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NEW EXISTENCE RESULTS OF ANTI-PERIODIC SOLUTIONS OF NONLINEAR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper is a continuation of Y. Liu, Anti-periodic solutions of nonlinear first order impulsive functional differential equations, Math. Slovaca 62 (2012), 695–720. By using Schaefer's fixed point theorem, new existence results on anti-periodic solutions of a class of nonlinear impulsive functional differential equations are established. The techniques to get the priori estimates of the possible solutions of the mentioned equations are different from those used in known papers. An example is given to illustrate the main theorems obtained. One sees easily that Example 3.1 can not be solved by Theorems 2.1–2.3 obtained in Liu's paper since (G2) in Theorem 2.1, (G4) in Theorem 2.2 and (G6) in Theorem 2.3 are not satisfied.

Keywords: anti-periodic solution; impulsive functional differential equation; fixed-point theorem; growth condition

MSC 2010: 34B16, 34C25

1. Introduction

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equation can be interpreted as the standard Malthus population model $y' = -a(t)y$ subject to a perturbation with periodical delay, this is $y'(t) = -a(t)y(t) + \lambda h(t)f(y(t-\tau(t))$ (see [9]).

It is well known that a function $x\colon \mathbb{R} \to \mathbb{R}$ is called anti-periodic function with anti-period $T > 0$ if $x(t+T) = -x(t)$ for all $t \in \mathbb{R}$. Furthermore, x is a periodic

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function with period T if x is an anti-periodic function with anti-period $T/2 > 0$. So we can get periodic solutions of a functional differential equation by obtaining anti-periodic solutions of the corresponding functional differential equation.

Anti-periodic boundary value problems for ordinary differential equations with or without impulses effects have been studied extensively in the last ten years since these problems appear in a variety of applications. For example, for first order ordinary differential equations without impulses effects, a Massera criterion is presented in [6], quasilinearization methods are applied in [27] and in [10], [12], [13], [14], [15], [21], [24], [26], [27] and [28] the validity of lower and upper solution methods coupled with the monotone iterative technique is shown.

The anti-periodic boundary problems for partial differential equations, abstract differential equations, evolution equations or higher order ordinary differential equations were considered in [1]–[8] and [25] and the references cited there.

We note that, in the above mentioned papers, the problems are discussed on a finite interval. For example, it is easy to see that the anti-periodic BVP on finite interval of the form

$$\begin{cases} x'(t) = -1, & t \in (0,1), \\ x(0) = -x(1), \end{cases}$$

has a unique solution $x(t) = -t + 1/2$. But one sees that the equation

$$x'(t) = -1$$

has no solution x satisfying $x(t) = -x(t+1)$ for all $t \in \mathbb{R}$ what we call an anti-periodic solution with anti-period 1. This shows us that the existence of solutions of an anti-periodic boundary value problem for a first order differential equation does not imply, in general, the existence of anti-periodic solutions of the corresponding differential equation.

In fact, the study of anti-periodic solutions for nonlinear evolution equations is closely related to the study of periodic solutions, and it was initiated by Okochi [23]. In the recent papers [11] and [17], the authors studied the existence of anti-periodic solutions for a class of differential equations.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction to the basic theory of impulsive differential equations, we refer the reader to [16].

One important question is whether the impulsive functional differential equation

$$\begin{cases} y'(t) + a(t)y(t) = h(t)f(y(t-\tau(t)), & t \in \mathbb{R}, \\ \Delta x(t_k) = I_k(x(t_k)), & k \in Z, \end{cases}$$

can support periodic solutions or anti-periodic solutions. This question has been studied extensively by a number of authors, see for example [19], [20] and [18] and the references therein.

In this paper, we study the nonlinear impulsive functional differential equation of the form

$$
(1.1)\quad \begin{cases} x'(t) + a(t)x(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))), \ t \in \mathbb{R}, \ t \neq t_k, \ k \in \mathbb{Z}, \\ \Delta x(t_k) = I_k(x(t_k)), \ k \in \mathbb{Z}, \end{cases}
$$

where $\mathbb{Z}, \mathbb{R}$ denote the integer set and the real number set, respectively, $T > 0$ is a constant, $\ldots < t_{-m} < \ldots < t_0 < t_1 < \ldots < t_m < \ldots$ are constants, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $I_k \colon \mathbb{R} \to \mathbb{R}$ is continuous, and $a \colon \mathbb{R} \to \mathbb{R}$, $f \colon \mathbb{R}^{n+2} \to \mathbb{R}$ and $\alpha_i \colon \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, n)$ are functions.

This paper is a continuation of [18]. The purpose is to establish new results on the existence of anti-periodic solutions of the equation (1.1). This is the first time that the Schaefer fixed point theorem [15] or [22] is used for studying the existence of anti-periodic solutions of an impulsive functional differential equation.

The remainder of this paper is divided into two sections, the main results are established in Section 2 and an example is given in Section 3 to illustrate the main results.

2. Main results

Let X be defined by

$$
X = \left\{ \begin{array}{l} x \colon \mathbb{R} \to \mathbb{R}, \ x|_{(t_k, t_{k+1})} \in C^0(t_k, t_{k+1}), \ k \in \mathbb{Z} \\ \text{there exist the limits } \lim\limits_{t \to t_k^-} x(t) = x(t_k), \\ \lim\limits_{t \to t_k^+} x(t), \ k \in \mathbb{Z} \text{ and } x(t) = -x(t+T) \ \text{ for all } t \in \mathbb{R} \end{array} \right\}.
$$

Define the norm $\|u\| = \sup\limits_{t \in \mathbb{R}} |u(t)|$ for all $u \in X$. It is easy to show that X is a real Banach space.

A function $f \colon \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is called an impulsive continuous function if

▷ $f(\cdot, u_0, u_1, \ldots, u_n) \in X$ for each $u = (u_0, \ldots, u_n) \in \mathbb{R}^{n+1}$;

▷ $f(t, \cdot, \ldots, \cdot)$ is continuous for all $t \in \mathbb{R}$.

By a solution of the equation (1) we mean a function $x \colon \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

▷ $x \in X$ is differentiable in (t_k, t_{k+1}) $(k \in \mathbb{Z})$, there exist the limits $\lim\limits_{t \to t_k^-} x'(t) = x'(t_k)$ and $\lim\limits_{t \to t_k^+} x'(t)$ $(k \in \mathbb{Z})$;

▷ $x' \in X$;

▷ $x(t) = -x(t+T)$ for all $t \in \mathbb{R}$;

▷ the equations in (1) are satisfied.

Let us list some assumptions:

(A1) there exists a positive integer l such that $t_k + T = t_{k+l}$ and $I_k(x) = -I_{k+l}(-x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$; denote

$$l_0 = \min\{l > 0\colon\ t_k + T = t_{k+l} \text{ and } I_k(x) = -I_{k+l}(-x) \\ \text{for all } k \in \mathbb{Z},\ x \in \mathbb{R}\};$$

(A2) $a|_{(t_k,t_{k+1})} \in C^0(t_k, t_{k+1})$ satisfies $a(t+T) = a(t)$ for all $t \in \mathbb{R}$ and there exist the limits $\lim\limits_{t\to t_k^-} a(t)$ and $\lim\limits_{t\to t_k^-} a(t)$ for all $k \in \mathbb{Z}$;

(A3) $\alpha_k \in C^1(\mathbb{R})$, the inverse function of α_k is denoted by β_k and there exists the numbers

$$\mu_k = \max_{t\in\mathbb{R}} \frac{|\alpha_k(t+T) - \alpha_k(t)|}{T}, \quad k = 1, \ldots, n;$$

(A4) f is an impulsive continuous function satisfying

$$f(t+T, -x_0, -x_1, \ldots, -x_n) = -f(t, x_0, x_1, \ldots, x_n)$$

for all $t \in \mathbb{R}$ and $(x_0, x_1, \ldots, x_n)) \in \mathbb{R}^{n+1}$;

(A5) I_k $(k \in \mathbb{Z})$ are continuous functions.

For $x \in X$, we define the nonlinear operator L by

$$\begin{aligned}(Lx)(t) = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \biggl[\int_t^{t+T} \exp\biggl(\int_t^s a(u)\,\mathrm{d}u\biggr) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \\ &+ \sum_{t\leqslant t_k<t+T} \exp\biggl(\int_t^{t_k} a(u)\,\mathrm{d}u\biggr) I_k(x(t_k))\biggr], \quad t \in \mathbb{R}.\end{aligned}$$

Lemma 2.1. *Suppose that* (A1)–(A5) *hold and* $x \in X$*. Then* $Lx \in X$*.*

P r o o f. It is easy to see for $t \in \mathbb{R}$ that

$$
\begin{aligned}
(Lx)(t+T) = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
&\times \bigg[\int_{t+T}^{t+2T} \exp\bigg(\int_{t+T}^{s} a(u)\,\mathrm{d}u\bigg) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \\
&\qquad + \sum_{t+T \leqslant t_k < t+2T} \exp\bigg(\int_{t+T}^{t_k} a(u)\,\mathrm{d}u\bigg) I_k(x(t_k))\bigg] \\
= &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \bigg[\int_{t}^{t+T} \exp\bigg(\int_{t+T}^{T+v} a(u)\,\mathrm{d}u\bigg) \\
&\times f(T+v, x(T+v), x(\alpha_1(T+v)), \ldots, x(\alpha_n(T+v)))\,\mathrm{d}s \\
&\qquad + \sum_{t \leqslant t_k - T < t+T} \exp\bigg(\int_{t+T}^{t_k} a(u)\,\mathrm{d}u\bigg) I_k(x(t_k))\bigg] \\
= &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \bigg[\int_{t}^{t+T} \exp\bigg(\int_{t+T}^{T+v} a(u)\,\mathrm{d}u\bigg) \\
&\times f(T+v, -x(v), x(T+\alpha_1(v)), \ldots, x(T+\alpha_n(v)))\,\mathrm{d}v \\
&\qquad + \sum_{t \leqslant t_{k-l_0} < t+T} \exp\bigg(\int_{t+T}^{t_{k-l_0}+T} a(u)\,\mathrm{d}u\bigg) I_k(x(t_{k-l_0}+T))\bigg] \\
= &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
&\times \bigg[\int_{t}^{t+T} \exp\bigg(\int_{t}^{v} a(u)\,\mathrm{d}u\bigg) [-f(v, x(v), x(\alpha_1(v)), \ldots, x(\alpha_n(v)))]\,\mathrm{d}v \\
&+ \sum_{t \leqslant t_{k-l_0} < t+T} \exp\bigg(\int_{t}^{t_{k-l_0}} a(u)\,\mathrm{d}u\bigg) I_k(-x(t_{k-l_0}))\bigg] = \frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
&\times \bigg[\int_{t}^{t+T} \exp\bigg(\int_{t}^{v} a(u)\,\mathrm{d}u\bigg) f(v, x(v), x(\alpha_1(v)), \ldots, x(\alpha_n(v)))\,\mathrm{d}v \\
&\qquad - \sum_{t \leqslant t_s < t+T} \exp\bigg(\int_{t}^{t_s} a(u)\,\mathrm{d}u\bigg) I_k(x(t_s))\bigg] = -(Lx)(t).
\end{aligned}
$$

On the other hand, one can easily show that $(Lx)|_{(t_k,t_{k+1})} \in C^0(t_k, t_{k+1})$, $k \in \mathbb{Z}$, and there exist the limits $\lim\limits_{t\to t_k^-}(Lx)(t) = (Lx)(t_k)$ and $\lim\limits_{t\to t_k^+}(Lx)(t)$ for all $k \in \mathbb{Z}$. This completes the proof. □

Lemma 2.2. *Suppose that* (A1)–(A5) *hold. Then $x \in X$ is a anti-periodic solution of the equation* (1.1) *if and only if x is a solution of the operator equation $x = Lx$.*

P r o o f. Suppose that $x \in X$ satisfies $x = Lx$. Then

$$\begin{aligned} x(t) = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \Big[\int_t^{t+T} \exp\Big(\int_t^s a(u)\,\mathrm{d}u\Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ &+ \sum_{t\leqslant t_k<t+T} \exp\Big(\int_t^{t_k} a(u)\,\mathrm{d}u\Big) I_k(x(t_k))\Big], \quad t \in \mathbb{R}. \end{aligned}$$

For $t \neq t_k$, since f and $x \in X$ are continuous at t, we know that x is differentiable at t and

$$\begin{aligned} x'(t) = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \Big[\exp\Big(\int_t^{t+T} a(u)\,\mathrm{d}u\Big) f(t+T, x(t+T), x(\alpha_1(T+t)), \dots, x(\alpha_n(t+T))) \\ &- a(t)\int_t^{t+T} \exp\Big(\int_t^s a(u)\,\mathrm{d}u\Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ &- f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))) - a(t)\sum_{t\leqslant t_k<t+T} \exp\Big(\int_t^{t_k} a(u)\,\mathrm{d}u\Big) I_k(x(t_k))\Big]. \end{aligned}$$

Then

$$\begin{aligned} x'(t) + a(t)x(t) = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \Big[\exp\Big(\int_t^{t+T} a(u)\,\mathrm{d}u\Big) f(t+T, -x(t), -x(\alpha_1(t)), \dots, -x(\alpha_n(t))) \\ &\qquad - f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t)))\Big] \\ = &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \Big[\exp\Big(\int_0^{T} a(u)\,\mathrm{d}u\Big) f(t+T, -x(t), -x(\alpha_1(t)), \dots, -x(\alpha_n(t))) \\ &\qquad - f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t)))\Big] \\ = &\ f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \quad t \in \mathbb{R}. \end{aligned}$$

On the other hand, it is easy to show that $x(t+T) = -x(t)$ for all $t \in \mathbb{R}$ and $\lim\limits_{t\to t_k^-} x(t) = x(t_k)$ and $\Delta x(t_k) = \lim\limits_{t\to t_k^+} x(t) - x(t_k) = I_k(x(t_k))$ for all $k \in \mathbb{Z}$.

Now suppose that x is an anti-periodic solution of the equation (1.1). We get that

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \ t \in \mathbb{R}, \\ \Delta x(t_k) = I_k(x(t_k)), \ k \in \mathbb{Z}. \end{cases}$$

Then

(2.1)
$$\Big(x(t) \exp\Big(\int_0^t a(s)\,\mathrm{d}s\Big)\Big)' = f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))) \exp\Big(\int_0^t a(s)\,\mathrm{d}s\Big).$$

Integrating (2.1) from t to $t+T$, one gets that

$$\begin{aligned} x(t+T) \exp\Big(\int_0^{t+T} a(u)\,\mathrm{d}u\Big) &- x(t) \exp\Big(\int_0^t a(u)\,\mathrm{d}u\Big) \\ = \int_t^{t+T} &\exp\Big(\int_0^s a(u)\,\mathrm{d}u\Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ &+ \sum_{t \leqslant t_k < t+T} \exp\Big(\int_0^{t_k} a(u)\,\mathrm{d}u\Big) I_k(x(t_k)). \end{aligned}$$

Then $x(t+T) = -x(t)$ implies that

$$\begin{aligned} x(t) = &-\frac{1}{1 + \exp(\int_0^T a(u)\,\mathrm{d}u)} \\ &\times \Big[\int_t^{t+T} \exp\Big(\int_t^s a(u)\,\mathrm{d}u\Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ &+ \sum_{t \leqslant t_k < t+T} \exp\Big(\int_t^{t_k} a(u)\,\mathrm{d}u\Big) I_k(x(t_k))\Big] = (Lx)(t). \end{aligned}$$

The proof is complete. □

Lemma 2.3. *Assume that* (A1)–(A5) *hold. Then L is a completely continuous operator.*

Proof. Let n_0 be the number of impulse points on $[0, T)$. It suffices to prove that L is continuous and L is compact. We divide the proof into two steps:

Step 1. Let $x_0 \in X$. Prove that L is continuous x_0.

Suppose $x_n \in X$ and $x_n \to x_0 \in X$. Then

$$\sup_{n=0,1,2,\dots} \sup_{t \in \mathbb{R}} |x_n(t)| = r < \infty.$$

Since $f(t,\cdot\otimes,\ldots,\cdot)$ and I_k are continuous, we get that $f(t,\cdot\otimes,\ldots,\cdot)$ and I_k are uniformly continuous on $[-r,r]^{n+1}$ and $[-r,r]$, respectively.

For any $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(t,u_0,\ldots,u_n)-f(t,v_0,\ldots,v_n)|<\varepsilon,\quad t\in\mathbb{R},\ |u_i-v_i|<\delta,\ i=0,1,\ldots,n$$

and

$$I_k(u)-I_k(v)|<\varepsilon,\quad |u-v|<\delta,\ k\in\mathbb{Z}.$$

Since $x_n\to x_0$ as $n\to\infty$, there exists $N>0$ such that

$$|x_n(t)-x_0(t)|<\delta,\quad t\in\mathbb{R},\ n>N.$$

Use

$$\begin{aligned}(Lx_n)(t)=\ &-\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)}\\ &\times\biggl[\int_t^{t+T}\exp\biggl(\int_t^s a(u)\,\mathrm{d}u\biggr)f(s,x_n(s),x_n(\alpha_1(s)),\ldots,x_n(\alpha_n(s)))\,\mathrm{d}s\\ &+\sum_{t\leqslant t_k<t+T}\exp\biggl(\int_t^{t_k}a(u)\,\mathrm{d}u\biggr)I_k(x_n(t_k))\biggr],\quad n=0,1,\ldots.\end{aligned}$$

It follows for $n>N$ and $t\in\mathbb{R}$ that

$$\begin{aligned}&|(Lx_n)(t)-(Lx_0)(t)|\leqslant\frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)}\\ &\times\biggl[\int_t^{t+T}\exp\biggl(\int_t^s a(u)\,\mathrm{d}u\biggr)|f(s,x_n(s),x_n(\alpha_1(s)),\ldots,x_n(\alpha_n(s)))\\ &-f(s,x_0(s),x_0(\alpha_1(s)),\ldots,x_0(\alpha_n(s)))|\,\mathrm{d}s\\ &+\sum_{t\leqslant t_k<t+T}\exp\biggl(\int_t^{t_k}a(u)\,\mathrm{d}u\biggr)|I_k(x_n(t_k))-I_k(x_0(t_k))|\biggr]\\ &<\frac{\varepsilon}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)}\biggl[\int_t^{t+T}\exp\biggl(\int_t^s a(u)\,\mathrm{d}u\biggr)\mathrm{d}s+\sum_{t\leqslant t_k<t+T}\exp\biggl(\int_t^{t_k}a(u)\,\mathrm{d}u\biggr)\biggr]\\ &\leqslant\frac{\varepsilon}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)}\biggl[\int_t^{t+T}\exp\biggl(\int_t^{t+T}a(u)\,\mathrm{d}u\biggr)\mathrm{d}s+\sum_{t\leqslant t_k<t+T}\exp\biggl(\int_t^{t+T}a(u)\,\mathrm{d}u\biggr)\biggr]\\ &=\frac{\varepsilon}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)}[T+n_0]\exp\biggl(\int_0^T a(u)\,\mathrm{d}u\biggr).\end{aligned}$$

So $Lx_n\to Lx_0$ as $n\to\infty$. Thus the continuity of L follows.

Step 2. Prove that L is compact.

Let $\Omega \subseteq X$ be a bounded set. Suppose that $\Omega \subseteq \{x \in X\colon\ \|x\| \leqslant M\}$. For $x \in \Omega$, we have

$$\begin{aligned}
|(Lx)(t)| = {} & \frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
& \times \bigg| \int_t^{t+T} \exp\bigg(\int_t^s a(u)\,\mathrm{d}u\bigg) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \\
& + \sum_{t\leqslant t_k<t+T} \exp\bigg(\int_t^{t_k} a(u)\,\mathrm{d}u\bigg) I_k(x(t_k))\bigg| \\
\leqslant {} & \frac{\max\limits_{t\in\mathbb{R},\,|x_i|\leqslant M,\,i=0,1,\ldots,n} |f(t,x_0,x_1,\ldots,x_n)|}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \bigg| \int_t^{t+T} \exp\bigg(\int_t^s a(u)\,\mathrm{d}u\bigg)\,\mathrm{d}s \\
& + \max_{|x|\leqslant M} |I_k(x)| \sum_{t\leqslant t_k<t+T} \exp\bigg(\int_t^{t_k} a(u)\,\mathrm{d}u\bigg)\bigg| \\
\leqslant {} & \frac{\max\limits_{t\in\mathbb{R},\,|x_i|\leqslant M,\,i=0,1,\ldots,n} |f(s,x_0,x_1,\ldots,x_n)|}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} T \exp\bigg(\int_0^T |a(u)|\,\mathrm{d}u\bigg)\,\mathrm{d}s \\
& + \frac{\max\limits_{|x|\leqslant M,\,k\in[0,n_0]} |I_k(x)|}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \sum_{0\leqslant t_k<T} \exp\bigg(\int_0^T |a(u)|\,\mathrm{d}u\bigg).
\end{aligned}$$

Hence L maps bounded sets into bounded sets.

Note that Lx is periodic with period $2T$. For $t_1, t_2 \in [0, 2T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
|(Lx)(t_1) - (Lx)(t_2)| \leqslant {} & \frac{1}{1+\exp\big(\int_0^T a(u)\,\mathrm{d}u\big)} \\
& \times \bigg[\bigg| \int_{t_1}^{t_1+T} \exp\bigg(\int_{t_1}^s a(u)\,\mathrm{d}u\bigg) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \\
& - \int_{t_2}^{t_2+T} \exp\bigg(\int_{t_2}^s a(u)\,\mathrm{d}u\bigg) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s\bigg| \\
& + \bigg| \sum_{t_1\leqslant t_k<t_1+T} \exp\bigg(\int_{t_1}^{t_k} a(u)\,\mathrm{d}u\bigg) I_k(x(t_k)) \\
& - \sum_{t_2\leqslant t_k<t_2+T} \exp\bigg(\int_{t_2}^{t_k} a(u)\,\mathrm{d}u\bigg) I_k(x(t_k))\bigg|\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\max\limits_{t\in\mathbb{R},|x_i|\leqslant M,i=0,1,\ldots,n} |f(t,x_0,x_1,\ldots,x_n)|}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
&\quad\times\biggl[\int_{t_1}^{t_1+T}\biggl|\exp\biggl(\int_{t_1}^{s} a(u)\,\mathrm{d}u\biggr)-\exp\biggl(\int_{t_2}^{s} a(u)\,\mathrm{d}u\biggr)\biggr|\,\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}\exp\biggl(\int_{t_2}^{s} a(u)\,\mathrm{d}u\biggr)\mathrm{d}s\biggr]+\frac{\max\limits_{|x|\leqslant M,\,k\in[0,n_0]}|I_k(x)|}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\
&\quad\times\biggl|\sum_{t_1\leqslant t_k<t_1+T}\exp\biggl(\int_{t_1}^{t_k} a(u)\,\mathrm{d}u\biggr)-\sum_{t_2\leqslant t_k<t_2+T}\exp\biggl(\int_{t_2}^{t_k} a(u)\,\mathrm{d}u\biggr)\biggr| \\
&\to 0 \quad \text{as } t_1\to t_2.
\end{aligned}
$$

This shows that $(Lx)(t)$ is equi-continuous on $\mathbb{R}$. The Arzelà-Ascoli theorem guarantees that $L(\Omega)$ is relatively compact, which means that L is compact. Hence the continuity and the compactness of L imply that L is completely continuous. $\square$

The following abstract existence theorem will be used in the proof of the main results of this paper. Its proof can be found in [22].

Lemma 2.4. *Let X be a Banach space. Suppose $L\colon X\to X$ is a completely continuous operator. If there exists an open bounded subset Ω such that $0\in\Omega\subset X$ and $x\neq\lambda Lx$ for all $x\in D(L)\cap\partial\Omega$ and $\lambda\in[0,1]$, then there is at least one $x\in\Omega$ such that $x=Lx$.*

Now, we establish existence results for at least one solution of the equation (1.1).

Theorem 2.1. *Suppose that* (A1)–(A5) *hold and*

(B1) *$I_k(x)(2x+I_k(x))\leqslant 0$ and $x(x+I_k(x))\geqslant 0$ for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$;*

(B2) *there exist impulsive continuous functions $h\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $g\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, and $\alpha,\beta\colon\mathbb{R}\to[0,\infty)$ such that*

  (i) *$f(t,x_0,\ldots,x_n)=h(t,x_0,\ldots,x_n)+g(t,x_0,\ldots,x_n)$ holds for all $(t,x_0,\ldots,x_n)\in\mathbb{R}\times\mathbb{R}^{n+1}$;*

  (ii) *$h(t,x_0,\ldots,x_n)x_0\leqslant 0$ holds for all $(t,x_0,\ldots,x_n)\in[t_0,t_0+T]\times\mathbb{R}^{n+1}$;*

  (iii) *there exists $t_0\in\mathbb{R}$ such that $|g(t,x_0,\ldots,x_n)|\leqslant\alpha(t)|x_0|+\beta(t)$ holds for all $(t,x_0,\ldots,x_n)\in[t_0,t_0+T]\times\mathbb{R}^{n+1}$.*

Then the equation (1.1) *has at least one anti-periodic solution.*

P r o o f. Let $\lambda \in [0,1]$. Consider the operator equation $x = \lambda Lx$. If $x \in X$ is a solution of $x = \lambda Lx$, we get that

$$\begin{aligned} x(t) = & -\lambda \frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ & \times \Big[\int_t^{t+T} \exp\Big(\int_t^s a(u)\,\mathrm{d}u \Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ & + \sum_{t \leqslant t_k < t+T} \exp\Big(\int_t^{t_k} a(u)\,\mathrm{d}u \Big) I_k(x(t_k)) \Big] = (Lx)(t). \end{aligned}$$

Then

$$\begin{cases} x'(t) + a(t)x(t) = \lambda f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \ t \in \mathbb{R}, \\ \Delta x(t_k) = \lambda I_k(x(t_k)), \ k \in \mathbb{Z}. \end{cases}$$

It follows that

$$\Big(x(t) \exp\Big(\int_{t_0}^t a(s)\,\mathrm{d}s \Big) \Big)' = \lambda f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))) \exp\Big(\int_{t_0}^t a(s)\,\mathrm{d}s \Big).$$

Since (B1) implies that $x(x + I_k(x)) \geqslant 0$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, we get that $x(t_k^+)x(t_k) \geqslant 0$ for all $k \in \mathbb{Z}$. Then $x(t_0) = -x(t_0 + T)$ implies that there exists $\xi \in [t_0, t_0 + T]$ such that $x(\xi) = 0$. Hence for $t \in [\xi, t_0 + T]$, integrating above the equation from ξ to t, one sees that

$$\begin{aligned} \frac{1}{2}\Big(x(t) \exp\Big(\int_{t_0}^t a(s)\,\mathrm{d}s \Big) \Big)^2 & - \frac{1}{2} \sum_{\xi < t_k \leqslant t} [(x(t_k^+))^2 - (x(t_k))^2] \exp\Big(2 \int_{t_0}^{t_k} a(s)\,\mathrm{d}s \Big) \\ & = \lambda \int_\xi^t f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s))) x(s) \exp\Big(\int_{t_0}^s a(u)\,\mathrm{d}u \Big)\,\mathrm{d}s. \end{aligned}$$

One sees from (B1) that

$$\begin{aligned} (x(t_k^+))^2 - (x(t_k))^2 &= (x(t_k^+) - x(t_k))(x(t_k^+) + x(t_k)) \\ &= \Delta x(t_k)(2x(t_k) + \Delta x(t_k)) \\ &= \lambda I_k(x(t_k))(2x(t_k) + \lambda I_k(x(t_k))) \\ &\leqslant \lambda I_k(x(t_k))(2x(t_k) + I_k(x(t_k))) \leqslant 0. \end{aligned}$$

Then (B2) implies that

$$
\begin{aligned}
&\frac{1}{2}[x(t)]^2 \exp\left(2\int_{t_0}^t a(s)\,\mathrm{d}s\right)\\
&\qquad\leqslant \lambda\int_\xi^t f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\qquad= \lambda\int_\xi^t g(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\qquad\quad+ \lambda\int_\xi^t h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\qquad\leqslant \lambda\int_\xi^t h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}
$$

We get from (B2) that

$$
\begin{aligned}
\frac{1}{2}[x(t)]^2 &\leqslant \lambda\int_\xi^t h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\leqslant \int_\xi^t |h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))|\,|x(s)|\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\leqslant \int_\xi^t (\alpha(s)|x(s)|+\beta(s))|x(s)|\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}
$$

Let

$$
v(t) = \int_\xi^t (\alpha(s)|x(s)|+\beta(s))|x(s)|\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)\mathrm{d}s.
$$

Then $v(\xi)=0$, $\frac{1}{2}[x(t)]^2\leqslant v(t)$ and

$$
v'(t) = \alpha(t)|x(t)|^2+\beta(t)|x(t)|-a(t)v(t).
$$

Hence

$$
v'(t)\leqslant 2\alpha(t)v(t)+\beta(t)\sqrt{2v(t)}-a(t)v(t) = (2\alpha(t)-a(t))v(t)+\beta(t)\sqrt{2v(t)}.
$$

It follows that

$$
\left(\sqrt{v(t)}\exp\Bigl(-\int_\xi^t\Bigl(\alpha(s)-\frac{a(s)}{2}\Bigr)\,\mathrm{d}s\Bigr)\right)' = \frac{\beta(t)}{2}\exp\Bigl(-\int_\xi^t\Bigl(2\alpha(s)-\frac{a(s)}{2}\Bigr)\,\mathrm{d}s\Bigr).
$$

Integrating from ξ to t, we get that

$$
\sqrt{v(t)}\exp\Bigl(-\int_\xi^t\Bigl(\alpha(s)-\frac{a(s)}{2}\Bigr)\,\mathrm{d}s\Bigr)\leqslant\int_\xi^t\frac{\beta(s)}{2}\exp\Bigl(-\int_\xi^s\Bigl(2\alpha(u)-\frac{a(u)}{2}\Bigr)\,\mathrm{d}u\Bigr)\,\mathrm{d}s.
$$

Thus for $t \in [\xi, t_0 + T]$, we have that

$$v(t) \leqslant \exp\left(2\int_\xi^t \left(\alpha(s) - \frac{a(s)}{2}\right) \mathrm{d}s\right)\left(\int_\xi^t \frac{\beta(s)}{2} \exp\left(-\int_\xi^s \left(2\alpha(u) - \frac{a(u)}{2}\right) \mathrm{d}u\right) \mathrm{d}s\right)^2.$$

Hence there exists a constant $M > 0$ such that $v(t) \leqslant M$ for all $t \in [\xi, t_0 + T]$. Hence $|x(t)| \leqslant \sqrt{2M}$ for all $t \in [\xi, t_0 + T]$. Then we get that $|x(t_0)| = |x(t_0 + T)| \leqslant \sqrt{2M}$. Now, we consider $t \in [t_0, \xi]$. Integrating the equation

$$\left(x(t) \exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right)\right)' = \lambda f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))) \exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right)$$

from t_0 to t, one sees that

$$\begin{aligned}
&\frac{1}{2}[x(t)]^2 \exp\left(2\int_{t_0}^t a(s)\,\mathrm{d}s\right)\\
&= \frac{1}{2}[x(t_0)]^2 + \frac{1}{2}\sum_{\xi < t_k \leqslant t}\left[\left(x(t_k^+)\exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\right)\right)^2 - \left(x(t_k^-)\exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\right)\right)^2\right]\\
&\quad + \lambda\int_{t_0}^t f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&\leqslant M + \frac{1}{2}\sum_{t_0 < t_k \leqslant t}\left[\left(x(t_k^+)\exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\right)\right)^2 - \left(x(t_k^-)\exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\right)\right)^2\right]\\
&\quad + \lambda\int_{t_0}^t f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}$$

Similarly to the discussion above, we get that

$$\frac{1}{2}[x(t)]^2 \leqslant M\exp\left(\int_t^{t_0} a(u)\,\mathrm{d}u\right) + \int_{t_0}^t (\alpha(s)|x(s)| + \beta(s))|x(s)|\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)\mathrm{d}s.$$

Thus there exist constants $A, B, C > 0$ such that

$$[x(t)]^2 \leqslant A + \int_{t_0}^t (B|x(s)|^2 + C|x(s)|)\,\mathrm{d}s.$$

Let $w(t) = \int_{t_0}^t (B|x(s)|^2 + C|x(s)|)\,\mathrm{d}s$. Then

$$w'(t) = B[x(t)]^2 + C|x(t)| \leqslant B(A + w(t)) + C\sqrt{A + w(t)}.$$

It follows that

$$2(\sqrt{A + w(t)}\mathrm{e}^{-Bt})' \leqslant C\mathrm{e}^{-Bt}.$$

Integrating the last equation from t_0 to t, we get that

$$2\sqrt{A+w(t)}\mathrm{e}^{-Bt} \leqslant 2\sqrt{A}\mathrm{e}^{-Bt_0} + \int_{t_0}^{t} C\mathrm{e}^{-Bs}\,\mathrm{d}s.$$

Hence there exists a constant $M' > 0$ such that $w(t) \leqslant M'$ for all $t \in [t_0, \xi]$. Then $[x(t)]^2 \leqslant A + M'$ for all $t \in [t_0, \xi]$.

It follows from the above discussion that $|x(t)| \leqslant \max\{\sqrt{A+M'}, \sqrt{2M}\}$ for all $t \in [t_0, t_0+T]$.

Since x is anti-periodic, we get that $|x(t)| \leqslant \max\{\sqrt{A+M'}, \sqrt{2M}\}$ for all $t \in \mathbb{R}$. Thus $\|x\| \leqslant \max\{\sqrt{A+M'}, \sqrt{2M}\}$ for all $x \in \Omega = \{x \in X\colon\ x = \lambda Lx$ for some $\lambda \in [0,1]\}$.

Choose $M_1 = \max\{\sqrt{A+M'}, \sqrt{2M}\}$. Let $\Omega_0 = \{x \in X\colon\ \|x\| < M_1 + 1\}$. Then $x \neq \lambda Lx$ for all $\lambda \in [0,1]$ and all $x \in \partial\Omega_0$. Lemmas 2.1 and 2.3 imply that $L\colon\ X \to X$ is completely continuous. It follows from Lemma 2.4 that there is $x \in X$ such that $x = Lx$. Then Lemma 2.2 implies that the equation (1.1) has at least one anti-periodic solution $x \in X$. The proof is complete. □

Theorem 2.2. *Suppose that* $\int_0^T a(u)\,\mathrm{d}u \leqslant 0$ *and* (A1)–(A5) *hold and*

(H1) $xI_k(x) \geqslant 0$ *for all* $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$;

(H2) *there exist impulsive continuous functions* $h\colon\ \mathbb{R} \times \mathbb{R}^n \to R$, $g_i\colon\ \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $r \in X$ *such that*

(i) $f(t, x_0, \ldots, x_n) = h(t, x_0, \ldots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ *holds for all* $(t, x_0, \ldots, x_n) \in \mathbb{R} \times \mathbb{R}^{n+1}$;

(ii) *there exists* $t_0 \in \mathbb{R}$ *and constants* $m \geqslant 0$ *and* $\beta > 0$ *such that*

$$h(t, x_0, \ldots, x_n)x_0 \exp\biggl(\int_{t_0}^{t} a(u)\,\mathrm{d}u\biggr) \geqslant \beta|x_0|^{m+1}$$

holds for all $(t, x_0, \ldots, x_n) \in [t_0, t_0+T] \times \mathbb{R}^{n+1}$;

(iii) *there exist the limits*

$$\lim_{|x|\to\infty} \sup_{t\in[t_0,t_0+T]} \frac{|g_i(t,x)| \exp(\int_{t_0}^{t} a(u)\,\mathrm{d}u)}{|x|^m} = r_i \in [0,\infty), \quad i = 0, \ldots, n.$$

Then the equation (1.1) *has at least one anti-periodic solution if*

$$r_0 + \sum_{k=1}^{n} r_k \|\beta_k'\|^{m/(m+1)}([\mu_k]+1)^{m/(m+1)} < \beta, \tag{2.2}$$

where $[\mu_k]$ *denotes the maximum integer not greater than* μ_k.

P r o o f. Let $\lambda \in [0,1]$. Consider the operator equation $x = \lambda Lx$. If $x \in X$ is a solution of $x = \lambda Lx$, we get that

$$\begin{aligned} x(t) = & -\lambda \frac{1}{1+\exp(\int_0^T a(u)\,\mathrm{d}u)} \\ & \times \Big[\int_t^{t+T} \exp\Big(\int_t^s a(u)\,\mathrm{d}u\Big) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s \\ & + \sum_{t \leqslant t_k < t+T} \exp\Big(\int_t^{t_k} a(u)\,\mathrm{d}u\Big) I_k(x(t_k))\Big] = (Lx)(t). \end{aligned}$$

Then

$$\text{(2.3)} \qquad \begin{cases} x'(t) + a(t)x(t) = \lambda f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \quad t \in \mathbb{R}, \\ \Delta x(t_k) = \lambda I_k(x(t_k)),\ k \in \mathbb{Z}. \end{cases}$$

To complete the proof of the theorem, we do the following three steps.

Step 1. Prove that there is a constant $M > 0$ so that $\int_{t_0}^{t_0+T} |x(s)|^{m+1}\,\mathrm{d}s \leqslant M$.

Transform the first equation in (2.3) into

$$\Big(x(t)\exp\Big(\int_{t_0}^t a(s)\,\mathrm{d}s\Big)\Big)' = \lambda f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t)))\exp\Big(\int_{t_0}^t a(s)\,\mathrm{d}s\Big).$$

Multiplying both sides by $x(t)\exp(\int_{t_0}^t a(s)\,\mathrm{d}s)$ and integrating from t_0 to $t_0 + T$, we get using (H2) that

$$\begin{aligned} & \frac{1}{2}\Big(x(t_0+T)\exp\Big(\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s\Big)\Big)^2 - \frac{1}{2}(x(t_0))^2 \\ & \quad - \frac{1}{2}\sum_{t_0 < t_k \leqslant t_0+T}\Big[\Big(x(t_k^+)\exp\Big(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2 - \Big(x(t_k)\Big(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2\Big] \\ & = \lambda\Big[\int_{t_0}^{t_0+T} h(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))x(s)\exp\Big(\int_{t_0}^s a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ & \quad + \int_{t_0}^{t_0+T} g_0(s, x(s))x(s)\exp\Big(\int_{t_0}^s a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ & \quad + \sum_{i=1}^n \int_{t_0}^{t_0+T} g_i(s, x(\alpha_i(s))x(s)\exp\Big(\int_{t_0}^s a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ & \quad + \int_{t_0}^{t_0+T} r(s)x(s)\exp\Big(\int_{t_0}^s a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s\Big]. \end{aligned}$$

It follows from (H1) that

$$\begin{aligned}(x(t_k^+))^2-(x(t_k))^2&=(x(t_k^+)-x(t_k))(x(t_k^+)+x(t_k))\\&=\Delta x(t_k)(2x(t_k)+\Delta x(t_k))=\lambda I_k(x(t_k))(2x(t_k)+\lambda I_k(x(t_k)))\\&\geqslant 2\lambda x(t_k)I_k(x(t_k))\geqslant 0.\end{aligned}$$

Together with $\int_0^T a(u)\,\mathrm{d}u\leqslant 0$ and $x(t_0+T)=-x(t_0)$, we get

$$\frac{1}{2}\biggl(x(t_0+T)\exp\biggl(\int_{t_0}^{t_0+T}a(s)\,\mathrm{d}s\biggr)\biggr)^2-\frac{1}{2}(x(t_0))^2\leqslant 0$$

and

$$\biggl(x(t_k^+)\exp\biggl(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\biggr)\biggr)^2-\biggl(x(t_k)\exp\biggl(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\biggr)\biggr)^2\geqslant 0.$$

Then

$$\begin{aligned}&\int_{t_0}^{t_0+T}h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_0+T}g_0(s,x(s))x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\quad+\sum_{i=1}^{n}\int_{t_0}^{t_0+T}g_i(s,x(\alpha_i(s))x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_0+T}r(s)x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\leqslant 0.\end{aligned}$$

It follows from (H2) that

$$\begin{aligned}\beta\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s&\leqslant-\int_{t_0}^{t_0+T}g_0(s,x(s))x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\quad-\sum_{i=1}^{n}\int_{t_0}^{t_0+T}g_i(s,x(\alpha_i(s))x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\quad-\int_{t_0}^{t_0+T}r(s)x(s)\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\&\leqslant\int_{t_0}^{t_0+T}|g_0(s,x(s))|\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)|x(s)|\,\mathrm{d}s\\&\quad+\sum_{i=1}^{n}\int_{t_0}^{t_0+T}|g_i(s,x(\alpha_i(s))|\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)|x(s)|\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_0+T}|r(s)|\exp\biggl(\int_{t_0}^{s}a(u)\,\mathrm{d}u\biggr)|x(s)|\,\mathrm{d}s.\end{aligned}$$

From (2.2), choose $\varepsilon > 0$ so that

$$(r_0+\varepsilon)+\sum_{k=1}^{n}(r_k+\varepsilon)\|\beta_k'\|^{m/(m+1)}([\mu_k]+1)^{m/(m+1)} < \beta. \tag{2.4}$$

For such $\varepsilon > 0$, together with (H2), there is $\delta > 0$ such that

$$\exp\left(\int_{t_0}^{t} a(u)\,\mathrm{d}u\right)|g_i(t,x)| < (r_i+\varepsilon)|x|^m \tag{2.5}$$

uniformly for $t \in [0,T]$ and $|x| > \delta,\ i = 0,1,\ldots,n$.

Let, for $i = 1,\ldots,n$,

$$\begin{aligned}
\Delta_{1,i} &= \{t\colon\ t\in[0,T],\ |x(\alpha_i(t))| \leqslant \delta\},\ i=1,\ldots,n,\\
\Delta_{2,i} &= \{t\colon\ t\in[0,T],\ |x(\alpha_i(t))| > \delta\},\ i=1,\ldots,n,\\
g_{\delta,i} &= \max_{t\in[0,T],\,|x|\leqslant\delta}|g_i(t,x)|,\ i=0,1,\ldots,n,\\
\Delta_1 &= \{t\in[0,T],\ |x(t)|\leqslant\delta\},\\
\Delta_2 &= \{t\in[0,T],\ |x(t)|>\delta\},\\
\delta &= \max\{g_{\delta,i}\colon\ i=0,\ldots,n\}.
\end{aligned}$$

Then we get

$$\begin{aligned}
\beta\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s &\leqslant (r_0+\varepsilon)\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\\
&\quad+\sum_{k=1}^{n}(r_k+\varepsilon)\int_{t_0}^{t_0+T}|x(\alpha_i(s))|^m|x(s)|\,\mathrm{d}s+\|r\|\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)\\
&\quad\times\int_{t_0}^{t_0+T}|x(s)|\,\mathrm{d}s+\delta\int_{t_0}^{t_0+T}|x(s)|\,\mathrm{d}s+\delta\sum_{k=1}^{n}\int_{t_0}^{t_0+T}|x(s)|\,\mathrm{d}s\\
&\leqslant (r_0+\varepsilon)\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\\
&\quad+\sum_{k=1}^{n}(r_k+\varepsilon)\left[\int_{t_0}^{t_0+T}|x(\alpha_i(s))|^{m+1}\,\mathrm{d}s\right]^{m/(m+1)}\left[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\right]^{1/(m+1)}\\
&\quad+\left[(n+1)\delta+\|r\|\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)\right]\int_{t_0}^{t_0+T}|x(s)|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (r_0+\varepsilon)\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\\
&\quad+\sum_{k=1}^{n}(r_k+\varepsilon)\biggl[\int_{\alpha_k(t_0)}^{\alpha_k(t_0+T)}|x(u)|^{m+1}|\beta_k'(u)|\,\mathrm{d}u\biggr]^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}\\
&\quad+\biggl[(n+1)\delta+\|r\|\exp\biggl(\int_0^T a^+(u)\,\mathrm{d}u\biggr)\biggr]T^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}\\
&\leqslant (r_0+\varepsilon)\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s+\sum_{k=1}^{n}(r_k+\varepsilon)\|\beta_k'\|^{m/(m+1)}\\
&\quad\times\biggl[\int_{\alpha_k(t_0)}^{\alpha_k(t_0+T)}|x(u)|^{1+m}\,\mathrm{d}u\biggr]^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}\\
&\quad+\biggl[(n+1)\delta+\|r\|\exp\biggl(\int_0^T a^+(u)\,\mathrm{d}u\biggr)\biggr]T^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}.
\end{aligned}
$$

Since (A3) implies that

$$\mu_k=\max_{t\in R}\frac{|\alpha_k(t+T)-\alpha_k(t)|}{T},\quad k=1,\dots,n,$$

we have

$$[\mu_k]T\leqslant\alpha_k(t_0+T)-\alpha_k(t_0)\leqslant([\mu_k]+1)T,$$

where $[y]$ denotes the maximum integer not greater than y. The fact that $|x(t)|$ is T-periodic implies that

$$
\begin{aligned}
&\beta\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\leqslant(r_0+\varepsilon)\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\\
&\quad+\sum_{k=1}^{n}(r_k+\varepsilon)\|\beta_k'\|^{m/(m+1)}([\mu_k]+1)^{m/(m+1)}\\
&\quad\times\biggl[\int_{t_0}^{t_0+T}|x(u)|^{1+m}\,\mathrm{d}u\biggr]^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}\\
&\quad+\biggl[(n+1)\delta+\|r\|\exp\biggl(\int_0^T a^+(u)\,\mathrm{d}u\biggr)\biggr]T^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}\\
&=\biggl[(r_0+\varepsilon)+\sum_{k=1}^{n}(r_k+\varepsilon)\|\beta_k'\|^{m/(m+1)}([\mu_k]+1)^{m/(m+1)}\biggr]\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\\
&\quad+\biggl[(n+1)\delta+\|r\|\exp\biggl(\int_0^T a^+(u)\,\mathrm{d}u\biggr)\biggr]T^{m/(m+1)}\biggl[\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\biggr]^{1/(m+1)}.
\end{aligned}
$$

It follows from (2.4) that there is a constant $M>0$ so that $\int_{t_0}^{t_0+T}|x(s)|^{m+1}\,\mathrm{d}s\leqslant M$.

Step 2. Prove that there is a constant $M_1 > 0$ so that $\|x\|_\infty \leqslant M_1$.

It follows from Step 1 that there is $\xi \in [t_0, t_0 + T]$ so that $|x(\xi)| \leqslant (M/T)^{1/(m+1)}$.

Case 1. If $t_0 \leqslant t < \xi$, multiplying both sides of the equation (4) by $x(t) \times \exp(\int_0^t a(s)\,\mathrm{d}s)$ and integrating it from t to ξ, we get, using (B1) and (B2), that

$$\begin{aligned}
\frac{1}{2}\Big[x(t)\exp\Big(\int_{t_0}^{t} a(s)\,\mathrm{d}s\Big)\Big]^2 &= \frac{1}{2}\Big[x(\xi)\exp\Big(\int_{t_0}^{\xi} a(s)\,\mathrm{d}s\Big)\Big]^2 \\
&- \frac{1}{2}\sum_{t<t_k\leqslant\xi}\Big[\Big(x(t_k^+)\exp\Big(\int_{t_0}^{t_k} a(u)\,\mathrm{d}u\Big)\Big)^2 - \Big(x(t_k^-)\exp\Big(\int_{t_0}^{t_k} a(u)\,\mathrm{d}u\Big)\Big)^2\Big] \\
&- \lambda\int_t^{\xi} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\exp\Big(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2}|x(t)|^2 &= \frac{1}{2}\Big[x(\xi)\exp\Big(\int_{t}^{\xi} a(s)\,\mathrm{d}s\Big)\Big]^2 \\
&- \frac{1}{2}\sum_{t<t_k\leqslant\xi}\Big[\Big(x(t_k^+)\exp\Big(\int_{t}^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2 - \Big(x(t_k^-)\exp\Big(\int_{t}^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2\Big] \\
&- \lambda\int_t^{\xi} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\exp\Big(\int_{t}^{s} a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\
\leqslant \frac{1}{2}\Big(\frac{M}{T}\Big)^{2/(m+1)} &\exp\Big(\int_0^T |a(s)|\,\mathrm{d}s\Big) \\
&+ \Big[\Big((r_0+\varepsilon) + \sum_{k=1}^{n}(r_k+\varepsilon)\|\beta_k\|^{m/(1+m)}([\mu_k]+1)^{m/(m+1)}\Big)M \\
&+ \Big(\int_{t_0}^{t_0+T}|r(s)|\exp\Big(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s\Big)^{m/(m+1)} M^{1/(m+1)}\Big]\exp\Big(2\int_0^T |a(u)|\,\mathrm{d}u\Big) \\
&+ (n+1)T^{m/(m+1)}M^{1/(m+1)}\exp\Big(2\int_0^T |a(u)|\,\mathrm{d}u\Big) =: M_2.
\end{aligned}$$

Hence one sees that

$$x^2(t) \leqslant 2M_2 =: M_3 \quad \text{for } t \in [t_0, \xi].$$

This implies $x^2(0) \leqslant M_3$. So $x^2(T) = x^2(0) \leqslant M_3$.

Case 2. For $t \in [\xi, t_0 + T]$, we have

$$\frac{1}{2}\Big[x(t)\exp\Big(\int_{t_0}^{t} a(s)\,\mathrm{d}s\Big)\Big]^2 = \frac{1}{2}\Big[x(t_0+T)\exp\Big(\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s\Big)\Big]^2$$
$$-\frac{1}{2}\sum_{t<t_k<t_0+T}\Big[\Big(x(t_k^+)\exp\Big(\int_T^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2 - \Big(x(t_k^-)\exp\Big(\int_T^{t_k} a(s)\,\mathrm{d}s\Big)\Big)^2\Big]$$
$$-\lambda\int_t^{t_0+T} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\exp\Big(\int_T^s a(u)\,\mathrm{d}u\Big)\,\mathrm{d}s.$$

Similarly to the above discussion, we get that there is $M_4 > 0$ so that $x^2(t) \leqslant M_4$ for $t \in [\xi, t_0 + T]$. All the above discussion implies that there is $M_1 > 0$ so that $|x(t)| \leqslant M_1$ for all $t \in [t_0, t_0 + T]$.

Since x is anti-periodic, we get that $|x(t)| \leqslant M_1$ for all $t \in \mathbb{R}$. Thus $\|x\| \leqslant M_1$ for all $x \in \Omega = \{x \in X\colon\ x = \lambda Lx$ for some $\lambda \in [0,1]\}$.

Step 3. Apply Lemma 2.4 to get a solution of the equation (1.1).

Let $\Omega_0 = \{x \in X\colon\ \|x\| < M_1 + 1\}$. Then $x \neq \lambda Lx$ for all $\lambda \in [0,1]$ and all $x \in \partial\Omega_0$. Lemmas 2.1 and 2.3 imply that $L\colon\ X \to X$ is completely continuous. It follows from Lemma 2.4 that there is $x \in X$ such that $x = Lx$. Then Lemma 2.2 implies that the equation (1.1) has at least one anti-periodic solution $x \in X$. The proof is complete. □

Theorem 2.3. *Suppose that* $\int_0^T a(u)\,\mathrm{d}u \geqslant 0$ *and* (A1)–(A5) *hold and*

(H3) $I_k(x)(2x + I_k(x)) \leqslant 0$ *for all* $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$;

(H4) *there exist impulsive continuous functions* $h\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $g_i\colon\ \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $r \in X$ *such that* (H2)(i) *and* (H2)(iii) *hold and*

(ii) *there exist* $t_0 \in \mathbb{R}$ *and constants* $m \geqslant 0$ *and* $\beta > 0$ *such that*

$$h(t, x_0, \ldots, x_n)x_0\exp\Big(\int_{t_0}^t a(u)\,\mathrm{d}u\Big) \leqslant -\beta|x_0|^{m+1}$$

holds for all $(t, x_0, \ldots, x_n) \in [t_0, t_0 + T] \times \mathbb{R}^{n+1}$.

Then the equation (1.1) *has at least one anti-periodic solution if*

$$r_0 + \sum_{k=1}^n r_k\|\beta_k\|^{m/(m+1)}([\mu_k] + 1)^{m/(m+1)} < \beta, \tag{2.6}$$

where $[\mu_k]$ *denotes the maximum integer not greater than* μ_k.

P r o o f. The proof is similar to that of Theorem 2.2. We get (2.3). (A2) implies that $\int_t^{t+T} a(s)\,\mathrm{d}s = \int_0^T a(u)\,\mathrm{d}u \geqslant 0$.

Multiplying both sides of the equation (2.3) by $x(t)\exp(\int_{t_0}^{t} a(s)\,\mathrm{d}s)$ and integrating it from t_0 to t_0+T, we get using (H4) that

$$\frac{1}{2}\Bigl[x(t_0+T)\exp\Bigl(\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s\Bigr)\Bigr]^2-\frac{1}{2}\Bigl[x(t_0)\exp\Bigl(\int_{t_0}^{t_0} a(s)\,\mathrm{d}s\Bigr)\Bigr]^2$$
$$-\frac{1}{2}\sum_{t_0<t_k\leqslant t_0+T}\Bigl[\Bigl(x(t_k^+)\exp\Bigl(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\Bigr)\Bigr)^2-\Bigl(x(t_k^-)\exp\Bigl(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s\Bigr)\Bigr)^2\Bigr]$$
$$=\lambda\int_{t_0}^{t_0+T} f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)\exp\Bigl(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s.$$

From the assumption (H3), we get that

$$\begin{aligned}(x(t_k^+))^2-(x(t_k^-))^2&=(x(t_k^+)-x(t_k^-))(x(t_k^+)+x(t_k^-))\\&=\Delta x(t_k^-)(2x(t_k^-)+\Delta x(t_k^-))=\lambda I_k(x(t_k^-))(2x(t_k^-)+\lambda I_k(x(t_k^-)))\\&\leqslant 2\lambda I_k(x(t_k^-))x(t_k^-)+\lambda[I_k(x(t_k^-))]^2=\lambda I_k(x(t_k^-))(2x(t_k^-)+I_k(x(t_k^-)))\leqslant 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{t_0}^{t_0+T} h(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)\exp\Bigl(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_0+T} g_0(s,x(s))x(s)\exp\Bigl(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s\\&\quad+\sum_{i=1}^{n}\int_{t_0}^{t_0+T} g_i(s,x(\alpha_i(s))x(s)\exp\Bigl(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_0+T} r(s)x(s)\exp\Bigl(\int_{t_0}^{s} a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s\geqslant 0.\end{aligned}$$

The remainder of the proof is similar to that of the proof of Theorem 2.1 and is omitted. □

R e m a r k 2.1. One can easily see that the assumptions imposed on α_k, I_k ($k=1,2,\ldots,n$) and f are weaker than those in [18], see (B1), (B2), (H1)–(H4) in this paper and (G1)–(G6) in [18]. So the results in this paper are new.

3. An example

Now, we present an example, whose solutions can not be obtained by theorems in other known papers, to illustrate the main results.

Example 3.1. Consider the following equation

$$(3.1)\qquad \begin{cases} x'(t) + (1+\sin 2t)x(t) \\ \qquad = a[x(t)]^{2q+1} + \sum\limits_{k=1}^{n} b_i[x(t-i^{-2})]^{2q+1} + \sin t,\ t\in\mathbb{R}, \\ \Delta x(t_k) = c_k[x(t_k)]^3,\ k\in\mathbb{Z}, \end{cases}$$

where $q \geqslant 0$ is an integer, $t_k = k\pi + \pi/2,\ k\in\mathbb{Z}$, a, b_k $(k = 1,\dots,n)$ are constants. The question is, under what conditions the equation (8) has at least one anti-periodic solution with anti-period π.

Corresponding to the equation (1.1), we find that

$$a(t) = 1+\sin 2t,\quad I_k(x) = c_k x^3\quad (k\in\mathbb{Z}),$$

$$f(t,x_0,x_1,\dots,x_n) = ax_0^{2q+1} + \sum_{k=1}^{n} b_i x_i^{2q+1} + \sin t\quad\text{and}\quad \alpha_k(t) = t-k^{-2}\quad (k\in\mathbb{Z}).$$

It is easy to check that (A1)–(A5) hold.

It is easy to see that $xI_k(x)\geqslant 0$ if $c_k\geqslant 0$ for all $k\in\mathbb{Z}$.

Choose $h(t,x_0,x_1,\dots,x_n) = ax_0^{2q+1}$, $g_i(t,x_i) = b_i x_i^{2q+1}$, $r(t) = \sin t$. Then

$$f(t,x_0,x_1,\dots,x_n) = h(t,x_0,x_1,\dots,x_n) + \sum_{i=1}^{n} g_i(t,x_i) + r(t).$$

It is easy to check that (H2) holds if $a>0$ with $t_0 = 0$, $\beta = a$ and $m = 2q+1$ and $r_i = |b_i|(2+\pi)$ $(i = 1,\dots,n)$. Since $\alpha_k(t) = t-k^{-2}$, we get that $\beta_k(t) = t+k^{-2}$ and $\mu_k = 1$. Then Theorem 2.2 implies that the equation (3.1) has at least one anti-periodic solution if

$$(3.2)\qquad \sum_{k=1}^{n} |b_k| 2^{(2q+1)/(2q+2)} < a.$$

Remark 3.1. This paper is a continuation of [18]. But the techniques used to get the a priori estimates of solutions in this paper are different from those used in [18]. One sees easily that Example 3.1 can not be solved by Theorems 2.1–2.3 obtained in [18] since (G2) in Theorem 2.1 [18], (G4) in Theorem 2.2 [18] and (G6) in Theorem 2.3 [18] are not satisfied.

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