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ON AN OVER-DETERMINED PROBLEM OF FREE BOUNDARY OF A DEGENERATE PARABOLIC EQUATION

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Abstract. This work is concerned with the inverse problem of determining initial value of the Cauchy problem for a nonlinear diffusion process with an additional condition on free boundary. Considering the flow of water through a homogeneous isotropic rigid porous medium, we have such desire: for every given positive constants K and T_0, to decide the initial value u_0 such that the solution $u(x,t)$ satisfies $\sup\limits_{x\in H_u(T_0)} |x| \geqslant K$, where $H_u(T_0) = \{x \in \mathbb{R}^N\colon\ u(x,T_0) > 0\}$. In this paper, we first establish a priori estimate $u_t \geqslant C(t)u$ and a more precise Poincaré type inequality $\|\varphi\|^2_{L^2(B_\varrho)} \leqslant \varrho\|\nabla\varphi\|^2_{L^2(B_\varrho)}$, and then, we give a positive constant C_0 and assert the main results are true if only $\|u_0\|_{L^2(\mathbb{R}^N)} \geqslant C_0$.

Keywords: inverse problem; parabolic equation; absorption

MSC 2010: 35K10, 35K65

1. Introduction

Consider the flow of water through a homogeneous isotropic rigid porous medium. If we assume the density of water to be constant, the volumetric moisture content u and the seepage velocity v of water are governed by the continuity equation $u_t + \nabla v = 0$. Employing Darcy's law, we can obtain the well-known porous media equation (see [6])

$$\text{(1.1)} \qquad \begin{cases} u_t = \Delta(u^m) - \kappa u^p & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where $m > p \geqslant 1$, $\kappa > 0$, $Q_T = \mathbb{R}^N \times (0,T)$ and

$$\text{(1.2)} \qquad 0 \leqslant u_0 \leqslant L, \quad \int_{\mathbb{R}^N} u_0 \,\mathrm{d}x > 0$$

for $L > 0$. The term $-\kappa u^p$ in the equation of (1.1) means that the system admits absorption when $\kappa > 0$.

The equation (1.1) is the simplest model of degenerate parabolic equation and many well-known qualitative properties have been shown in last decades. For example, some authors (see [12]) discussed the large-time behavior of the solution to the Cauchy problem (1.1) and got an estimate $\|u\|_{L^2} \leqslant Ct^{-\alpha}$ for $\alpha > 0$. This inequality shows that the total mass of the system will be extinguished by the absorption $-\kappa u^p$ as $t \to \infty$, but it does not tell us how far away the diffusing substance will reach at a given time T_0. That is to say, from such estimates we cannot know where the free boundary of the solution is.

It is well-known that the study of free boundary has a long history. Certainly, if we consider a uniform parabolic equation without absorption, for example, the linear heat equation $u_t = \Delta u$, we see that $u(x,t) > 0$ everywhere in Q if only its initial value u_0 satisfies (1.2), thus, the speed of propagation of u is infinite in this case. However, this fact is not true for degenerate parabolic equations. For example, L. A. Peletier and B. H. Gilding (see [5], [11]) discussed the free boundary problems of degenerate parabolic equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} + \frac{\partial u^n}{\partial x},$$

respectively. They proved that the speeds of propagation of the solutions are finite. But they got no explicit formulas. For the case of the dimension $N \geqslant 1$, the Barenblatt function (see [2])

$$B(x,t,C) = t^{-\lambda}\Big[C - \sigma\frac{|x|^2}{t^{2\mu}}\Big]_+^{1/(m-1)} \tag{1.3}$$

gives a source-type solution to the Cauchy problem

$$\begin{cases} B_t = \Delta(B^m) & \text{in } Q, \\ B(x,0) = M\delta(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where $m > 1$, $[h]_+ = \max\{h, 0\}$,

$$\lambda = \frac{N}{N(m-1)+2}, \quad \mu = \frac{\lambda}{N}, \quad \sigma = \frac{\lambda(m-1)}{2mN}, \tag{1.4}$$

and C is a positive constant such that $\int_{\mathbb{R}^N} B \, \mathrm{d}x = M$. For every $t > 0$ denote

$$H_u(t) = \{x \in \mathbb{R}^N \colon\ u(x,t) > 0\}. \tag{1.5}$$

$H_u(t)$ is the positivity set of the solution. Then (1.3) implies $H_B(t) = \{x \in \mathbb{R}^N : |x| < \sqrt{C/\kappa}\, t^\mu\}$, and therefore, we get the exact expanding behavior of free boundary:

$$|x| = \sqrt{\frac{C}{\kappa}} t^\mu, \quad x \in \partial H_B(t). \tag{1.6}$$

This fact tells us that the solution $B(x,t)$ may retain its positivity at any given point when t increases. To extend the result of [2], J. L. Vazquez (see [12]), employing the Barenblatt function and comparison theorem, proved that the solution to the Cauchy problem of the equation

$$\begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^N \end{cases}$$

also has a bounded positivity set $H_u(t)$:

$$c_1 t^\mu \leqslant |x| \leqslant c_2 t^\mu, \quad x \in \partial H_u(t), \tag{1.7}$$

where $u_0(x)$ satisfies (1.2) and is supported in a bounded set of $\mathbb{R}^N$. Here we see that the speed of propagation of $H_u(t)$ is similar to the one of $H_B(t)$. Moreover, if the initial value u_0 subjects to some restrictions (see [6]), so that the solution $u(x,t)$ is continuous, then (1.7) shows that every point of the space is eventually reached by the diffusing substance. However, for a general parabolic equation $u_t = \sum\limits_{i,j=1}^{N} (\partial/\partial x_j) a_{i,j}\, \partial u/\partial x_i + \sum\limits_{i=1}^{N} b_i\, \partial u/\partial x_i + cu$, whether the property will be retained or not, there are yet no other explicit results to the knowledge of the author. Although the equation (1.1) has an absorption $-\kappa u^p$, we can easily see (in Section 2 of the present work) that the solution of (1.1) will not extinguish for $t \in (0,\infty)$. Thus, we guess that the positivity set $H_u(t)$ does not always become smaller as t increases. So we have such a desire: for every positive constant K and T_0 to decide the initial value u_0 such that the solution $u(x,t,u_0)$ satisfies

$$\sup_{x \in H_u(T_0)} |x| \geqslant K. \tag{1.8}$$

We see that the problem (1.1)–(1.2) with the additional condition (1.8) is an overdetermined problem. We know that there are many works devoted to different kinds of such ill-posed problems on parabolic equations in the recent years (see [9], [7], [8], [14]). But most of them discussed the solvability of these problems and few of them are concerned with free boundary problems.

We say that a nonnegative function $u(x,t) \in C([0,\infty)\colon\ L^1(\mathbb{R}^N))$ is a solution of the Cauchy problem (1.1) with the initial value (1.2) in Q if

(i) u_t, u^m, $\Delta u^m \in L^1_{\rm loc}((0,\infty)\colon\ L^1(\mathbb{R}^N))$;

(ii) $u_t = \Delta u^m - \theta u^p$ in the sense of distributions in Q;

(iii) $u(x,t) \to u_0(x)$ in $L^1(\mathbb{R}^N)$ as $t \to 0$.

Our main result reads:

Theorem 1. *The problem* (1.1)–(1.2) *has a unique global nonnegative weak solution* $u(x,t)$ *with the properties*

$$u_t \geqslant \frac{\kappa(m-p)L^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)L^{p-1}t}-1)}u$$

in the sense of distributions in Q_T, *and*

$$\int_{\mathbb{R}^N} u^s \,\mathrm{d}x \geqslant L^{s-1}\mathrm{e}^{-L^{p-1}\kappa t}\int_{\mathbb{R}^N} u_0 \,\mathrm{d}x \tag{1.9}$$

for $s \in (0,1]$ *and* $t > 0$.

Theorem 2. *Suppose*

$$\operatorname{supp} u_0 = B_\varepsilon = \{x \in \mathbb{R}^N\colon\ |x| < \varepsilon\}$$

with $\varepsilon > 0$. *For every given* $K > 0$ *and* T_0, *there exists a positive constant* C_0 *depending on* T_0 *and* K, *such that the solution to* (1.1), (1.2) *satisfies*

$$\sup_{x\in H_u(T_0)} |x| \geqslant K$$

when $\|u_0\|_{L(\mathbb{R}^N)} \geqslant C_0$.

Remark. If there is no absorption in the system, that is to say, $\kappa = 0$ in the equation (1.1), we will show that the positive constant C_0 is defined more clearly. This fact will be shown by a corollary in Section 4 of the present work, where we see that $\sup\limits_{x\in H_u(T_0)} |x| \geqslant cT_0^\mu$ for some $c > 0$, which is just the left of (1.7).

2. Some estimates

We prove our Theorem 1 in this section. To do this, we need to establish some lemmas firstly. Although the proof of the existence and uniqueness of the solution to the problem (1.1)–(1.2) has been established by others (see [12], [10]) with a standard procedure, we also want to show the main steps which will be used to prove our main conclusion.

Lemma 2.1 (The existence of a solution). *For every given $T > 0$, the Cauchy problem* (1.1)–(1.2) *has a nonnegative solution $u(x,t)$ in Q_T.*

Proof. For every $k > 2$ and $T > 0$, set

$$Q_{k,T} = B_k \times (0,T), \quad S_{k,T} = \partial B_k \times (0,T),$$

and

$$u_{0,\eta} = \int_{\mathbb{R}^N} u_0(y) J_\eta(x,y)\,\mathrm{d}y, \quad u_{0\eta,k} = u_{0\eta}\zeta_k,$$

where $B_k = \{x \in \mathbb{R}^N \colon |x| < k\}$ and

$$J(x) = \begin{cases} \mathrm{e}^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases}$$

and $J_\eta(x) = (1/(\gamma\eta^N))J(x/\eta)$ with $\gamma = \int_{|x|<1} \mathrm{e}^{1/(|x|^2-1)}\,\mathrm{d}x$ for $\eta > 0$, and $\{\zeta_k\}_{k>2}$ is a smooth cutoff sequence with the following properties: $\zeta_k(x) \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{cases} \zeta_k(x) = 1, & |x| \leqslant k-1, \\ 0 < \zeta_k(x) < 1, & k-1 < |x| < k, \\ \zeta_k(x) = 0, & |x| \geqslant k. \end{cases}$$

Clearly, $u_{0\eta,k}(x) \to u_0(x)$ in $L^1(\mathbb{R}^N)$ as $\eta \to 0$ and $k \to \infty$. Moreover, it is not difficult to see that the derivatives of the functions ζ_k up to second order are bounded with respect to $x \in \mathbb{R}^N$ uniformly. Specially, there is a positive constant γ such that

$$|\nabla\zeta_k| \leqslant \frac{\gamma}{k} \quad \text{and} \quad |\Delta\zeta_k| \leqslant \frac{\gamma}{k^2}.$$

We next consider the Dirichlet problem

$$\begin{cases} u_t = \Delta(u^m) - \theta u^p & \text{in } Q_{k,T}, \\ u(x,t) = \eta^* & \text{in } S_{k,T}, \\ u(x,0) = u_{0\eta,k}(x) + \eta & \text{in } B_k, \end{cases} \tag{2.1}$$

where $\eta^* = (\eta^{1-p} + (p-1)\theta T)^{1/(1-p)}$. A similar procedure (see Theorem 4 Ch. II in [12]) yields that the Dirichlet problem (2.1) has a unique solution $u_{\eta,k} \in C^\infty(Q_{k,T}) \cap C(\overline{Q}_{k,T})$. Letting $k \to \infty$, $\eta \to 0$ and employing a procedure similar to the one used in Chapter III in [12], we see that there exists a nonnegative function $u(x,t)$, which is the solution of the Cauchy problem (1.1)–(1.2) in Q and

$$0 \leqslant u(x,t) \leqslant L \quad \text{in } Q_T.$$

□

To prove the uniqueness, we need to give the L^1-contraction principle first.

Lemma 2.2 (L^1-contraction principle). *Suppose u and $\tilde{u}$ to be two solutions to the problem* (1.1)–(1.2) *corresponding to the initial data u_0 and $\tilde{u}_0$. Then*

$$\int_{\mathbb{R}^N} |u^p - \tilde{u}^p| \,\mathrm{d}x \leqslant \left(pL^{p-1} \int_{\mathbb{R}^N} |u_0 - \tilde{u}_0| \,\mathrm{d}x\right) \cdot \mathrm{e}^{-p\theta L^{p-1} t} \quad t > 0. \tag{2.2}$$

Proof. Take a function $h(x) \in C^\infty(\mathbb{R}^1)$ such that

$$h(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp\left[\dfrac{-1}{x^2} \exp \dfrac{-1}{(x-1)^2}\right], & 0 < x < 1, \\ 1, & x \geqslant 1. \end{cases}$$

Clearly, $0 \leqslant h(x) \leqslant 1$ and $h'(x) \geqslant 0$. Denote $h_\varepsilon(x) = h(x/\varepsilon)$ for $\varepsilon > 0$ and set

$$w = u^m - \tilde{u}^m.$$

We have

$$\begin{aligned} \int_{\mathbb{R}^N} (u - \tilde{u})_t h_\varepsilon(w) \,\mathrm{d}x &= \int_{\mathbb{R}^N} \Delta w h_\varepsilon(w) \,\mathrm{d}x - \theta \int_{\mathbb{R}^N} (u^p - \tilde{u}^p) h_\varepsilon(w) \,\mathrm{d}x \\ &\leqslant -\theta \int_{\mathbb{R}^N} (u^p - \tilde{u}^p) h_\varepsilon(w) \,\mathrm{d}x \quad t > 0. \end{aligned}$$

Since $w > 0$ iff $u > \tilde{u}$, Lemma 3.1 of [3] yields

$$\int_{\mathbb{R}^N} (u - \tilde{u})_t p_\varepsilon(w) \,\mathrm{d}x \to \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} [u - \tilde{u}]_+ \,\mathrm{d}x, \quad \text{as } \varepsilon \to 0,$$

where $[u - \tilde{u}]_+ = \max(u - \tilde{u}, 0)$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} [u - \tilde{u}]_+ \,\mathrm{d}x \leqslant -\theta \int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_+ \,\mathrm{d}x, \quad t > 0.$$

This yields

$$\int_{\mathbb{R}^N} [u - \tilde{u}]_+ \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} [u_0 - \tilde{u}_0]_+ \,\mathrm{d}x - \theta \int_0^t \int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_+ \,\mathrm{d}x \,\mathrm{d}\tau \quad t > 0. \tag{2.3}$$

Clearly, $[u^p - \tilde{u}^p]_+ \leqslant pL^{p-1}[u - \tilde{u}]_+$ thanks to $0 \leqslant u, \tilde{u} \leqslant L$ and $p \geqslant 1$. Using this inequality in (2.3) yields

$$\int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_+ \,\mathrm{d}x \leqslant pL^{p-1} \int_{\mathbb{R}^N} [u_0 - \tilde{u}_0]_+ \,\mathrm{d}x - \theta pL^{p-1} \int_0^t \int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_+ \,\mathrm{d}x \,\mathrm{d}\tau.$$

Finally, the Gronwall inequality gives

$$\int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_+ \,\mathrm{d}x \leqslant \Big(pL^{p-1} \int_{\mathbb{R}^N} [u_0 - \tilde{u}_0]_+ \,\mathrm{d}x\Big) \mathrm{e}^{-\theta p L^{p-1} t}. \tag{2.4}$$

Similarly,

$$\int_{\mathbb{R}^N} [u^p - \tilde{u}^p]_- \,\mathrm{d}x \leqslant \Big(pL^{p-1} \int_{\mathbb{R}^N} [u_0 - \tilde{u}_0]_- \,\mathrm{d}x\Big) \mathrm{e}^{-\theta p L^{p-1} t}. \tag{2.5}$$

Combining (2.4) and (2.5) gives (2.2). □

Lemma 2.2 implies the following result:

Corollary 1 (The uniqueness)**.** *The solution $u(x,t)$ obtained in Lemma* 2.1 *is unique.*

Lemma 2.3. *Let $u(x,t)$ be a nonnegative solution of the problem* (1.1) *with* (1.2) *in Q_T. Then*

$$u_t \geqslant \frac{\kappa(m-p)L^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)L^{p-1}t} - 1)} u$$

in the sense of distributions in Q_T and,

$$\int_{\mathbb{R}^N} u^s \,\mathrm{d}x \geqslant L^{s-1} \mathrm{e}^{-L^{p-1}\kappa t} \int_{\mathbb{R}^N} u_0 \,\mathrm{d}x \tag{2.6}$$

for $s \in (0,1]$ and $t \in (0,T)$.

P r o o f. For every given $T > 0$, suppose that $u_{\eta,k}$ is the solution of the Dirichlet problem (2.1) in $Q_{k,T}$. We first prove

$$\frac{\partial}{\partial t} u_{\eta,k} \geqslant \frac{\kappa(m-p)(L+\eta)^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)(L+\eta)^{p-1}t} - 1)} u_{\eta,k} \quad \text{in } Q_T. \tag{2.7}$$

To do this, we set

$$V = (u_{\eta,k})^m \quad \text{and} \quad q = \frac{V_t}{V}.$$

Thereby,

$$q(x,t) = 0 \quad \text{on } S_{k,T}. \tag{2.8}$$

For every given $t > 0$, set

$$\begin{aligned} \Omega_k^+ &= \{x \in \Omega\colon\ q(x,t) > 0\}, \\ \Omega_k^- &= \{x \in \Omega\colon\ q(x,t) < 0\}, \\ \Omega_k^0 &= \{x \in \Omega\colon\ q(x,t) = 0\}. \end{aligned}$$

Thereby,

$$Q_T = (\Omega_k^- \cup \Omega_k^0) \times (0,T) \cup \Omega_k^- \times (0,T).$$

Owing to $m > p \geqslant 1$, the right hand side of (2.7) is negative, so (2.7) is true for $(x,t) \in (\Omega_k^+ \cup \Omega_k^0) \times (0,T)$. Therefore, we next prove (2.7) for $(x,t) \in \Omega_k^- \times (0,T)$ only. It follows from $V_t = V'(\Delta V - \kappa u_{\eta,k}^p)$ that $q = (V'/V)[\Delta V - \kappa(u_{\eta,k}^p)]$. Thus,

$$\begin{aligned} q_t &= \frac{V'}{V}[\Delta V_t - \kappa(u_{\eta,k}^p)_t] + \frac{V'' \cdot (u_{\eta,k})_t \cdot V - V' \cdot V_t}{V^2}[\Delta V - \kappa(u_{\eta,k}^p)] \\ &= \frac{V'}{V}\Delta V_t + \frac{[\Delta V - \kappa(u_{\eta,k}^p)]^2}{V^2}[VV'' - (V')^2] - \kappa\frac{V'}{V}(u_{\eta,k}^p)_t. \end{aligned}$$

Since $\Delta V_t = \Delta(qV) = V\Delta q + 2\nabla V \cdot \nabla q + q\Delta V$,

$$\begin{aligned} q_t &= V'\Delta q + 2\frac{V'}{V}\nabla V \cdot \nabla q + \frac{V'}{V}q\Delta V + \frac{[\Delta V - \kappa(u_{\eta,k}^p)]^2}{V^2}[VV'' - (V')^2] \\ &\quad - \kappa p\frac{V'}{V}u_{\eta,k}^{p-1}(u_{\eta,k})_t \\ &= V'\Delta q + 2\frac{V'}{V}\nabla V \cdot \nabla q + q\Big[q + \kappa\frac{V'}{V}(u_{\eta,k})^p\Big] \\ &\quad + \frac{q^2}{(V')^2}[VV''(u) - (V')^2] - \kappa pqu_{\eta,k}^{p-1} \\ &= V'\Delta q + 2\frac{V'}{V}\nabla V \cdot \nabla q + q^2\frac{VV''}{(V')^2} + \kappa q\Big[\frac{V'}{V}(u_{\eta,k})^p - pu_{\eta,k}^{p-1}\Big] \\ &= V'\Delta q + 2\frac{V'}{V}\nabla V \cdot \nabla q + \frac{m-1}{m}q^2 + \kappa(m-p)u_{\eta,k}^{p-1}q. \end{aligned} \tag{2.9}$$

Recalling $q < 0$ in this case, $u_{\eta,k} \leqslant L + \eta$ and $m > p \geqslant 1$, we have $(m-p)u_{\eta,k}^{p-1}q \geqslant (m-p)(L+\eta)^{p-1}q$. Thus,

$$q_t \geqslant \varphi'\Delta q + 2\frac{\varphi'}{V}\nabla V \cdot \nabla q + \frac{m-1}{m}q^2 + \kappa(m-p)(L+\eta)^{p-1}q. \tag{2.10}$$

Moreover,

$$q = 0 \quad \text{on } \partial\Omega_k^- \times (0,T).$$

Consider the equation

$$\bar{q}_t = \varphi'\Delta\bar{q} + 2\frac{\varphi'}{V}\nabla V \cdot \nabla\bar{q} + \frac{m-1}{m}\bar{q}^2 + \kappa(m-p)(L+\eta)^{p-1}\bar{q}. \tag{2.11}$$

It is easy to see that the function

$$\bar{q}_* = \frac{m\kappa(m-p)(L+\eta)^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)(L+\eta)^{p-1}t} - 1)}$$

satisfies the equation (2.11) in $\Omega_k^- \times (0,T)$ and

$$\bar{q}_*(x,0) = -\infty,$$
$$\bar{q}_*(x,t) < 0 \quad \text{on } \partial\Omega_k^- \times (0,T).$$

Although the domain $\Omega_k^- \times (0,T)$ may not be a cylinder of $\mathbb{R}^N \times \mathbb{R}^+$, the comparison theorem (see Th. 16 in Ch. 2 of [4]) is also applicable in this situation. The comparison theorem claims $q \geqslant \bar{q}_*$, and this fact means

$$\frac{\partial u_{\eta,k}}{\partial t} \geqslant \frac{\kappa(m-p)(L+\eta)^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)(L+\eta)^{p-1}t}-1)} u_{\eta,k} \quad \text{in } \Omega_k^- \times (0,T). \tag{2.12}$$

Thus (2.7) holds in $\Omega_k^- \times (0,T)$. Finally, letting $\eta \to 0$ and $k \to \infty$ in (2.7) gives the first result of our Lemma 2.3.

To get the estimate (2.6), we first take a cutoff function ζ_k defined in Lemma 2.1, and integrate by parts as follows:

$$\begin{aligned}\int_{\mathbb{R}^N} (u-u_0)\zeta_k \,\mathrm{d}x &= \int_0^t \int_{\mathbb{R}^N} [\Delta(u^m) - \kappa u^p]\zeta_k \,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_0^t \int_{\mathbb{R}^N} [u^m \Delta\zeta_k - \kappa u^p \zeta_k] \,\mathrm{d}x\,\mathrm{d}\tau \\ &\geqslant \int_0^t \int_{\mathbb{R}^N} [u^m \Delta\zeta_k - \kappa L^{p-1} u \zeta_k] \,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Since the definition of the solution tells us $u^m \in L^1(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} u^m \Delta\zeta_k \,\mathrm{d}x \to 0$ as $k \to \infty$. So, the above inequality yields $\int_{\mathbb{R}^N}(u-u_0)\,\mathrm{d}x \geqslant -\kappa L^{p-1}\int_0^t\int_{\mathbb{R}^N} u \,\mathrm{d}x\,\mathrm{d}t$. The Gronwall inequality implies

$$\int_{\mathbb{R}^N} u \,\mathrm{d}x \geqslant \mathrm{e}^{-L^{p-1}\kappa t} \int_{\mathbb{R}^N} u_0 \,\mathrm{d}x \quad t > 0. \tag{2.13}$$

For the case of $0 < s < 1$, we see that

$$\begin{aligned}\int_{\mathbb{R}^N} u^s \,\mathrm{d}x &\geqslant L^{s-1} \int_{\mathbb{R}^N} u \,\mathrm{d}x \\ &\geqslant L^{s-1}\mathrm{e}^{-L^{p-1}\kappa t} \int_{\mathbb{R}^N} u_0 \,\mathrm{d}x \quad t > 0.\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) gives (2.6). □

Combining the above conclusions, we know our Theorem 1 holds.

3. The expanding behavior of $H_u(t)$

In this section, we prove our Theorem 2. Supposing $u(x,t)$ to be the solution of (1.1), we can first get a rough description on the expanding behavior of $H_u(t)$. In fact, for every $l \geqslant 1$ and $\Omega \subset \mathbb{R}^N$, Lemma 2.3 implies

$$\frac{1}{l}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^l\,\mathrm{d}x \geqslant \frac{\kappa(m-p)L^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)L^{p-1}t}-1)}\int_\Omega u^l\,\mathrm{d}x. \tag{3.1}$$

Consequently,

$$\ln\Big(\frac{\int_\Omega u^l(x,t_2)\,\mathrm{d}x}{\int_\Omega u^l(x,t_1)\,\mathrm{d}x}\Big) \geqslant \ln\Big(\frac{\mathrm{e}^{\kappa(m-p)L^{p-1}t_1}-1}{\mathrm{e}^{\kappa(m-p)L^{p-1}t_2}-1}\Big)^{\frac{l}{m-1}}.$$

This means

$$\begin{aligned}\int_\Omega u^l(x,t_2)\,\mathrm{d}x\cdot(\mathrm{e}^{\kappa(m-p)L^{p-1}t_2}-1)^{l/(m-1)}\\ \geqslant \int_\Omega u^l(x,t_1)\,\mathrm{d}x\cdot(\mathrm{e}^{\kappa(m-p)L^{p-1}t_1}-1)^{l/(m-1)}.\end{aligned} \tag{3.2}$$

In other words, if $u(x,t)$ is the solution to (1.1) with (1.2), then (3.2) claims the following fact:

$$\text{if}\quad \int_\Omega u^l(x,t_0)\,\mathrm{d}x > 0,\quad \text{then}\quad \int_\Omega u^l(x,t)\,\mathrm{d}x > 0\quad \text{for all } t > t_0. \tag{3.3}$$

Although the formula (3.3) tells us the solution $u(x,t)$ will never vanish even if the equation (1.1) has the absorption $-\kappa u^p$, thereby, the positive set $H_u(t)$ will never be empty, we are interested in giving an explicit formula. To do this, we need to establish a more precise Poincaré type inequality. It is well-known that there exists a positive constant k such that $k\|\varphi\|^2_{L^2(\Omega)} \leqslant \|\nabla\varphi\|^2_{L^2(\Omega)}$ for $\varphi \in H^1_0(\Omega)$. Recently, Wu (see p. 13 in [13]) proved that

$$k \leqslant \varrho^{-2} \tag{3.4}$$

if $\Omega = \{(x_1,x_2,\dots,x_N) \in \mathbb{R}^N\colon\ a_i < x_i < a_i+\varrho\}$. In order to prove Theorem 2 we first show that the choice (3.4) is also right if Ω is a sphere in $\mathbb{R}^N$.

Lemma 3.1. *Assume* $B_\varrho = \{x \in \mathbb{R}^N \colon |x| < \varrho\}$. *If* $u \in H_0^1(B_\varrho)$. *Then*

$$\|u\|_{L^2(B_\varrho)} \leqslant \varrho \|\nabla u\|_{L^2(B_\varrho)}. \tag{3.5}$$

Proof. We first suppose $u \in C_0^\infty(B_\varrho)$. For every $x \in B_\varrho$, there is a $x_* \in \partial B_\varrho$, such that the three points $0, x$ and x_* lie on a radius $\overline{ox_*}$. Denote the vector from x_* to x by r. We have

$$\begin{aligned} u(x) &= u(x) - u(x_*) \\ &= \int_{x_*}^{x} \frac{\partial u}{\partial r}\,\mathrm{d}r. \end{aligned}$$

Using the Hölder inequality we get

$$|u(x)|^2 \leqslant \varrho \int_0^\varrho \Big|\frac{\partial u}{\partial r}\Big|^2\,\mathrm{d}r.$$

This gives

$$\begin{aligned} \int_{B_\varrho} |u(x)|^2\,\mathrm{d}x &\leqslant \varrho \int_{B_\varrho}\int_0^\varrho \Big|\frac{\partial u}{\partial r}\Big|^2\,\mathrm{d}r\,\mathrm{d}x \\ &\leqslant \varrho^2 \int_{B_\varrho} |\nabla u|^2\,\mathrm{d}x. \end{aligned}$$

The general case is done by approximation. □

Proof of Theorem 2. For a given $T_0 > 0$, if $H_u(T_0)$ is unbounded, then the proof is finished. Thereby, we next suppose $H_u(T_0)$ to be bounded. Denote

$$\varrho_{(T_0)} = \sup_{x \in H_u(T_0)} |x|, \quad t > 0.$$

For every $\lambda > 0$, set $\varrho = \lambda + \varrho_{(T_0)}$. Clearly, $u = 0$ on ∂B_ϱ. By Lemma 2.3,

$$\int_{B_\varrho} u^m[\Delta(u^m) - \kappa u^p]\,\mathrm{d}x \geqslant \frac{\kappa(m-p)L^{p-1}}{(m-1)(\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0} - 1)} \int_{B_\varrho} u^{1+m}\,\mathrm{d}x.$$

It follows from (3.5) that

$$\int_{B_\varrho} u^{2m}\,\mathrm{d}x + \varrho^2\kappa \int_{B_\varrho} u^{m+p}\,\mathrm{d}x \leqslant \varrho^2 \frac{\kappa(m-p)L^{p-1}}{(m-1)(1 - \mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})} \int_{B_\varrho} u^{1+m}\,\mathrm{d}x. \tag{3.6}$$

By the Hölder inverse inequality (see p. 24 in [1]), we have

$$\int_{B_\varrho} u^{m+p}\,\mathrm{d}x \geqslant \Big(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Big)^{(m+p)/(1+m)} \cdot |B_\varrho|^{(1-p)/(1+m)} \tag{3.7}$$

and

$$\int_{B_\varrho} u^{2m}\,\mathrm{d}x \geqslant \Bigl(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Bigr)^{2m/(1+m)} \cdot |B_\varrho|^{(1-m)/(1+m)}. \tag{3.8}$$

Using (3.7) and (3.8) in (3.6) gives

$$\begin{aligned}\Bigl(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Bigr)^{\frac{2m}{1+m}} \cdot |B_\varrho|^{\frac{1-m}{1+m}} + \varrho^2\kappa\Bigl(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Bigr)^{\frac{m+p}{1+m}} \cdot |B_\varrho|^{\frac{1-p}{1+m}} \\ \leqslant \varrho^2 \frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})} \int_{B_\varrho} u^{m+1}\,\mathrm{d}x.\end{aligned}$$

Owing to $u_0 > 0$ on B_ε, (3.3) claims $\int_{B_\varrho} u^{m+1}\,\mathrm{d}x > 0$, hence

$$\begin{aligned}\Bigl(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Bigr)^{\frac{m-1}{1+m}} \cdot |B_\varrho|^{\frac{1-m}{1+m}} + \varrho^2\kappa\Bigl(\int_{B_\varrho} u^{m+1}\,\mathrm{d}x\Bigr)^{\frac{p-1}{1+m}} \cdot |B_\varrho|^{\frac{1-p}{1+m}} \\ \leqslant \varrho^2 \frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}.\end{aligned} \tag{3.9}$$

On the other hand, using the Hölder inverse inequality again yields

$$\int_{B_\varrho(x^*)} u^{m+1}\,\mathrm{d}x \geqslant \Bigl(\int_{B_\varrho(x^*)} u^{s}\,\mathrm{d}x\Bigr)^{\frac{1+m}{s}} \cdot |B_\varrho(x^*)|^{\frac{s-1-m}{s}} \quad \text{for } 0 < s \leqslant 1+m.$$

Thus,

$$\begin{aligned}\Bigl(\int_{B_\varrho} u^{s_1}\,\mathrm{d}x\Bigr)^{\frac{m-1}{s_1}} |B_\varrho|^{\frac{(s_1-1-m)(m-1)}{s_1(1+m)}+\frac{1-m}{1+m}} + \varrho^2\kappa\Bigl(\int_{B_\varrho} u^{s_2}\,\mathrm{d}x\Bigr)^{\frac{p-1}{s_2}} |B_\varrho|^{\frac{(s_2-1-m)(p-1)}{s_2(1+m)}+\frac{1-p}{1+m}} \\ \leqslant \varrho^2 \frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}t})}\end{aligned}$$

for $0 < s_1, s_2 \leqslant 1$. Letting $s_1 = 1$ and $s_2 = N(p-1)/(N(m-1)+2)$ (and recalling the fact $|B_\varrho| = \pi^{N/2}\Gamma(1+N/2)^{-1}\varrho^N$), we have

$$\begin{aligned}\Bigl[\int_{\mathbb{R}^N} u\,\mathrm{d}x \cdot \pi^{-\frac{N}{2}}\Gamma\Bigl(1+\frac{N}{2}\Bigr)\Bigr]^{m-1} + \kappa\Bigl[\int_{\mathbb{R}^N} u^{\frac{N(p-1)}{N(m-1)+2}}\,\mathrm{d}x \cdot \pi^{-\frac{N}{2}}\Gamma\Bigl(1+\frac{N}{2}\Bigr)\Bigr]^{\frac{N(m-1)+2}{N}} \\ \leqslant \varrho^{2+N(m-1)} \frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}.\end{aligned}$$

Using (2.13) and (2.14), we have

$$\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\int_{\mathbb{R}^N} u_0\,\mathrm{d}x\cdot \pi^{-N/2}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{m-1} + \kappa\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\int_{\mathbb{R}^N} u_0\,\mathrm{d}x\cdot L^{\frac{N(p-m)-2}{N(m-1)+2}}\pi^{-N/2}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{\frac{N(m-1)+2}{N}} \leqslant \varrho^{2+N(m-1)}\frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}.$$

Recalling the definition of μ in (1.4), we get

$$\varrho \geqslant \chi^{\mu}, \tag{3.10}$$

where

$$\chi = \frac{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}{\kappa(m-p)L^{p-1}} \times\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\int_{\mathbb{R}^N} u_0\,\mathrm{d}x\cdot \pi^{-\frac{N}{2}}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{m-1} + \frac{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}{(m-p)L^{p-1}} \times\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\int_{\mathbb{R}^N} u_0\,\mathrm{d}x\cdot L^{\frac{N(p-m)-2}{N(m-1)+2}}\pi^{-\frac{N}{2}}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{\frac{N(m-1)+2}{N}}.$$

Letting $\lambda \to 0$ in (3.10) gives

$$\sup_{x\in H_u(T_0)} |x| \geqslant \chi^{\mu}.$$

Setting

$$\chi^{\mu} = K$$

and

$$y = \Big(\int_{\mathbb{R}^N} u_0\,\mathrm{d}x\Big)^{m-1},$$

we get the following equation with respect to y:

$$y + I_1 y^{1+\frac{2}{N(m-1)}} - I_2 K^{\frac{1}{\mu}} = 0, \tag{3.11}$$

where

$$I_1 = \kappa L^{\frac{N(p-m)-2}{N}}\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\pi^{-\frac{N}{2}}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{\frac{2}{N}},$$

$$I_2 = \frac{\kappa(m-p)L^{p-1}}{(m-1)(1-\mathrm{e}^{-\kappa(m-p)L^{p-1}T_0})}\Big[\mathrm{e}^{-L^{p-1}\kappa T_0}\cdot\pi^{-\frac{N}{2}}\Gamma\Big(1+\frac{N}{2}\Big)\Big]^{1-m}.$$

Since the function $F(y) = y + I_1 y^{1+2/(N(m-1))} - I_2 K^{1/\mu}$ increases with respect to $y > 0$, we see that there exists only one positive constant C_0^{m-1}, which satisfies the equation (3.11). Therefore, we see that $\sup\limits_{x \in H_u(T_0)} |x| \geqslant K$ if $\int_{\mathbb{R}^N} u_0 \, \mathrm{d}x \geqslant C_0$. $\square$

4. A special case

Here we show a special case of $\kappa = 0$ in the equation (1.1). From Lemma 2.3, we may easily get $u_t \geqslant -u/((m-1)t)$ in this case, which is the well-known estimate for the equation $u_t = \Delta u^m$ (see [12]). Thus, we can employ a procedure similar to (3.10) and get

$$\chi = (m-1)T_0 \left[\pi^{-\frac{N}{2}} \Gamma\Big(1 + \frac{N}{2}\Big) \cdot \int_{\mathbb{R}^N} u_0 \, \mathrm{d}x \right]^{m-1}.$$

Hence, we have the following results:

Corollary 2. *Let* $\kappa = 0$ *in the equation* (1.1). *For every given* K *and* T_0, *if* $\int_{\mathbb{R}^N} u_0 \, \mathrm{d}x \geqslant C_0$, *where*

$$C_0 = K^{\frac{1}{\mu(m-1)}} [(m-1)T_0]^{\frac{1}{1-m}} \left[\pi^{-\frac{N}{2}} \Gamma\Big(1 + \frac{N}{2}\Big) \right]^{-1},$$

then $\sup\limits_{x \in H_u(T_0)} |x| \geqslant K$.

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