2$. Since

$$\sqrt{\mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{X}\right]^{-1}\mathbf{H}'\right\}} = 1.946272,$$

$$\sqrt{\frac{1}{9999}\sum_{i=1}^{10000}\left\{\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right] - \frac{1}{10000}\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\}^2} = 2.165647,$$

$$\sqrt{\mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\right]^{-1}\mathbf{H}'\right\} + \frac{2}{3}\mathrm{Tr}\left[\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\mathbf{S}^{-1}_{[M_X\Sigma(\vartheta_0)M_X]^+}\right]} = 2.037746.$$

Also in this case

$$\frac{2.037746}{2.165647} = 0.940941,$$

the approximate standard deviation of the plug-in estimator attains in mean 94 % of the actual value.

The probability density functions of the random variable $\mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]$ see for $\vartheta_1 = 0.1^2$, $\vartheta_2 = 0.3^2$ on Fig. 1, for the $\vartheta_1 = 1^2$, $\vartheta_2 = 3^2$ on Fig. 2.

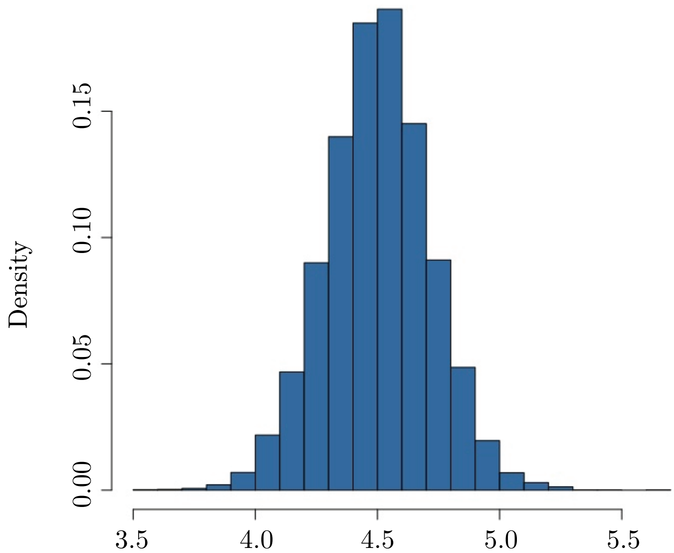

Fig. 1: $E\left\{ \mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\} = 4.5$

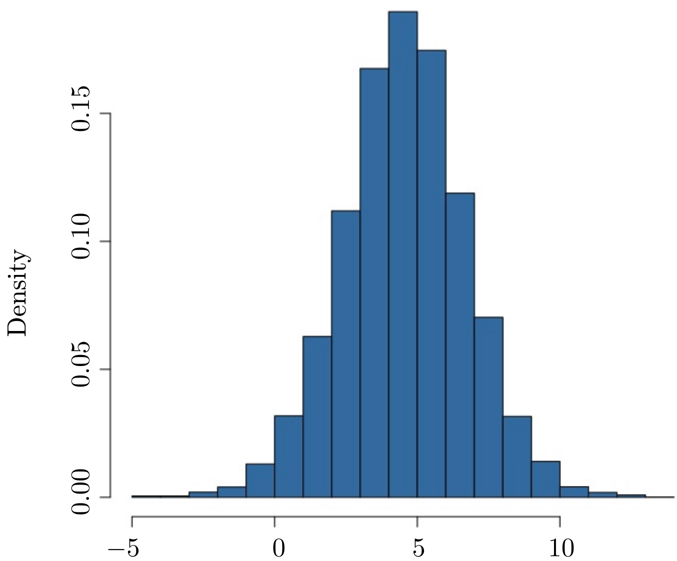

Fig. 2: $E\left\{ \mathrm{Tr}\left[\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})\right]\right\} = 4.5$

The probability density of the random variable

$$\tau(\widehat{\boldsymbol{\vartheta}}) = \mathrm{Tr}\left\{\mathbf{H}\left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})\mathbf{X}\right]^{-1}\mathbf{H}'\right\} + \frac{2}{3}\,\mathrm{Tr}\left[\mathbf{Z}_H(\widehat{\boldsymbol{\vartheta}})\mathbf{S}^{-1}_{[M_X\Sigma(\widehat{\vartheta})M_X]^+}\right]$$

see for $\vartheta_1 = 0.1^2$, $\vartheta_2 = 0.3^2$ on Fig. 3, for the $\vartheta_1 = 1^2$, $\vartheta_2 = 3^2$ on Fig. 4.

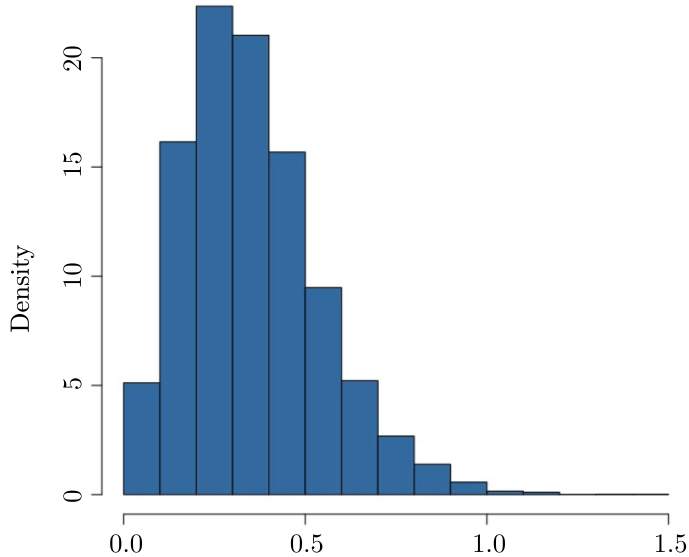

Fig. 3: $E\{\tau(\widehat{\boldsymbol{\vartheta}})\} = 0.03787975$

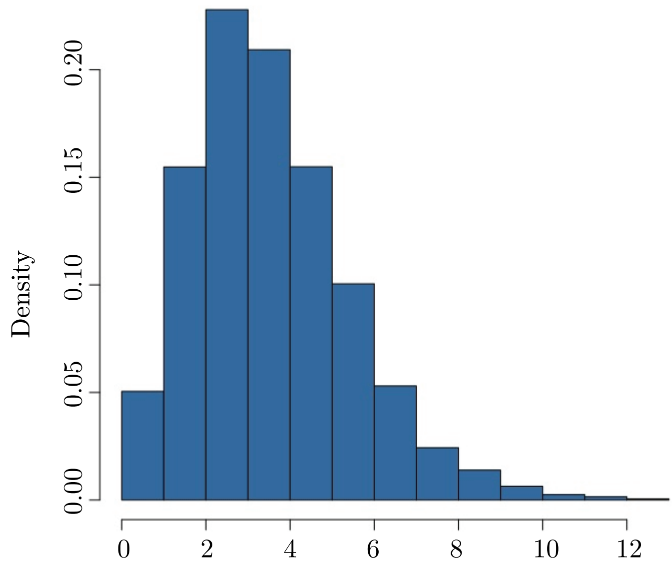

Fig. 4: $E\{\tau(\widehat{\boldsymbol{\vartheta}})\} = 3.787975$

References

[1] Fišerová, E., Kubáček, L., Kunderová, P.: Linear Statistical Models: Regularity and Singularities. *Academia*, Praha, 2007.

[2] Kubáček, L.: Multivariate Statistical Models Revisited. *Vyd. University Palackého*, Olomouc, 2008.

[3] Rao, C. R., Mitra, S. K.: Generalized Inverse of Matrices and its Applications. *Wiley*, New York–London–Sydney–Toronto, 1971.

[4] Rao, C. R., Kleffe, J.: Estimation of Variance Components and Applications. *North–Holland*, Amsterdam–New York–Oxford–Tokyo, 1988.