Applications of Mathematics

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Applications of Mathematics, Vol. 58 (2013), No. 5, 555–593

Persistent URL: `http://dml.cz/dmlcz/143432`

58 (2013) APPLICATIONS OF MATHEMATICS No. 5, 555–593

GENERALIZED n-LAPLACIAN: SEMILINEAR NEUMANN PROBLEM WITH THE CRITICAL GROWTH

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(Received July 8, 2011)

Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$. Applying the generalized Moser-Trudinger inequality without boundary condition, the Mountain Pass Theorem and the Ekeland Variational Principle, we prove the existence and multiplicity of nontrivial weak solutions to the problem

$$u \in W^1L^\Phi(\Omega), \quad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) + V(x)\Phi'(|u|)\frac{u}{|u|} = f(x,u) + \mu h(x) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where Φ is a Young function such that the space $W^1L^\Phi(\Omega)$ is embedded into exponential or multiple exponential Orlicz space, the nonlinearity $f(x,t)$ has the corresponding critical growth, $V(x)$ is a continuous potential, $h \in (L^\Phi(\Omega))^*$ is a nontrivial continuous function, $\mu \geqslant 0$ is a small parameter and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$.

Keywords: Orlicz-Sobolev space, Mountain Pass Theorem, Palais-Smale sequence, Ekeland Variational Principle

MSC 2010: 46E35, 46E30, 26D10

1. Introduction

Throughout the paper ω_{n-1} denotes the surface of the unit sphere and the n-dimensional Lebesgue measure is denoted by $\mathcal{L}_n$.

In this paper, we show that the techniques for proving the existence and multiplicity of weak solutions to the Dirichlet problem concerning the generalized n-Laplace

The author was supported by the research project MSM 0021620839 of the Czech Ministry MŠMT.

equation with the nonlinearity in the critical growth range (see [12] and [8]) can be used also for the Neumann problem. In particular, we are dealing with the differential equation

(1.1)
$$u \in WL^{\Phi}(\Omega), \quad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) + V(x)\Phi'(|u|)\frac{u}{|u|} = f(x,u) + \mu h(x) \quad \text{in } \Omega$$
$$\text{and } \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega.$$

Here $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$, Φ is a Young function with the growth corresponding to the Trudinger-type embedding of the Orlicz-Sobolev space $WL^{\Phi}(\Omega)$ into an exponential or multiple exponential Orlicz space, $\mu \geqslant 0$ is a small parameter, $h \in (L^{\Phi}(\Omega))^*$ is a nontrivial continuous function, $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, V is a continuous potential and f is a nonlinearity with the critical growth with respect to Φ. The precise assumptions on Φ, V and f are given below.

It is an often studied problem to find solutions to the Laplace equation

$$u \in W_0^{1,2}(\Omega) \quad \text{and } -\Delta u = f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{1.2}$$

For $n \geqslant 3$ and f satisfying $\lim\limits_{t\to\infty}(f(x,t)/t^q) = 0$ uniformly on Ω with $q < (n+2)/(n-2)$, there are many results using the compactness of the embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r \in [1, 2n/(n-2))$ (see a review article by Lions [23] and the references given there). Problem (1.2) under condition $\lim\limits_{t\to\infty}(f(x,t)/t^{(n+2)/(n-2)}) = 0$ becomes much more difficult thanks to the fact that the embedding of $W_0^{1,2}(\Omega)$ into $L^{2n/(n-2)}(\Omega)$ is no longer compact. This difficulty has been overcome by Brézis and Nirenberg [6]. Their method uses the Mountain Pass Theorem by Ambrosetti and Rabinowitz [4].

When $n = 2$, we do not only have the Sobolev embedding into $L^r(\Omega)$ for any $r \in [0,\infty)$ but there is also the embedding of $W_0^{1,2}(\Omega)$ into the Orlicz space $\exp L^2(\Omega)$. This is a special case of the Trudinger embedding [28] of the Sobolev space of $W_0^{1,n}(\Omega)$, $n \geqslant 2$, into the Orlicz space $\exp L^{n/(n-1)}(\Omega)$. In particular, there is so called Moser-Trudinger inequality by Moser [24]

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)} \leqslant 1} \int_\Omega \exp(K|u|^{n/(n-1)})\,\mathrm{d}x \leqslant C(n, \mathcal{L}_n(\Omega))$$
$$\text{if and only if} \quad K \leqslant n\omega_{n-1}^{1/(n-1)}.$$

For $n = 2$, Adimurthi [2] using the Moser-Trudinger inequality modified the variational approach by Brézis and Nirenberg [6] so that he was able to prove the existence of a nontrivial weak solution to (1.2) also in the case of the nonlinearity with an exponential growth.

For $n \geqslant 2$, Adimurthi's method works for the n-Laplace equation

$$u \in W_0^{1,n}(\Omega) \quad \text{and} \quad -\Delta_n u = f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

where $\Delta_n u := \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ and $f(x,t) \approx \exp(b|t|^{n/(n-1)})$ for some $b > 0$. See for example [1].

In the recent paper [12], the above techniques are modified for a differential equation corresponding to the embedding of the Orlicz-Sobolev space $W_0 L^n \log^\alpha L(\Omega)$, $n \geqslant 2$, $\alpha < n-1$, into the Orlicz space $\exp L^{n/(n-1-\alpha)}(\Omega)$ (this embedding is due to Fusco, Lions, Sbordone [21] and Edmunds, Gurka, Opic [17]). The result is the existence of a nontrivial weak solution to the equation

$$u \in W_0 L^\Phi(\Omega) \quad \text{and} \quad -\operatorname{div}\left(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x,u) \quad \text{in } \Omega, \tag{1.4}$$

with Φ being a Young function that behaves like $t^n \log^\alpha(t)$, $\alpha < n-1$, for large t and with the nonlinearity f having so called critical growth (corresponding to the choice of the Young function Φ).

The results from the paper [12] were further generalized in the papers [11], [9] and [8] in several ways (generalized n-Laplace equation corresponding to the embedding into multiple exponential spaces, singular nonlinearity, the case of $WL^\Phi(\mathbb{R}^n)$, multiplicity of solutions) motivated by some recent results concerning the n-Laplace equation, see for example [3], [15], [16], and [27].

The present article is motivated by the paper [25] which alters the methods from [1] so that they can be applied to the Neumann problem concerning the n-Laplace equation. In our case, we modify the methods from [12] and [8].

Assumptions on Φ, V and f. For $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n-1$, we set

$$\gamma = \frac{n}{n-1-\alpha} > 0, \quad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0, \tag{1.5}$$
$$K_{l,n,\alpha} = \begin{cases} B^{1/B} n \omega_{n-1}^{\gamma/n} & \text{for } l = 1, \\ B^{1/B} \omega_{n-1}^{\gamma/n} & \text{for } l \geqslant 2. \end{cases}$$

The following notation enables us to work with the multiple exponential spaces comfortably. For $k \in \mathbb{N}$, let us write

$$\log_{[k]}(t) = \log(\log_{[k-1]}(t)), \quad \text{where } \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[k]}(t) = \exp(\exp_{[k-1]}(t)), \quad \text{where } \exp_{[1]}(t) = \exp(t).$$

We suppose that $\Phi\colon [0,\infty) \to [0,\infty)$ is a C^1-Young function satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n\bigl(\prod_{j=1}^{l-1}\log_{[j]}^{n-1}(t)\bigr)\log_{[l]}^{\alpha}(t)} = 1 \tag{1.6}$$

(for $l = 1$ we read (1.6) as $\lim\limits_{t\to\infty}(\Phi(t)/(t^n\log_{[1]}^{\alpha}(t))) = 1$). Next, we suppose that there is $C > 0$ such that

$$\frac{1}{C}t^n \leqslant \Phi(t) \leqslant Ct^n \quad \text{for } t \in \Bigl[0, \frac{1}{C}\Bigr) \tag{1.7}$$

and

$$t \mapsto \Phi'(t)t \quad \text{is a Young function.} \tag{1.8}$$

Let us also recall a condition that is often used when discussing the critical case concerning the generalized Moser-Trudinger inequality (see for example [10, Theorem 1.1(v) and Theorem 1.2(v)])

$$\Phi(t) \leqslant t^n\Biggl(\prod_{j=1}^{l-1}\log_{[j]}^{n-1}(t)\Biggr)\log_{[l]}^{\alpha}(t)(1-\log_{[l]}^{-\beta}(t)) \quad \text{for } t \in [t_\Phi,\infty) \tag{1.9}$$

for some $\beta \in (0,\min\{1,B\})$ and $t_\Phi \geqslant 1$. Notice that the assumptions (1.6) and (1.7) together with the fact that Φ is a C^1-Young function imply the existence of $c_\Phi > 0$ such that

$$c_\Phi\Phi'(t)t \leqslant \Phi(t), \quad t > 0. \tag{1.10}$$

The potential $V\colon \Omega \to \mathbb{R}$ satisfies

$$V \text{ is continuous} \quad \text{and} \quad 0 < V_0 \leqslant V(x) \leqslant V_1 < \infty. \tag{1.11}$$

The function $f\colon \Omega\times\mathbb{R} \mapsto \mathbb{R}$ is supposed to satisfy the following conditions. There are $M > 1$, $t_M > 0$, $C_b > 0$ and $b > 0$ satisfying

$$f \text{ is uniformly continuous on } \Omega\times[-t_0,t_0] \quad \text{for every } t_0 > 0, \tag{1.12}$$

$$f(x,0) = 0 \quad \text{and} \quad tf(x,t) > 0 \quad \text{for all } x\in\Omega \text{ and } t \neq 0,$$

$$0 < F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s \leqslant M|t|^{1-1/M}|f(x,t)| \tag{1.13}$$

$$\text{provided } |t| > t_M \text{ and } x \in \Omega,$$

$$|f(x,t)| \leqslant C_b\exp_{[l]}(b|t|^\gamma) \quad \text{for every } t\in\mathbb{R} \text{ and } x\in\Omega, \tag{1.14}$$

$$\limsup_{t\to 0}\frac{F(x,t)}{V_0\Phi(|t|)} < 1 \quad \text{uniformly on } \Omega \tag{1.15}$$

and

$$\liminf_{t\to\infty} \frac{tf(x,t)}{\exp_{[l]}(b|t|^{\gamma})} > 0 \quad \text{uniformly on } \Omega. \tag{1.16}$$

Variational formulation. We define

$$J_\mu(u) = \int_\Omega (\Phi(|\nabla u|) + V(x)\Phi(|u|) - F(x,u) - \mu h(x)u)\,\mathrm{d}x, \quad u \in WL^\Phi(\Omega), \tag{1.17}$$

where $\mu \geqslant 0$ and $h \in (L^\Phi(\Omega))^*$ is assumed to be a nontrivial function. In a standard way (see for example [12, Section 6]; use (1.14) and Theorem 3.1(i) given below when dealing with the part of the functional corresponding to $F(x,u)$) it can be shown that this is a C^1-functional on $WL^\Phi(\Omega)$ and its Fréchet derivative is

$$\langle J'_\mu(u), v\rangle = \int_\Omega \Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla v + V(x)\Phi'(|u|)\frac{u}{|u|}v - f(x,u)v - \mu h(x)v\Big)\,\mathrm{d}x, \quad u, v \in WL^\Phi(\Omega), \tag{1.18}$$

where the symbol $\langle J'_\mu(u), v\rangle$ denotes the value of the linear functional $J'_\mu(u)$ of v.

We say that $u \in WL^\Phi(\Omega)$ is a weak solution to the problem (1.1) if

$$\langle J'_\mu(u), v\rangle = 0 \quad \text{for every } v \in WL^\Phi(\Omega). \tag{1.19}$$

Now, we can state our main result.

Theorem 1.1. *Let $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n-1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$. Suppose that a C^1-Young function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfies* (1.6), (1.7), (1.8), *and* (1.9) *with $\beta \in (0, \min\{1, B\})$. Let $V\colon \Omega \to \mathbb{R}$ satisfy* (1.11) *and let $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a function satisfying* (1.12), (1.13), (1.14), (1.15), *and* (1.16). *Let $h \in (L^\Phi(\Omega))^*$ be a nontrivial continuous function. Then there is $\mu_0 > 0$ with the following property:*
If $\mu \in [0, \mu_0)$, then the problem (1.1) *has at least two distinct weak solutions in $WL^\Phi(\Omega)$. Moreover, if $\mu \in (0, \mu_0)$, then all the weak solutions are nontrivial. If $\mu = 0$, then there is a trivial weak solution (included in the above multiplicity result).*

The paper is organized as follows. After Preliminaries we focus on the generalized Moser-Trudinger inequality. The fourth section is devoted to the proof of the fact that the functional J_μ satisfies the assumptions of the Mountain Pass Theorem. The properties of the Palais-Smale sequences are given in the fifth section. Finally, in the sixth section we apply the Mountain Pass Theorem and the Ekeland Variational

Principle to obtain two convergent Palais-Smale sequences. Then we show that the limit functions are distinct provided $\mu \in (0, \mu_0)$.

For the convenience of the reader acquainted with the article [8], our paper is organized in a similar way, we use the same strategies of the proofs when possible and we also use the same notation.

2. Preliminaries

By χ_A we mean the characteristic function of $A \subset \mathbb{R}^n$. By $B(x_0, R)$ we denote the open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with radius $R > 0$. If $x_0 = 0$ we simply write $B(R)$.

For two functions $g, h\colon [0,\infty) \mapsto [0,\infty)$ we write $g \lesssim h$ if there is $C > 0$ such that $g(t) \leqslant Ch(t)$ for every $t \in [0,\infty)$. If u is a measurable function on A, then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on A. If $u = 0$ (or $u \neq 0$), we call it trivial (or nontrivial).

By C we denote a generic positive constant which may depend on l, n, α, and Φ. This constant may vary from expression to expression as usual. The symbol $\mathrm{o}(1)$ stands for a sequence indexed by $k \in \mathbb{N}$ and converging to zero as $k \to \infty$.

By $\mathcal{M}(A)$ we denote the set of all Radon measures on a compact set A. We write that $\nu_k \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(A)$ if $\int_A \psi \,\mathrm{d}\nu_k \to \int_A \psi \,\mathrm{d}\nu$ for every $\psi \in C(A)$.

Properties of $\exp_{[l]}$**.** For given $l \in \mathbb{N}$ and $p \geqslant 1$, one can easily prove that there is $C \geqslant 1$ such that

$$\exp_{[l]}^p(t) \leqslant C \exp_{[l]}(pt) \quad \text{for all } t \geqslant 0. \tag{2.1}$$

Young functions and Orlicz spaces. A function $\Phi\colon [0,\infty) \to [0,\infty)$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} (\Phi(t)/t) = \infty$.

Denote by $L^\Phi(A, \mathrm{d}\nu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure ν. If $\nu = \mathcal{L}_n$ we simply write $L^\Phi(A)$. The space $L^\Phi(A, \mathrm{d}\nu)$ is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(A,\mathrm{d}\mu)} = \inf\left\{\lambda > 0\colon \int_A \Phi\Bigl(\frac{|u(x)|}{\lambda}\Bigr)\,\mathrm{d}\nu(x) \leqslant 1\right\}. \tag{2.2}$$

By Ψ we denote the associated Young function to Φ. The dual space to $L^\Phi(A, \mathrm{d}\nu)$ can be identified as the Orlicz space $L^\Psi(A, \mathrm{d}\nu)$. We further have the generalized Hölder's inequality

$$\int_A |u(y)v(y)|\,\mathrm{d}\nu(y) \leqslant 2\|u\|_{L^\Phi(A,\mathrm{d}\nu)} \|v\|_{L^\Psi(A,\mathrm{d}\nu)}. \tag{2.3}$$

Δ_2-condition. We say that a function Φ satisfies the Δ_2-condition if there is $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \quad \text{whenever } t \geqslant 0 \ .$$

It is not difficult to check the Δ_2-condition for our Young functions satisfying (1.6) and (1.7). Therefore, one easily proves

$$\int_A \Phi\Big(\frac{|u|}{\|u\|_{L^\Phi(A,\mathrm{d}\nu)}}\Big)\,\mathrm{d}\nu = 1 \quad \text{whenever } \|u\|_{L^\Phi(A,\mathrm{d}\nu)} > 0 \tag{2.4}$$

and

$$\|u_k\|_{L^\Phi(A,\mathrm{d}\nu)} \overset{k\to\infty}{\longrightarrow} 0 \quad \Longleftrightarrow \quad \int_A \Phi(|u_k|)\,\mathrm{d}\nu \overset{k\to\infty}{\longrightarrow} 0. \tag{2.5}$$

Similarly as in [8] we need the following results (see [8, Lemmas 2.2 and 2.4]).

Lemma 2.1. *If a Young function Φ satisfies* (1.6) *and* (1.7), *then* $\Psi(\Phi') \lesssim \Phi$.

Lemma 2.2. *Let Φ be a Young function satisfying* (1.6) *and* (1.7). *Then for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|u\|^{n+\varepsilon}_{L^\Phi(A,\mathrm{d}\nu)} \leqslant \int_A \Phi(|u|)\,\mathrm{d}\nu \leqslant \|u\|^{n-\varepsilon}_{L^\Phi(A,\mathrm{d}\nu)} \quad \textit{provided} \quad \|u\|_{L^\Phi(A,\mathrm{d}\nu)} < \delta.$$

Further, we need the Brézis-Lieb lemma from [5, Theorem 2 and Examples (b)].

Lemma 2.3. *Let $\{f_k\}$ be a sequence of ν-measurable functions on $A \subset \mathbb{R}^n$ such that $f_k \to f$ a.e. in A. Let Φ be a Young function. Suppose that $f \in L^\Phi(A,\mathrm{d}\nu)$ and $\|f_k\|_{L^\Phi(A,\mathrm{d}\nu)} \leqslant C$. Then*

$$\int_A |\Phi(|f_k|) - \Phi(|f_k - f|) - \Phi(|f|)|\,\mathrm{d}\nu \overset{k\to\infty}{\longrightarrow} 0.$$

Orlicz-Sobolev spaces. Let $A \subset \mathbb{R}^n$ be a nonempty bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$ and let Φ be a Young function satisfying (1.6). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u\colon\ u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm

$$\|u\|_{WL^{\Phi}(A)} := \|u\|_{L^{\Phi}(A)} + \|\nabla u\|_{L^{\Phi}(A)},$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

Hence, $WL^{\Phi}(A)$ is a Banach space satisfying

$$WL^{\Phi}(A) \subset L^r(A),\ r \in [1,\infty), \quad \text{and} \quad WL^{\Phi}(A) \subset L^{\Phi}(\Omega),$$

where both embeddings are compact. Moreover, bounded $C^{\infty}(A)$-functions are dense in $WL^{\Phi}(A)$. We write that $u_k \rightharpoonup u$ in $WL^{\Phi}(A)$ if

$$\int_A f_k v \,\mathrm{d}x \to \int_A f v \,\mathrm{d}x \quad \text{and} \quad \int_A \frac{\partial u_k}{\partial x_i} v \,\mathrm{d}x \to \int_A \frac{\partial u}{\partial x_i} v \,\mathrm{d}x$$
$$\text{for every } v \in L^{\Psi}(A) \text{ and } i \in \{1,\dots,n\}.$$

We denote by $W_0L^{\Phi}(A)$ the closure of $C_0^{\infty}(A)$ in $WL^{\Phi}(A)$.

Tools from the Measure Theory. We make use of the following result from [12, Lemma 2.5] (see also [14, Lemma 2.1]).

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $\theta \in [0,1)$ and let $\{u_k\}$ be a sequence of functions from $L^1(\Omega)$ converging to $u \in L^1(\Omega)$ a.e. in Ω. Let $f\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function bounded on $\Omega \times [-t_0, t_0]$ for every $t_0 > 0$. Suppose that $f(x,u_k)|u_k|^{\theta}$ and $f(x,u)|u|^{\theta}$ belong to $L^1(\Omega)$ and*

$$\int_{\Omega} |f(x,u_k)u_k| \leqslant C.$$

Then $f(x,u_k)|u_k|^{\theta} \to f(x,u)|u|^{\theta}$ in $L^1(\Omega)$.

Remark 2.5. Using the same methods as in the proof of [14, Lemma 2.1] one easily proves the following observation. Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set, $p > 1$ and let $\{v_k\}$ be a sequence of functions from $L^p(\Omega)$ converging to $v \in L^p(\Omega)$ a.e. in Ω. Suppose that

$$\int_{\Omega} |v_k|^p \leqslant C \quad \text{for every } k \in \mathbb{N}.$$

Then $v_k \to v$ in $L^q(\Omega)$ for every $q \in [1,p)$.

We use the Generalized Lebesgue Dominated Convergence Theorem (see [26, Exercise 5.4.13]).

Proposition 2.6. *Let* $\{u_k\}$, $\{v_k\}$ *be sequences of measurable functions on* $A \subset \mathbb{R}^n$ *such that* $|u_k| \leqslant v_k$ *for all* $k \in \mathbb{N}$. *Let* u *and* v *be measurable functions on* A *such that* $u_k \to u$ *a.e. in* A *and* $v_k \to v$ *a.e. in* A. *Then*

$$\lim_{k\to\infty} \int_A v_k = \int_A v \quad \Longrightarrow \quad \lim_{k\to\infty} \int_A u_k = \int_A u.$$

Tools from the Calculus of Variations. Our key instrument is the following version of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [4].

Theorem 2.7. *Let* X *be a real Banach space and* $J \in C^1(X, \mathbb{R})$. *Suppose that there exist a neighborhood* U *of* 0 *in* X *and* $\xi \in \mathbb{R}$ *satisfying the following conditions:*

(i) $J(0) < \xi$,

(ii) $J(u) \geqslant \xi$ *on the boundary of* U,

(iii) *there is* $w \notin U$ *such that* $J(w) < \xi$.

Set

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J(u) \geqslant \xi,$$

where $\Gamma = \{g \in C([0,1], X)\colon\ g(0) = 0, g(1) = w\}$. *Then there is a sequence* $\{u_k\} \subset X$ *such that*

$$J(u_k) \to c \quad \text{and} \quad J'(u_k) \to 0 \quad \text{in } X^*. \tag{2.6}$$

By $C^1(X, \mathbb{R})$ we denote the class of functionals (possibly nonlinear) on X with the continuous Fréchet derivative.

The sequence satisfying (2.6) is called the Palais-Smale sequence and the constant c is a Palais-Smale level. Notice that this version of the Mountain Pass Theorem is slightly different from that which is often used and which requires the Palais-Smale condition (the Palais-Smale sequence has a subsequence convergent in the norm) and asserts that there is a critical point $x_0 \in X$ satisfying $J(x_0) = c$. The reason is that we need a bit less from the Palais-Smale sequence than the convergence in the norm. Our approach is taken from [6]. See [6, page 459] for the discussion concerning the proof of Theorem 2.7.

The second weak solution to (1.1) is obtained by the following version of the Ekeland Variational Principle [20].

Theorem 2.8. *Let X be a complete metric space and Y be its nonempty closed subset. Suppose that $\Lambda\colon Y \to \mathbb{R}$ is lower semicontinuous and bounded from below. Then for every $\delta > 0$ there is $u_\delta \in Y$ such that*

$$\Lambda(u_\delta) \leqslant \Lambda(u) + \delta \operatorname{dist}(u, u_\delta) \quad \textit{for every } u \in Y. \tag{2.7}$$

3. Generalized Trudinger embedding and generalized Moser-Trudinger inequality

On embedding into exponential and multiple exponential spaces. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$. The space $WL^n \log^\alpha L(\Omega)$, $\alpha < n-1$, of the Sobolev type, modeled on the Zygmund space $L^n \log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t (see [21] and [17]). Moreover it is shown in [17] (see also [18]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e. the space $WL^n \log^{n-1} L \log^\alpha \log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Further in the limiting case $\alpha = n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n \log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where the Young function Φ satisfies

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1,$$

the space $L^n \log^{n-1} L \log^\alpha \log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^\alpha(\log(t))} = 1,$$

and so on. For other results concerning these spaces we refer the reader to [17], [18], and [19].

On generalized Moser-Trudinger inequality. We need a version of the Moser-Trudinger inequality for the space $WL^\Phi(\Omega)$ from [10, Theorem 1.2]. First, we define the median of given measurable function $u\colon \Omega \to \mathbb{R}$ by

$$\operatorname{med}(u) = \sup\Big\{t \in \mathbb{R}\colon \mathcal{L}_n(\{x \in \Omega\colon u(x) > t\}) > \frac{\mathcal{L}_n(\Omega)}{2}\Big\}.$$

Theorem 3.1. *Let $K \geqslant 0$, $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n-1$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain from the class $C^{1,\theta}$ for some $\theta \in (0,1]$. Let Φ be a Young function satisfying* (1.6).

(i) *If $u \in WL^{\Phi}(\Omega)$, then*

$$\int_\Omega \exp_{[l]}(K|u(x)|^\gamma)\,\mathrm{d}x < \infty.$$

(ii) *If $K < K_{l,n,\alpha}(1/2)^{\gamma/n}$, $M \geqslant 0$, $u \in WL^{\Phi}(\Omega)$ with $\|\nabla u\|_{L^\Phi(\Omega)} \leqslant 1$ and $|\mathrm{med}(u)| \leqslant M$, then*

$$\int_\Omega \exp_{[l]}(K|u(x)|^\gamma)\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),K,M).$$

(iii) *If $K > K_{l,n,\alpha}(1/2)^{\gamma/n}$ and $M > 0$, then there is a smooth domain $\widetilde{\Omega} \subset \mathbb{R}^n$ such that*

$$\sup_{u\in WL^\Phi(\widetilde{\Omega}),\|\nabla u\|_{L^\Phi(\widetilde{\Omega})}\leqslant 1,|\mathrm{med}(u)|\leqslant M} \int_{\widetilde{\Omega}} \exp_{[l]}(K|u(x)|^\gamma)\,\mathrm{d}x = \infty.$$

Let us note that there is also a version of the Moser-Trudinger inequality for the space $W_0L^\Phi(\Omega)$. In such a version the borderline parameter $K_{l,n,\alpha}(1/2)^{\gamma/n}$ is replaced by $K_{l,n,\alpha}$ and moreover, it is not necessary to control the medians (see for example [8, Theorem 3.1]).

In our applications of Theorem 3.1(ii), the boundedness of medians is always ensured, since we work with sequences $\{u_k\}$ bounded in $L^\Phi(\Omega)$ and we have an estimate

$$\Phi(|\mathrm{med}(v)|)\frac{\mathcal{L}_n(\Omega)}{2} \leqslant \int_\Omega \Phi(|v|), \quad v \in L^\Phi(\Omega). \tag{3.1}$$

We make use of another version of Theorem 3.1(ii).

Proposition 3.2. *Let $l \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, and $M \geqslant 0$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain from the class $C^{1,\theta}$ for some $\theta \in (0,1]$. Let Φ be a Young function satisfying* (1.6). *Let $v \in WL^\Phi(\Omega)$ with $|\mathrm{med}(v)| \leqslant M$ and*

$$\int_\Omega \Phi(|\nabla v|) \leqslant \tilde{c} < \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}. \tag{3.2}$$

Then there is $q > 1$ independent of the choice of v such that

$$\int_\Omega \exp_{[l]}(bq|v|^\gamma) \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),\tilde{c},M).$$

P r o o f. The proof is obtained modifying the proof of Theorem 3.1(ii) (i.e. [10, Theorem 1.2(ii)]) in the same way as [12, Proposition 3.2] is obtained from the proof of [22, Theorem 1.1]. Let us recall the main ideas.

From the proof of Theorem 3.1(ii) we see that the assumption $\|\nabla u\|_{L^\Phi(\Omega)} \leqslant 1$ can be replaced by

$$\|\nabla u\|_{L^\Phi(\widetilde{\Omega})} \leqslant 1, \quad \text{where } \widetilde{\Omega} = \{x \in \Omega \colon\ |\nabla u| > G\}, \tag{3.3}$$

with $G > 0$ being a fixed arbitrarily large number (in such a case C also depends on G). Next, using (3.2), (1.6) for $t > G$ (with G very large) and the definition of the Luxemburg norm one finds $\hat{c} \in (0,1)$ such that

$$\|\nabla v\|_{L^\Phi(\widetilde{\Omega})} \leqslant C_1 := \hat{c}\Big(\frac{1}{2}\Big)^{1/n}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{1/\gamma}$$

(if G is very large, then the norm is very close to the n-th root of the modular, similarly as in the proof of [8, Lemma 4.2]). Finally, if $q > 1$ is so close to 1 that $q\hat{c}^\gamma < 1$, then we can apply the version of Theorem 3.1(ii) with the assumption (3.3), since

$$bq|v|^\gamma = bqC_1^\gamma\Big(\frac{|v|}{C_1}\Big)^\gamma = K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} q\hat{c}^\gamma\Big(\frac{|v|}{C_1}\Big)^\gamma$$

(notice that $\|\nabla(v/C_1)\|_{L^\Phi(\widetilde{\Omega})} \leqslant 1$ and $|\mathrm{med}(v/C_1)| \leqslant M/C_1$). $\square$

We also need a version of [8, Proposition 4.1].

Proposition 3.3. *Let $l \in \mathbb{N}$ and $n \geqslant 2$, $\alpha < n-1$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$. Let Φ be a Young function satisfying (1.6). Let $\{u_k\}_{k=1}^\infty \subset WL^\Phi(\Omega)$ satisfy*

$$\text{there is finite } \lim_{k\to\infty}\int_\Omega \Phi(|\nabla u_k|), \quad u_k \rightharpoonup u \text{ in } WL^\Phi(\Omega) \quad \text{and} \quad \nabla u_k \to \nabla u \text{ a.e. in } \Omega$$

for some $u \in WL^\Phi(\Omega)$. Then for every

$$p < P := \Big(\lim_{k\to\infty}\int_\Omega \Phi(|\nabla u_k|) - \int_\Omega \Phi(|\nabla u|)\Big)^{-\gamma/n}$$

(where we define $P = \infty$ if the difference in the brackets is zero) we have

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p|u_k|^\gamma\Big) \quad \text{is bounded in } L^1(\Omega).$$

P r o o f. We can use the proof of [8, Proposition 4.1] with the following changes. Instead of the Moser-Trudinger inequality for the space $W_0L^\Phi(\Omega)$, we use the version of the Moser-Trudinger inequality given by Theorem 3.1(i) and (ii) (with (3.3) instead of $\|\nabla u\|_{L^\Phi(\Omega)} \leqslant 1$). Notice that the boundedness of $\operatorname{med}(u_k - u)$ easily follows from the weak convergence in $WL^\Phi(\Omega)$ and (3.1). The constant $K_{l,n,\alpha}$ is always replaced by $K_{l,n,\alpha}(1/2)^{\gamma/n}$. $\square$

4. The geometry of the functional J_μ

In this section we check that our functional J_μ has the Mountain Pass Geometry (i.e. assumptions (i), (ii), and (iii) from Theorem 2.7 are satisfied).

Lemma 4.1. *If $u \in WL^\Phi(\Omega)$ is such that $u \geqslant 0$ and $u \neq 0$, then*

$$J_\mu(tu) \overset{t\to\infty}{\longrightarrow} -\infty.$$

Moreover, this convergence is uniform with respect to μ taken from a bounded set.

Proof. Since $u \neq 0$ and $u \geqslant 0$, there is $\tau > 0$ such that

$$\mathcal{L}_n(\{u \geqslant \tau\}) \geqslant \tau.$$

Moreover, we observe that it follows from (1.12) and (1.13) that there is $C_1 > 0$ such that

$$F(x,t) \geqslant C_1 \exp(C_1 |t|^{1/M}), \quad |t| \geqslant t_M.$$

Thus, by (1.12) we have a similar inequality on $[\tau, \infty)$ with a constant $C_2 > 0$

$$F(x,t) \geqslant C_2 \exp(C_1 t^{1/M}) \quad \text{for } t \in [\tau, \infty).$$

Further, for every $t > 1$ we can find $m \in \mathbb{N}$ such that $2^m < t \leqslant 2^{m+1}$. Thus using the Δ_2-condition, the above estimates, $\int_\Omega |hu| \leqslant C$ and the fact that Φ and $F(x,\cdot)$ are increasing on $[0,\infty)$, we obtain

$$\begin{aligned} J_\mu(tu) &= \int_\Omega (\Phi(t|\nabla u|) + V(x)\Phi(t|u|) - F(x,tu) - \mu t h(x) u) \\ &\leqslant \int_\Omega (\Phi(2^{m+1}|\nabla u|) + V_1 \Phi(2^{m+1}|u|) + \mu 2^{m+1}|h(x)u|) - \int_{\{u \geqslant \tau\}} F(x, 2^m u) \\ &\leqslant C_\Delta^{m+1} \int_\Omega (\Phi(|\nabla u|) + V_1 \Phi(|u|)) + C\mu 2^{m+1} - \tau C_2 \exp(C_1 2^{m/M} \tau^{1/M}) \\ &\qquad \overset{m\to\infty}{\longrightarrow} -\infty. \end{aligned}$$

$\square$

Lemma 4.2. *There is $\mu_0 > 0$ such that for every $\mu \in [0, \mu_0)$ there is $\varrho_\mu > 0$ satisfying*

$$\inf_{\|u\|_{WL^\Phi(\Omega)}=\varrho_\mu} J_\mu(u) > 0$$

(i.e. there is $\xi_\mu > 0$ such that if $u \in WL^\Phi(\Omega)$ with $\|u\|_{WL^\Phi(\Omega)} = \varrho_\mu$, then $J_\mu(u) \geqslant \xi_\mu$) and

$$c_0 = c_0(\mu) := \inf_{\|u\|_{WL^\Phi(\Omega)}\leqslant\varrho_\mu} J_\mu(u)$$

has the following properties:

If $\mu = 0$, then we have $c_0 = 0$.

If $\mu \in (0, \mu_0)$, then the constant $\varrho_\mu > 0$ is chosen so that $\varrho_\mu \to 0_+$ as $\mu \to 0_+$ and

$$c_0 \geqslant C(\mu, \varrho_\mu), \quad \text{where } C(\mu, \varrho_\mu) \overset{\mu\to 0_+}{\longrightarrow} 0_-.$$

Proof. Fix $q > n$. By the assumptions (1.13), (1.14), and (1.15) we can find $\eta > 0$ so that

$$F(x,t) \leqslant (1-2\eta)V_0\Phi(|t|) + C\exp_{[l]}(b|t|^\gamma)|t|^q = F_1(t) + F_2(t).$$

By (1.11) we obtain

$$\int_\Omega F_1(u) = (1-2\eta)V_0\int_\Omega \Phi(|u|) \leqslant (1-2\eta)\int_\Omega(\Phi(|\nabla u|) + V(x)\Phi(|u|)). \tag{4.1}$$

Next, fix $p > 1$. If ϱ is so small that $bp\varrho^\gamma < K_{l,n,\alpha}(1/2)^{\gamma/n}$, then from Hölder's inequality, Theorem 3.1(ii) (the medians of $(1/\varrho)u$ are bounded, see (3.1)), (2.1), the fact that $WL^\Phi(\Omega)$ is continuously embedded into $L^r(\Omega)$, for every $r \in [1,\infty)$, and from the equivalence of the norms $\|\cdot\|_{L^\Phi(\Omega,V(x)\,\mathrm{d}x)}$ and $\|\cdot\|_{L^\Phi(\Omega)}$ (see (1.11)), for every $u \in WL^\Phi(\Omega)$ such that $\|u\|_{WL^\Phi(\Omega)} = \varrho$ we obtain

$$\begin{aligned}\int_\Omega F_2(u) &= C\int_\Omega \exp_{[l]}(b|u|^\gamma)|u|^q \\ &\leqslant C\Big(\int_\Omega \exp_{[l]}\Big(bp\varrho^\gamma\Big(\frac{|u|}{\varrho}\Big)^\gamma\Big)\Big)^{1/p}\Big(\int_\Omega |u|^{qp'}\Big)^{1/p'} \\ &\leqslant C\|u\|^q_{L^{qp'}}(\Omega) \leqslant C\|u\|^q_{WL^\Phi(\Omega)} \leqslant C\|\nabla u\|^q_{L^\Phi(\Omega)} + C\|u\|^q_{L^\Phi(\Omega,V(x)\,\mathrm{d}x)}.\end{aligned}$$

Hence, for $\varrho > 0$ small enough Lemma 2.2 with $\varepsilon \in (0, q-n)$ gives

$$\begin{aligned}\int_\Omega F_2(u) &\leqslant C\|\nabla u\|^{q-n-\varepsilon}_{L^\Phi(\Omega)}\|\nabla u\|^{n+\varepsilon}_{L^\Phi(\Omega)} + C\|u\|^{q-n-\varepsilon}_{L^\Phi(\Omega,V(x)\,\mathrm{d}x)}\|u\|^{n+\varepsilon}_{L^\Phi(\Omega,V(x)\,\mathrm{d}x)} \\ &\leqslant \eta\int_\Omega(\Phi(|\nabla u|) + V(x)\Phi(|u|)).\end{aligned} \tag{4.2}$$

Thus, we obtain from (4.1) and (4.2) and the generalized Hölder's inequality

$$\begin{aligned}J_\mu(u) &= \int_\Omega (\Phi(|\nabla u|) + V(x)\Phi(|u|) - F(x,u) - \mu h(x)u) \\ &\geqslant \eta \int_\Omega (\Phi(|\nabla u|) + V(x)\Phi(|u|)) - 2\mu \|h\|_{L^\Psi(\Omega)} \|u\|_{L^\Phi(\Omega)}.\end{aligned}$$

Next $\|u\|_{WL^\Phi(\Omega)} = \varrho$ implies that $\|\nabla u\|_{L^\Phi(\Omega)} \geqslant \varrho/2$ or $\|u\|_{L^\Phi(\Omega)} \geqslant \varrho/2$ (thus, $\|\nabla u\|_{L^\Phi(\Omega, V(x)\,\mathrm{d}x)} \geqslant \varrho/C_1$). Hence, Lemma 2.2 with $\varepsilon = 1$ and $\|u\|_{L^\Phi(\Omega)} \leqslant \|u\|_{WL^\Phi(\Omega)}$ imply for all $\varrho > 0$ small enough that

$$J_\mu(u) \geqslant \eta \Big(\frac{\varrho}{C_1}\Big)^{n+1} - 2\mu \|h\|_{L^\Psi(\Omega)} \varrho.$$

Now, if $\mu = 0$, then we conclude easily. Otherwise we set

$$\varrho_\mu = C_1 \Big(\frac{4C_1 \|h\|_{L^\Psi(\Omega)} \mu}{\eta}\Big)^{1/n}.$$

We plainly have $\varrho_\mu \to 0_+$ as $\mu \to 0_+$. Furthermore, if $\|u\|_{WL^\Phi(\Omega)} = \varrho_\mu$, then

$$J_\mu(u) \geqslant \eta \Big(\frac{4C_1 \|h\|_{L^\Psi(\Omega)} \mu}{\eta}\Big) \frac{\varrho_\mu}{C_1} - 2\mu \|h\|_{L^\Psi(\Omega)} \varrho_\mu = 2\mu \|h\|_{L^\Psi(\Omega)} \varrho_\mu =: \xi_\mu > 0,$$

while for every $u \in WL^\Phi(\Omega)$ satisfying $\|u\|_{WL^\Phi(\Omega)} \leqslant \varrho_\mu$ we have

$$J_\mu(u) \geqslant -2\mu \|h\|_{L^\Psi(\Omega)} \varrho_\mu \quad \text{and} \quad -2\mu \|h\|_{L^\Psi(\Omega)} \varrho_\mu \overset{\mu \to 0_+}{\longrightarrow} 0_-.$$

$\square$

Lemma 4.3. *There is $v \in W_0 L^\Phi(\Omega)$ with $\|v\|_{WL^\Phi(\Omega)} = 1$ such that for every $\mu > 0$ there is $t_\mu > 0$ with the following property: For every $t \in (0, t_\mu)$ we have $J_\mu(tv) < 0$.*

In particular, for every $\mu \in (0, \mu_0)$ (where μ_0 comes from Lemma 4.2) we have $c_0 < 0$.

Proof. Since h is continuous and nontrivial, we obtain an open set $G \subset \Omega$ such that h is bounded away from zero on G. We can easily construct a nontrivial $W_0 L^\Phi(\Omega)$-function $\tilde v$ supported on G with the same sign as h has on G. Further we can assume that $\tilde v$ and $\nabla \tilde v$ are bounded. Normalizing suitably, we get $v \in W_0 L^\Phi(\Omega)$ such that $\|v\|_{WL^\Phi(\Omega)} = 1$ and

$$\int_\Omega hv = \int_G hv = C_1 > 0.$$

Finally, as F is nonnegative, using the above construction, (1.7) and (1.11), we obtain for $t > 0$ small enough

$$\begin{aligned} J_\mu(tv) &= \int_\Omega (\Phi(t|\nabla v|) + V(x)\Phi(t|v|) - F(x,tv) - \mu t h(x) v) \\ &\leqslant Ct^n \int_\Omega (|\nabla v|^n + V_1|v|^n) - \mu C_1 t = Ct^n - C_1 \mu t \end{aligned}$$

and we conclude the proof easily. □

Upper estimate of the Palais-Smale level. In the rest of this section we show that the Palais-Smale level is not too high. Fix $x_0 \in \mathbb{R}^n$ and $R > 0$. We make use of the concentrating sequences of $W_0 L^\Phi(B(x_0,R))$-functions from [7] and [13] (these sequences are also used in [10] when showing that we cannot have $K = K_{l,n,\alpha}(1/2)^{\gamma/n}$ in Theorem 3.1(ii)). For $l = 1$ we set

$$w_k(x) = g_k(|x - x_0|), \tag{4.3}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R,\infty), \\ \Big(-\dfrac{2}{R}y + 2\Big) K_{1,n,\alpha}^{-1/\gamma} n^B \log^B(2) k^{1/\gamma - B}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in \Big[\dfrac{R}{2}, R\Big], \\ K_{1,n,\alpha}^{-1/\gamma} n^B \log^B\Big(\dfrac{R}{y}\Big) k^{1/\gamma - B}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \text{for } y \in \Big[R\mathrm{e}^{-k/n}, \dfrac{R}{2}\Big], \\ K_{1,n,\alpha}^{-1/\gamma} k^{1/\gamma}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \text{for } y \in [0, R\mathrm{e}^{-k/n}]. \end{cases}$$

In the case $l \geqslant 2$ we fix $T > \exp_{[l]}(1)$ and we define

$$w_k(x) = g_k(|x - x_0|), \tag{4.4}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R,\infty), \\ \Big(-\dfrac{2}{R}y + 2\Big) K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^B(T+2) k^{1/\gamma - B}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in \Big[\dfrac{R}{2}, R\Big], \\ K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^B\Big(T + \dfrac{R}{y}\Big) k^{1/\gamma - B}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in \Big[R\exp_{[l]}^{-1/n}(k), \dfrac{R}{2}\Big], \\ K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^B(T + \exp_{[l]}^{1/n}(k)) k^{1/\gamma - B}\Big(1 + \dfrac{\log(k)}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in [0, R\exp_{[l]}^{-1/n}(k)]. \end{cases}$$

We need the following estimate.

Lemma 4.4. *Let $x_0 \in \partial\Omega$, $R > 0$ and w_k be defined by (4.3) and (4.4), respectively. Then for every $\vartheta_0 > 0$ there is $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$ and $\vartheta \in [0, \vartheta_0]$ we have*

$$\int_\Omega \Phi(\vartheta|\nabla w_k|) + V(x)\Phi(\vartheta|w_k|) \leqslant \frac{1}{2}\vartheta^n. \tag{4.5}$$

Proof. The proof is obtained modifying the proofs of [7, Example 5.1] (the case of $l = 1$) and [13, Theorem 4.1] (the case of $l \geqslant 2$), respectively, and adding some rough estimates.

Without loss of generality, we can suppose that $x_0 = 0$. As $\partial\Omega$ is of class $C^{1,\theta}$, $\theta \in (0, 1]$, we can also suppose that there exists $R_0 \in (0, R)$ and a $C^{1,\theta}$-function $\eta\colon B_{R_0}(0) \cap \{x_n = 0\} \to \mathbb{R}$ such that

$$\begin{gathered} \eta(0) = 0, \quad \nabla\eta(0) = 0, \\ \Omega \cap B_{R_0}(0) = \{(x', x_n) \in B_{R_0}(0)\colon\ \eta(x') < x_n\}, \end{gathered}$$

and

$$\partial\Omega \cap B_{R_0}(0) = \{(x', x_n) \in B_{R_0}(0)\colon\ \eta(x') = x_n\},$$

where $(x_1, x_2, \ldots, x_n) = (x', x_n)$. Further, for each $r > 0$ we denote $B_+(r) = B(r) \cap \{x_n > 0\}$, $B_-(r) = B(r) \cap \{x_n < 0\}$. For every $k \in \mathbb{N}$ we set $M = M(k) = R/k^{\log(k)}$. We have

$$\int_\Omega \Phi(\vartheta|\nabla w_k|) + V(x)\Phi(\vartheta|w_k|) = \int_\Omega \Phi(\vartheta|\nabla w_k|) + \int_\Omega V(x)\Phi(\vartheta|w_k|) = J_1 + J_2.$$

Next, if k is large enough, we can write

$$J_1 \leqslant \int_{\Omega\cap B_-(M)} \ldots + \int_{B_+(M)} \ldots + \int_{B(\frac{R}{2})\setminus B(M)} \ldots + \int_{B(R)\setminus B(\frac{R}{2})} \ldots = I_0 + I_1 + I_2 + I_3.$$

Fix $\varepsilon \in (0, B - \beta)$ (recall that $\beta \in (0, \min\{1, B\})$ comes from (1.9)).

Now, we distinguish two cases. First, let $l = 1$. By an easy modification of the corresponding estimates in the proof of [7, Example 5.1] (it is essential that ϑ is bounded from above) we have that

$$\begin{gathered} I_3 \leqslant C\vartheta^n k^{-B}, \qquad I_2 \leqslant C\vartheta^n k^{-B} \log^{2Bn+4}(k) \\ \text{and} \quad I_1 = \frac{1}{2}\int_{B(M)} \Phi(\vartheta|\nabla w_k|) \leqslant \frac{1}{2}\vartheta^n\Big(1 - \frac{1}{k^{\beta+\varepsilon}}\Big). \end{gathered} \tag{4.6}$$

Next, as ϑ is bounded, $(1/\gamma - B)n = -B$ (see (1.5)) and we have

$$(4.7)\quad |g_k'(y)| \leqslant C \log^{B-1}\Big(\frac{R}{y}\Big)\frac{1}{y}k^{1/\gamma - B} \quad \text{on } (R\mathrm{e}^{-k/n}, M) \text{ for } k \text{ sufficiently large,}$$

$$M \overset{k\to\infty}{\longrightarrow} 0,$$

$$\eta(x') \leqslant C|x'|^{1+\theta} \quad \text{for } |x'| \text{ sufficiently small}$$

and

$$\Phi(t) \leqslant t^n(1+\log^n(t)) \quad \text{for every } t \geqslant 0,$$

we obtain for $k \geqslant k_1$, where k_1 is large enough

$$\begin{aligned}(4.8)\qquad I_0 &= \int_{\Omega\cap B_-(M(k))} \Phi(\vartheta|\nabla w_k|)\,\mathrm{d}x \\ &\leqslant C\int_{R\mathrm{e}^{-k/n}}^{M(k)} |\vartheta g_k'(y)|^n(1+\log^n(|\vartheta g_k'(y)|))y^{n-1+\theta}\,\mathrm{d}y \\ &\leqslant C\vartheta^n k^{(1/\gamma-B)n}\int_{R\mathrm{e}^{-k/n}}^{M(k)} \log^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(1+\log^n(k)+\log^n\Big(\frac{1}{y}\Big)\Big)y^{\theta-1}\,\mathrm{d}y \\ &\leqslant C\vartheta^n k^{(1/\gamma-B)n}\log^n(k)\int_0^{M(k_1)} \log^{(B-1)n}\Big(\frac{R}{y}\Big)\Big(1+\log^n\Big(\frac{1}{y}\Big)\Big)y^{\theta-1}\,\mathrm{d}y \\ &= C\vartheta^n k^{-B}\log^n(k).\end{aligned}$$

Hence, since $\beta+\varepsilon < B$, (4.6) and (4.8) yield for k large enough

$$(4.9)\qquad J_1 \leqslant \frac{1}{2}\vartheta^n\Big(1 - \frac{1}{2k^{\beta+\varepsilon}}\Big).$$

For $l \geqslant 2$, (4.9) is obtained in a similar way. Indeed, by a minor modification of the proof of [13, Theorem 4.1] we obtain a version of (4.6) (with a bit different power of $\log(k)$ in the estimate concerning I_2). When estimating I_0, the formula (4.7) becomes a bit more complicated, however thanks to $\theta > 0$ the last integral in (4.8) is still finite. Moreover, the power of k is $-B$ again.

Finally, we estimate J_2. Given $p > 0$ we fix $q \in (0,1)$. Now, we define a sequence of auxiliary radii by $R_k = R\exp_{[l]}^{-1}(k^q)$, $k \in \mathbb{N}$. Notice that for k large enough R_k satisfies

$$R\exp_{[l]}^{-1/n}(k) \leqslant R_k \leqslant \frac{R}{2}.$$

Thus, we observe from (4.3) and (4.4), respectively, that for k large enough we have

$$(4.10)\qquad \int_{B(R_k)} |w_k|^p \leqslant \mathcal{L}_n(B(R_k))g_k^p(0) = C\exp_{[l]}^{-n}(k^q)k^{p/\gamma}$$

and

$$\text{(4.11)} \qquad \int_{B(R/2)\setminus B(R_k)} |w_k|^p \leqslant \mathcal{L}_n\Big(B\Big(x_0, \frac{R}{2}\Big)\Big) g_k^p(R_k)$$
$$\leqslant C \log_{[l]}^{Bp}(\exp_{[l]}(k^q)) k^{(1/\gamma - B)p} = C k^{Bpq + (1/\gamma - B)p}.$$

Since on $B(R)\setminus B(R/2)$ we trivially have $|w_k| \leqslant C k^{1/\gamma - B}$, (4.10) and (4.11) yield

$$\text{(4.12)} \qquad \int_{B(R)} |w_k|^p \leqslant C k^{Bpq + (1/\gamma - B)p}.$$

Now, as $\Phi(t) \leqslant Ct^n + Ct^{n+1}$ (by (1.6) and (1.7)), $V(x)$ is bounded (by (1.11)), ϑ is bounded, $Bq > 0$, $1/\gamma - B < 0$ and $(B - 1/\gamma)n = B$ (by (1.5)), from (4.12) we obtain

$$\text{(4.13)} \qquad \int_{B(R)} V(x)\Phi(|\vartheta w_k|) \leqslant C\vartheta^n k^{B(n+1)q + (1/\gamma - B)n} = C\vartheta^n k^{B(n+1)q - B}.$$

Finally, we see that if q and ε are small enough, then (4.9), (4.13) and $\beta < B$ imply (4.5). $\square$

Lemma 4.5. *If $\mu_0 > 0$ is small enough, then there is $w \in WL^\Phi(\Omega)$ such that*

$$J_\mu(tw) < c_0 + \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma} \quad \textit{for every } t \in [0,\infty) \textit{ and } \mu \in [0,\mu_0).$$

P r o o f. First, we prove the assertion for $\mu = 0$. Since, by Lemma 4.2, in this case $c_0 = 0$, our aim is to prove that there are $\varepsilon > 0$ and a function $w \in WL^\Phi(\Omega)$ such that for every $t \in [0,\infty)$ we have

$$\text{(4.14)} \qquad J_0(tw) \leqslant \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma} - 2\varepsilon.$$

To do this, fix $x_0 \in \partial\Omega$ and $R > 0$. Suppose that $x_0 = 0$ to simplify our notation. By (1.16) we can find $C > 0$ satisfying

$$\text{(4.15)} \qquad \liminf_{t\to\infty} \frac{tf(x,t)}{\exp_{[l]}(b|t|^\gamma)} > C \quad \text{uniformly with respect to } x \in \Omega.$$

Our aim is to show that there is $k \in \mathbb{N}$ such that the assertion of the lemma holds for w_k given by (4.3) and (4.4), respectively. Aiming at contradiction suppose that for all $k \in \mathbb{N}$ we have

$$\sup_{\vartheta \geqslant 0} J_0(\vartheta w_k) \geqslant \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}.$$

In view of Lemma 4.1 there are $\vartheta_k > 0$, $k \in \mathbb{N}$, such that

$$J_0(\vartheta_k w_k) = \max_{\vartheta \geqslant 0} J_0(\vartheta w_k).$$

Since F is nonnegative (see (1.12)), we arrive at

$$\int_\Omega \Phi(\vartheta_k|\nabla w_k|) + V(x)\Phi(\vartheta_k|w_k|) \geqslant J_0(\vartheta_k w_k) \geqslant \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}. \tag{4.16}$$

Now, we claim that ϑ_k are bounded away from zero. Indeed, for $k \in \mathbb{N}$ such that $\vartheta_k \leqslant 1$ and large enough so that (4.5) holds with $\vartheta = 1$, we have by (4.16), (4.5) and by the fact that Φ is a Young function (hence $\Phi(ts) \leqslant t\Phi(s)$ for every $t \in [0,1]$ and $s \geqslant 0$)

$$\begin{aligned}\frac{1}{2}\vartheta_k &\geqslant \vartheta_k \int_\Omega \Phi(|\nabla w_k|) + V(x)\Phi(|w_k|)\\ &\geqslant \int_\Omega \Phi(\vartheta_k|\nabla w_k|) + V(x)\Phi(\vartheta_k|w_k|) \geqslant \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}.\end{aligned}$$

Further, as $\frac{\mathrm{d}}{\mathrm{d}\vartheta} J_0(\vartheta w_k)|_{\vartheta=\vartheta_k} = 0$, it follows that

$$\int_\Omega \Phi'(\vartheta_k|\nabla w_k|)|\nabla w_k| + V(x)\Phi'(\vartheta_k|w_k|)|w_k| = \int_\Omega w_k f(x, \vartheta_k w_k).$$

Multiplying both sides by ϑ_k, using (1.10), (4.15) (recall that ϑ_k are bounded away from zero), the fact that

$$\mathcal{L}_n(\Omega \cap B(R\exp_{[l]}^{-1/n}(k))) \geqslant C\mathcal{L}_n(B(R\exp_{[l]}^{-1/n}(k))) = C\exp_{[l]}^{-1}(k) \quad \text{for } k \text{ large enough}$$

and the definition of w_k we obtain $k_1 \geqslant k_0$ such that for $k \geqslant k_1$ we have

$$\begin{aligned}&\int_\Omega \Phi(\vartheta_k|\nabla w_k|) + V(x)\Phi(\vartheta_k|w_k|)\\ &\quad\geqslant c_\Phi \int_\Omega \Phi'(\vartheta_k|\nabla w_k|)\vartheta_k|\nabla w_k| + V(x)\Phi'(\vartheta_k|w_k|)\vartheta_k|w_k|\\ &\quad= c_\Phi \int_\Omega \vartheta_k w_k f(x, \vartheta_k w_k)\\ &\quad\geqslant c_\Phi \int_{\Omega\cap B(R\exp_{[l]}^{-1/n}(k))} \vartheta_k w_k f(x, \vartheta_k w_k)\\ &\quad\geqslant C \int_{\Omega\cap B(R\exp_{[l]}^{-1/n}(k))} \exp_{[l]}(b|\vartheta_k w_k|^\gamma)\\ &\quad= C\exp_{[l]}^{-1}(k)\exp_{[l]}(b|\vartheta_k w_k(0)|^\gamma).\end{aligned} \tag{4.17}$$

Next, if $l = 1$, then (4.3) gives

$$\exp_{[l]}(b|\vartheta_k w_k(0)|^\gamma) = \exp\Big(\frac{b\vartheta_k^\gamma}{K_{1,n,\alpha}}(k+\log(k))\Big),$$

and if $l \geqslant 2$, then we have by [13, Proof of Theorem 4.1] for k large enough

$$\Big(\frac{\log_{[l]}(\exp_{[l]}^{1/n}(k))}{k}\Big)^{B\gamma}(k+\log(k)) \geqslant k + \frac{1}{2}\log(k)$$

which together with (4.4) implies

$$\exp_{[l]}(b|\vartheta_k w_k(0)|^\gamma) \geqslant \exp_{[l]}\Big(\frac{b\vartheta_k^\gamma}{K_{l,n,\alpha}}\Big(k+\frac{1}{2}\log(k)\Big)\Big).$$

Therefore, there exists $k_2 \geqslant k_1$ so that for $k \geqslant k_2$ we infer from (4.17)

$$\begin{aligned}\int_\Omega \Phi(\vartheta_k|\nabla w_k|) &+ V(x)\Phi(\vartheta_k|w_k|) \\ &\geqslant C\exp_{[l]}^{-1}(k)\exp_{[l]}\Big(\frac{b\vartheta_k^\gamma}{K_{l,n,\alpha}}\Big(k+\frac{1}{2}\log(k)\Big)\Big).\end{aligned} \tag{4.18}$$

Now, for each $k \in \mathbb{N}$ satisfying $\vartheta_k \geqslant 2$ let us find $a_k \in \mathbb{N}$ such that $\vartheta_k \in [2^{a_k}, 2^{a_k+1})$. Therefore, the Δ_2-condition, (4.5) (for $\vartheta = 1$) and (4.18) give us $k_3 \geqslant k_2$ such that for every $k \geqslant k_3$ satisfying $\vartheta_k \geqslant 2$ we have

$$\begin{aligned}CC_\Delta^{a_k} &\geqslant \frac{1}{2}C_\Delta^{a_k+1} \\ &\geqslant C_\Delta^{a_k+1}\int_\Omega \Phi(|\nabla w_k|) + V(x)\Phi(|w_k|) \geqslant \int_\Omega \Phi(\vartheta_k|\nabla w_k|) + V(x)\Phi(\vartheta_k|w_k|) \\ &\geqslant C\exp_{[l]}^{-1}(k)\exp_{[l]}\Big(\frac{b}{K_{l,n,\alpha}}2^{a_k\gamma}\Big(k+\frac{1}{2}\log(k)\Big)\Big).\end{aligned}$$

Therefore, the a_k are bounded and thus there is $\vartheta_0 \geqslant 1$ such that $\vartheta_k \leqslant \vartheta_0$ for every $k \in \mathbb{N}$.

Hence, there is $k_4 \geqslant k_3$ so that for every $k \geqslant k_4$ we have a version of the estimate (4.5) with $\vartheta = \vartheta_k$. This has the following consequences. First, (4.16) and (4.5) give

$$\vartheta_k \geqslant \Big(\frac{K_{l,n,\alpha}}{b}\Big)^{1/\gamma} \quad \text{for } k \geqslant k_4. \tag{4.19}$$

Second, (4.5), (4.18) and (4.19) imply

$$\begin{aligned}C = \frac{1}{2}\vartheta_0^n \geqslant \frac{1}{2}\vartheta_k^n &\geqslant \int_\Omega \Phi(\vartheta_k|\nabla w_k|) + V(x)\Phi(\vartheta_k|w_k|) \\ &\geqslant C\exp_{[l]}^{-1}(k)\exp_{[l]}\Big(k+\frac{1}{2}\log(k)\Big) \xrightarrow{k\to\infty} \infty.\end{aligned}$$

Thus, we have a contradiction and we have proved (4.14).

Finally, since $\int_\Omega |hw| \leqslant C$, using Lemma 4.1, (1.17) and (4.14) we obtain for every $\mu > 0$ small enough

$$\max_{t\geqslant 0} J_\mu(tw) \leqslant \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma} - \varepsilon.$$

Moreover, by Lemma 4.2 we can guarantee that $c_0 > -\varepsilon$ provided $\mu > 0$ is small enough, and the result follows. $\square$

5. Properties of the Palais-Smale sequence

In this section we study the properties of the Palais-Smale sequence corresponding to the functional J_μ. Our main aim is to show that it contains a subsequence with the gradients converging a.e. in Ω (see Lemma 5.2) and that the limit (in the sense of (5.10)) is a weak solution to the problem (1.1) (see Lemma 5.3).

Let $\{u_k\}$ be a Palais-Smale sequence from $WL^\Phi(\Omega)$, that is by (2.6),

$$J_\mu(u_k) = \int_\Omega \Big(\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) - F(x,u_k) - \mu h(x)u_k\Big) \overset{k\to\infty}{\longrightarrow} c, \tag{5.1}$$

and by (1.18) there are $\varepsilon_k \to 0$ such that for every $v \in WL^\Phi(\Omega)$ we have

$$\begin{aligned}|\langle J'_\mu(u_k), v\rangle| = \bigg|\int_\Omega \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla v + V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}v \\ - f(x,u_k)v - \mu h(x)v\Big)\bigg| \leqslant \varepsilon_k\|v\|_{WL^\Phi(\Omega)}.\end{aligned} \tag{5.2}$$

Lemma 5.1. *There is a constant $C > 0$ independent of $k \in \mathbb{N}$ such that*

$$\|\nabla u_k\|_{L^\Phi(\Omega)} \leqslant C, \quad \int_\Omega \Phi(|\nabla u_k|) \leqslant C, \tag{5.3}$$

$$\|u_k\|_{L^\Phi(\Omega)} \leqslant C, \quad \int_\Omega \Phi(|u_k|) \leqslant C, \tag{5.4}$$

and

$$0 \leqslant \int_\Omega f(x,u_k)u_k \leqslant C. \tag{5.5}$$

Proof. Using (1.6) and (1.7), it can be easily shown that there is $\lambda_0 > 0$ large enough so that

$$\Phi(\lambda t) \geqslant \lambda^{n-1/2}\Phi(t) \quad \text{for every } t \geqslant 0, \lambda \geqslant \lambda_0. \tag{5.6}$$

From (1.13) we see that for every $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$F(x,t) \leqslant \varepsilon f(x,t)t, \quad |t| \geqslant t_\varepsilon. \tag{5.7}$$

We obtain from (5.1), (5.7), (5.2) with $v = u_k$ and (1.10)

$$\begin{aligned}
&\int_\Omega (\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|)) \\
&\quad \leqslant C + \int_\Omega F(x,u_k) + \int_\Omega \mu h(x) u_k \leqslant C_\varepsilon + \varepsilon \int_\Omega f(x,u_k)u_k + \int_\Omega \mu h(x)u_k \\
&\quad \leqslant C_\varepsilon + \varepsilon\biggl(\int_\Omega (\Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k| - \mu h(x)u_k) \\
&\qquad + \varepsilon_k \|u_k\|_{WL^\Phi(\Omega)}\biggr) + \int_\Omega \mu h(x)u_k \\
&\quad \leqslant C_\varepsilon + \frac{\varepsilon}{c_\Phi}\int_\Omega (\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|)) \\
&\qquad + \varepsilon\varepsilon_k \|u_k\|_{WL^\Phi(\Omega)} + (1-\varepsilon)\int_\Omega \mu h(x)u_k.
\end{aligned} \tag{5.8}$$

Next, as $h \in L^\Psi(\Omega)$ and $\|u_k\|_{L^\Phi(\Omega)} \leqslant \|u_k\|_{WL^\Phi(\Omega)}$, the generalized Hölder's inequality gives

$$\int_\Omega h(x)u_k \leqslant 2\|h\|_{L^\Psi(\Omega)}\|u_k\|_{L^\Phi(\Omega)} \leqslant C\|u_k\|_{WL^\Phi(\Omega)}.$$

Thus, if ε is sufficiently small, then (5.8) implies

$$\int_\Omega (\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|)) \leqslant C + C\|u_k\|_{WL^\Phi(\Omega)}.$$

Hence, the definition of the norm on $WL^\Phi(\Omega)$ and (1.11) yield

$$\int_\Omega (\Phi(|\nabla u_k|) + \Phi(|u_k|)) \leqslant C + C\|\nabla u_k\|_{L^\Phi(\Omega)} + C\|u_k\|_{L^\Phi(\Omega)}. \tag{5.9}$$

Finally, from (2.4) together with (5.6) we can easily see that all terms in (5.9) have to be bounded. This is (5.3) and (5.4).

The upper estimate in (5.5) now follows from (5.2) (with $v = u_k$, see also (1.10)). The integral in (5.5) is nonnegative by (1.12). $\square$

By (5.3), (5.4) and the reflexivity of $WL^\Phi(\Omega)$, there is a function $u \in WL^\Phi(\Omega)$ (passing to a suitable subsequence of $\{u_k\}$ if necessary) such that

$$\begin{aligned}
&u_k \rightharpoonup u \quad \text{in } WL^\Phi(\Omega), \\
&u_k \to u \quad \text{in } L^\Phi(\Omega), \\
&u_k \to u \quad \text{in } L^r(\Omega) \quad \text{for every } r \in [1,\infty), \\
&u_k \to u \quad \text{a.e. in } \Omega.
\end{aligned} \tag{5.10}$$

Next, by (1.14) and Theorem 3.1(i) we have $f(x,u), f(x,u_k) \in L^1(\Omega)$. Since we also have (5.5), Lemma 2.4 with $\theta = 0$ implies

$$\lim_{k\to\infty} \int_\Omega f(x,u_k) = \int_\Omega f(x,u). \tag{5.11}$$

Moreover, from (5.5) and Lemma 2.4 with $\theta = 1 - 1/M$ we also obtain

$$f(x,u_k)|u_k|^{1-1/M} \overset{k\to\infty}{\longrightarrow} f(x,u)|u|^{1-1/M} \quad \text{in } L^1(\Omega)$$

and thus by (1.13) and Proposition 2.6 we see that

$$\lim_{k\to\infty} \int_\Omega F(x,u_k) = \int_\Omega F(x,u). \tag{5.12}$$

Lemma 5.2. *Passing to a subsequence, we have*

$$\nabla u_k \to \nabla u \quad \textit{a.e. on } \Omega.$$

Proof. Our aim is to show that for every $\varepsilon > 0$ we can find $\Omega_\varepsilon \subset \Omega$ such that $\mathcal{L}_n(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and a subsequence of $\{u_k\}$ (still denoted $\{u_k\}$) such that $\nabla u_k \to \nabla u$ a.e. in Ω_ε.

Step 1. (Choice of a subsequence)

By (5.3) the sequence $\{\Phi(|\nabla u_k|)\}$ is bounded in $L^1(\Omega)$ and thus passing to a subsequence, we can suppose that

$$\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\overline{\Omega}). \tag{5.13}$$

Step 2. (Choice of Ω_ε)

Fix $\varepsilon > 0$. By Theorem 3.1(ii) and the fact that u_k are bounded in $WL^\Phi(\Omega)$ (hence their medians are bounded on any subset of Ω by a constant depending on its measure, see (3.1)), we can find $\tau > 0$ so small that for every $B(x,R) \subset \Omega$ we have

$$\|\nabla u_k\|_{L^\Phi(B(x,R))} < \tau \quad \Longrightarrow \quad \int_{B(x,R)} \exp_{[l]}(2b|u_k|^\gamma) \leqslant C = C(R). \tag{5.14}$$

Next, using the Δ_2-condition one can show that there is $\sigma > 0$ so that for each $\widetilde{\Omega} \subset \Omega$, we have

$$\int_{\widetilde{\Omega}} \Phi(|v|) < \sigma \quad \Longrightarrow \quad \|v\|_{L^\Phi(\widetilde{\Omega})} < \tau. \tag{5.15}$$

We define the set

$$A_\sigma = \{z \in \overline{\Omega}\colon\ \mu(z) \geqslant \sigma\}.$$

Since $|\mu|(\overline{\Omega}) < \infty$ (by (5.3) and (5.13)), we obtain that A_σ is a finite set, i.e. $A_\sigma = \{z_j\}_{j=1}^m$. Choose $\varrho > 0$ so small that $B(z_i, \varrho) \cap B(z_j, \varrho) = \emptyset$ for $i \neq j$ and

$$\sum_{j=1}^m \mathcal{L}_n(B(z_j, \varrho)) < \frac{\varepsilon}{2}.$$

Next, as $\mathcal{L}_n$ is a Radon measure, we can find compact sets $K_\varepsilon, L_\varepsilon \subset \Omega$ such that $K_\varepsilon \subset \operatorname{Int} L_\varepsilon$ and $\mathcal{L}_n(\Omega \setminus K_\varepsilon) < \varepsilon/2$. Let us define

$$\Omega_\varepsilon = K_\varepsilon \setminus \bigcup_{j=1}^m B(z_j, \varrho) \quad \text{and} \quad B_\varepsilon = \bigcup_{j=1}^m B\Big(z_j, \frac{\varrho}{2}\Big) \cap \Omega.$$

Clearly,

$$\mathcal{L}_n(\Omega \setminus \Omega_\varepsilon) < \varepsilon. \tag{5.16}$$

Fix $\psi_\varepsilon \in C_0^1(\overline{\Omega})$ such that $0 \leqslant \psi_\varepsilon \leqslant 1$, $\psi_\varepsilon = 1$ on Ω_ε and $\psi_\varepsilon = 0$ on $(\overline{\Omega} \setminus L_\varepsilon) \cup \overline{B}_\varepsilon$.

Step 3. (Proof of (5.17): decomposition of the integral)

We want to prove

$$0 \leqslant \int_{\Omega_\varepsilon} \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u) \overset{k\to\infty}{\longrightarrow} 0. \tag{5.17}$$

From (5.2) with $v = \psi_\varepsilon u_k$ and $v = \psi_\varepsilon u$ we obtain

(5.18)
$$\begin{aligned}\int_\Omega \Phi'(|\nabla u_k|)|\nabla u_k|\psi_\varepsilon \leqslant \int_\Omega \Big(& - u_k\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla\psi_\varepsilon - V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon u_k \\ &+ \psi_\varepsilon f(x, u_k)u_k + \mu\psi_\varepsilon h(x)u_k\Big) + \varepsilon_k\|\nabla(\psi_\varepsilon u_k)\|_{WL^\Phi(\Omega)}\end{aligned}$$

and

(5.19)
$$\begin{aligned}\int_\Omega -\Phi'(|\nabla u_k|)\psi_\varepsilon\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla u \leqslant \int_\Omega \Big(& u\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla\psi_\varepsilon + V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon u \\ &- \psi_\varepsilon f(x, u_k)u - \mu\psi_\varepsilon h(x)u\Big) + \varepsilon_k\|\nabla(\psi_\varepsilon u)\|_{WL^\Phi(\Omega)}.\end{aligned}$$

Next, we observe that if $g\colon \mathbb{R} \to \mathbb{R}$ is a differentiable convex function, then we trivially have

$$(g'(s_2) - g'(s_1))(s_2 - s_1) \geqslant 0 \quad \text{for all } s_1, s_2 \in \mathbb{R}.$$

In particular, for

$$g(s) = \Phi(|s\nabla u_k + (1-s)\nabla u|), \quad s_1 = 0,\ s_2 = 1,$$

we obtain the inequality

$$0 \leqslant \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u).$$

This, after integration, gives

$$\begin{aligned}
0 &\leqslant \int_{\Omega_\varepsilon} \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u) \\
&\leqslant \int_{\Omega} \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u)\psi_\varepsilon \\
&= \int_{\Omega} \Phi'(|\nabla u_k|)|\nabla u_k|\psi_\varepsilon - \Phi'(|\nabla u_k|)\psi_\varepsilon\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla u \\
&\quad + \psi_\varepsilon\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_k).
\end{aligned}$$

Therefore, we obtain from (5.18) and (5.19)

(5.20)

$$\begin{aligned}
0 &\leqslant \int_{\Omega_\varepsilon} \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u) \\
&\leqslant \int_{\Omega} \Big(- u_k\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla\psi_\varepsilon - V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon u_k + \psi_\varepsilon f(x,u_k)u_k \\
&\quad + \mu\psi_\varepsilon h(x)u_k\Big) + \int_{\Omega} \Big(u\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla\psi_\varepsilon + V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon u \\
&\quad - \psi_\varepsilon f(x,u_k)u - \mu\psi_\varepsilon h(x)u\Big) + \varepsilon_k\|\nabla(\psi_\varepsilon u_k)\|_{WL^\Phi(\Omega)} + \varepsilon_k\|\nabla(\psi_\varepsilon u)\|_{WL^\Phi(\Omega)} \\
&\quad + \int_{\Omega} \psi_\varepsilon\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_k) \\
&= \int_{\Omega} \Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla\psi_\varepsilon(u - u_k) + \int_{\Omega} \psi_\varepsilon\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_k) \\
&\quad + \int_{\Omega} V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon(u - u_k) + \int_{\Omega} \psi_\varepsilon f(x,u_k)(u_k - u) \\
&\quad + \mu\int_{\Omega} \psi_\varepsilon h(x)(u_k - u) + \varepsilon_k\|\nabla(\psi_\varepsilon u_k)\|_{WL^\Phi(\Omega)} + \varepsilon_k\|\nabla(\psi_\varepsilon u)\|_{WL^\Phi(\Omega)} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

Step 4. (Proof of (5.17): estimate concerning I_1)

From $\Psi(\Phi') \lesssim \Phi$ (see Lemma 2.1) and (5.3) we know that $\Phi'(|\nabla u_k|)$ are bounded in $L^\Psi(\Omega)$ and by (5.10) we see that $u_k \to u$ in $L^\Phi(\Omega)$. Hence, we can use the generalized Hölder's inequality to obtain

$$|I_1| \leqslant \int_\Omega \left|\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla\psi_\varepsilon(u-u_k)\right| \tag{5.21}$$
$$\leqslant \max_{x\in\overline{\Omega}} |\nabla\psi_\varepsilon(x)| 2\|\Phi'(|\nabla u_k|)\|_{L^\Psi(\Omega)}\|u-u_k\|_{L^\Phi(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

Step 5. (Proof of (5.17): estimate concerning I_2)

As $\Psi(\Phi') \lesssim \Phi$ by Lemma 2.1, we observe that

$$\psi_\varepsilon\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \in L^\Psi(\Omega, \mathbb{R}^n).$$

Next, we use the fact that $u_k \rightharpoonup u$ in $WL^\Phi(\Omega)$ (cf. (5.10)) and the duality between $L^\Phi(\Omega, \mathbb{R}^n)$ and $L^\Psi(\Omega, \mathbb{R}^n)$ to obtain

$$|I_2| = \left|\int_\Omega \psi_\varepsilon\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_k)\right| \overset{k\to\infty}{\longrightarrow} 0. \tag{5.22}$$

Step 6. (Proof of (5.17): estimate concerning I_3)

The boundedness of $V(x)\psi_\varepsilon$ (see (1.11)), $\Psi(\Phi') \lesssim \Phi$ (see Lemma 2.1), the generalized Hölder's inequality, (5.4) and (5.10) yield

$$|I_3| \leqslant \int_\Omega \left|V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi_\varepsilon(u-u_k)\right| \tag{5.23}$$
$$\leqslant C\|\Phi'(|u_k|)\|_{L^\Psi(\Omega)}\|u-u_k\|_{L^\Phi(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

Step 7. (Proof of (5.17): estimate concerning I_4)

First let us show that there is $C > 0$ such that for $k \in \mathbb{N}$ large enough we have

$$\int_{L_\varepsilon\setminus B_\varepsilon} f^2(x, u_k) \leqslant C. \tag{5.24}$$

Fix $x \in L_\varepsilon \setminus A_\sigma$. Then there is $r_x > 0$ such that $B(x, r_x) \subset \Omega$ and $\mu(B(x, r_x)) < \sigma$. Consider the test-function $\varphi \in C(\overline{\Omega})$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B(x, r_x/2) \cap \overline{\Omega}$, and $\varphi = 0$ on $\overline{\Omega} \setminus B(x, r_x)$. Thus by (5.13)

$$\int_{B(x,r_x)} \Phi(|\nabla u_k|)\varphi \overset{k\to\infty}{\longrightarrow} \int_{B(x,r_x)} \varphi\,\mathrm{d}\mu \leqslant \mu(B(x, r_x)) < \sigma.$$

Therefore, we have for all $k \in \mathbb{N}$ large enough

$$\int_{B(x,r_x/2)} \Phi(|\nabla u_k|) < \sigma$$

and hence (5.15) gives

$$\|\nabla u_k\|_{L^\Phi(B(x,r_x/2))} < \tau.$$

Next, we apply (1.14), (2.1) and (5.14) to obtain

$$\int_{B(x,r_x/2)} f^2(x,u_k) \leqslant C \int_{B(x,r_x/2)} \exp_{[l]}(2b|u_k|^\gamma) \leqslant C. \tag{5.25}$$

Since $L_\varepsilon \setminus B_\varepsilon$ is a compact set, we obtain $x_1, x_2, \ldots, x_m \in L_\varepsilon \setminus B_\varepsilon$ such that

$$L_\varepsilon \setminus B_\varepsilon \subset \bigcup_{j=1}^{m} B\Big(x_j, \frac{r_{x_j}}{2}\Big)$$

and thus (5.25) applied to each $B(x_j, r_{x_j}/2)$, $j = 1, \ldots, m$, implies (5.24).

Finally, Hölder's inequality, (5.10) and (5.24) imply

$$|I_4| \leqslant \int_{L_\varepsilon \setminus B_\varepsilon} |f(x,u_k)(u_k - u)| \leqslant \Big(\int_{L_\varepsilon \setminus B_\varepsilon} f^2(x,u_k)\Big)^{1/2} \|u_k - u\|_{L^2(\Omega)} \overset{k\to\infty}{\longrightarrow} 0. \tag{5.26}$$

Step 8. (Proof of (5.17): estimate concerning I_5)

Since $h \in L^\Psi(\Omega)$ and $u_k \to u$ in $L^\Phi(\Omega)$ (by (5.10)), the generalized Hölder's inequality gives

$$|I_5| \leqslant \mu \int_\Omega |\psi_\varepsilon h(x)(u_k - u)| \leqslant 2\mu \|h\|_{L^\Psi(\Omega)} \|u_k - u\|_{L^\Phi(\Omega)} \overset{k\to\infty}{\longrightarrow} 0. \tag{5.27}$$

Step 9. (Proof of (5.17): estimates concerning I_6 and I_7)

By (5.3) and (5.4) we have

$$\begin{aligned}\|\nabla(u_k\psi_\varepsilon)\|_{L^\Phi(\Omega)} &= \|u_k \nabla\psi_\varepsilon + \psi_\varepsilon \nabla u_k\|_{L^\Phi(\Omega)} \\ &\leqslant \max_{x\in\overline{\Omega}} |\nabla\psi_\varepsilon(x)| \|u_k\|_{L^\Phi(\Omega)} + \max_{x\in\overline{\Omega}} |\psi_\varepsilon(x)| \|\nabla u_k\|_{L^\Phi(\Omega)} \leqslant C\end{aligned}$$

and by (5.4) we also see that

$$\|u_k\psi_\varepsilon\|_{L^\Phi(\Omega)} \leqslant \|u_k\|_{L^\Phi(\Omega)} \leqslant C.$$

Hence,

$$\|u_k\psi_\varepsilon\|_{WL^\Phi(\Omega)} = \|\nabla(u_k\psi_\varepsilon)\|_{L^\Phi(\Omega)} + \|u_k\psi_\varepsilon\|_{L^\Phi(\Omega)} \leqslant C.$$

In the same way we obtain

$$\|u\psi_\varepsilon\|_{WL^\Phi(\Omega)} \leqslant C.$$

Hence, from $\varepsilon_k \to 0$ we infer

$$I_6 + I_7 \overset{k\to\infty}{\longrightarrow} 0. \tag{5.28}$$

Step 10. (Convergence a.e.)

From (5.20), (5.21), (5.22), (5.23), (5.26), (5.27) and (5.28) we obtain (5.17). Using (5.17) and the fact that the function Φ is strictly convex, we see that passing to a subsequence we have

$$\nabla u_k \to \nabla u \quad \text{a.e. in } \Omega_\varepsilon.$$

This can be done for every $\varepsilon = 1/m$, $m \in \mathbb{N}$. Thus, the diagonal subsequence has the desired property. □

Lemma 5.3. *The function $u \in WL^\Phi(\Omega)$ given by* (5.10) *is a weak solution to the problem* (1.1), *i.e. we have* (1.19).

Proof. The proof consists of two steps. First, we show (1.19) for test-functions from $WL^\Phi(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ only. In the second step, we use the density of these functions in $WL^\Phi(\Omega)$.

Step 1. We want to prove that for every function $\psi \in WL^\Phi(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\psi + \int_\Omega V(x)\Phi'(|u|)\frac{u}{|u|}\psi - \int_\Omega f(x,u)\psi - \mu\int_\Omega h(x)\psi = 0. \tag{5.29}$$

In view of (5.2) with $v = \psi$ it is enough to prove that

$$\int_\Omega \Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla\psi \overset{k\to\infty}{\longrightarrow} \int_\Omega \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\psi, \tag{5.30}$$

$$\int_\Omega V(x)\Phi'(|u_k|)\frac{u_k}{|u_k|}\psi \overset{k\to\infty}{\longrightarrow} \int_\Omega V(x)\Phi'(|u|)\frac{u}{|u|}\psi \tag{5.31}$$

and

$$\int_\Omega f(x,u_k)\psi \overset{k\to\infty}{\longrightarrow} \int_\Omega f(x,u)\psi. \tag{5.32}$$

Let us prove (5.30). By the reflexivity of $L^\Psi(\Omega,\mathbb{R}^n)$, Lemma 2.1 and (5.3) we can pass to a subsequence to obtain

$$\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup w \quad \text{in } L^\Psi(\Omega,\mathbb{R}^n).$$

However, we know that $\nabla u_k \to \nabla u$ a.e. in Ω by Lemma 5.2, and thus the continuity of Φ' and the fact that the weak limit has to be the same as the a.e. pointwise limit imply

$$\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \quad \text{in } L^{\Psi}(\Omega, \mathbb{R}^n).$$

As $\nabla\psi$ can be used as a test-function, we obtain (5.30).

The proof of (5.31) follows easily from the Generalized Lebesgue Dominated Convergence Theorem (Proposition 2.6). Indeed, ψ is bounded, $V(x)$ is bounded by (1.11), $\Phi'(t) \leqslant Ct^{n-1} + Ct^n$ (by (1.6), (1.7) and (1.10)), and we have (5.10).

Finally, (5.32) follows from (5.11) and $\psi \in L^\infty(\Omega)$. Thus, we have proved (5.29).

Step 2. Fix $v \in WL^\Phi(\Omega)$. Then there are $\{\psi_k\} \subset WL^\Phi(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ such that $\psi_k \to v$ in $WL^\Phi(\Omega)$.

By Lemma 2.1 we observe that

$$\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \in L^\Psi(\Omega, \mathbb{R}^n).$$

Thus, as $\nabla\psi_k \to \nabla v$ in $L^\Phi(\Omega, \mathbb{R}^n)$, we obtain from the generalized Hölder's inequality (2.3)

$$\int_\Omega \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla v - \nabla\psi_k) \overset{k\to\infty}{\longrightarrow} 0. \tag{5.33}$$

Next, since $\psi_k \to v$ in $L^\Phi(\Omega)$, $\Phi'(|u|) \in L^\Psi(\Omega)$ (by Lemma 2.1 and $u \in WL^\Phi(\Omega)$) and $V(x)$ is bounded (by (1.11)), the generalized Hölder's inequality (2.3) gives

$$\left|\int_\Omega V(x)\Phi'(|u|)\frac{u}{|u|}(v - \psi_k)\right| \leqslant 2V_1\|v - \psi_k\|_{L^\Phi(\Omega)}\|\Phi'(|u|)\|_{L^\Psi(\Omega)} \overset{k\to\infty}{\longrightarrow} 0. \tag{5.34}$$

By Hölder's inequality, (1.14), (2.1) and Theorem 3.1(i) we easily obtain

$$\left|\int_\Omega f(x,u)(v - \psi_k)\right| \leqslant \|f(x,u)\|_{L^2(\Omega)}\|v - \psi_k\|_{L^2(\Omega)} \overset{k\to\infty}{\longrightarrow} 0. \tag{5.35}$$

Finally, as $h \in L^\Psi(\Omega)$ and $\psi_k \to v$ in $L^\Phi(\Omega)$ we have by the generalized Hölder's inequality (2.3)

$$\mu\int_\Omega h(x)(v - \psi_k) \overset{k\to\infty}{\longrightarrow} 0. \tag{5.36}$$

Now, (1.19) follows from (5.29), (5.33), (5.34), (5.35), and (5.36). $\square$

Lemma 5.4. *If the Palais-Smale sequence $\{u_k\} \subset WL^\Phi(\Omega)$ satisfies*

$$\liminf_{k\to\infty} \|\nabla u_k\|_{L^\Phi(\Omega)} < \Big(\frac{1}{2}\Big)^{1/n} \Big(\frac{K_{l,n,\alpha}}{b}\Big)^{1/\gamma}, \tag{5.37}$$

then (passing to a subsequence if necessary) $u_k \to u$ in $WL^\Phi(\Omega)$ (the strong convergence).

Proof. We can write $u_k = u + w_k$. Further, by (5.37), we see that passing to a subsequence we can suppose that there is $p > 1$ such that for every $k \in \mathbb{N}$ we have

$$p^{2/\gamma} \|\nabla u_k\|_{L^\Phi(\Omega)} \leqslant C_1 := \Big(\frac{1}{2}\Big)^{1/n} \Big(\frac{K_{l,n,\alpha}}{b}\Big)^{1/\gamma}. \tag{5.38}$$

Our aim is to show that $w_k \to 0$ in $WL^\Phi(\Omega)$.

First, (1.14), (2.1), (3.1), (5.4), (5.38), and Theorem 3.1(ii) give

$$\begin{aligned} \int_\Omega |f(x,u_k)|^p &\leqslant C \int_\Omega \exp_{[l]}(pb|u_k|^\gamma) \\ &= C \int_\Omega \exp_{[l]} \Big(\frac{1}{p}\Big(\frac{1}{2}\Big)^{\gamma/n} K_{l,n,\alpha}\Big(\frac{|u_k|}{C_1}\Big)^\gamma\Big) \leqslant C. \end{aligned} \tag{5.39}$$

Further, $f(x,u) \in L^p(\Omega)$ by (1.14), (2.1) and Theorem 3.1(i). By (1.12) and (5.10) we also see that $f(x,u_k) \to f(x,u)$ a.e. in Ω. Therefore, for fixed $q \in (1,p)$ we can use Remark 2.5 with the functions $v_k = f(x,u_k)$ and $v = f(x,u)$, and we obtain

$$f(x,u_k) \to f(x,u) \quad \text{in } L^q(\Omega).$$

Thus, as $u \in L^r(\Omega)$ for every $r \in [1,\infty)$, Hölder's inequality yields

$$\int_\Omega f(x,u_k)u \overset{k\to\infty}{\longrightarrow} \int_\Omega f(x,u)u. \tag{5.40}$$

Since, $w_k \to 0$ in $L^r(\Omega)$ for every $r \in [1,\infty)$, Hölder's inequality together with (5.39) also give that

$$\int_\Omega f(x,u_k)w_k \overset{k\to\infty}{\longrightarrow} 0. \tag{5.41}$$

Now, fix $\varepsilon > 0$. By the Brézis-Lieb lemma (Lemma 2.3) for the function $t \mapsto \Phi'(t)t$ (see also (1.8)), (5.2) (with $v = u$ and $v = u_k$, respectively), (5.3), (5.4), (5.10), (5.40) and (5.41) we have for $k \in \mathbb{N}$ large enough

$$
\begin{aligned}
&\int_\Omega (\Phi'(|\nabla w_k|)|\nabla w_k| + V(x)\Phi'(|w_k|)|w_k|)\\
&\quad \leqslant \varepsilon + \int_\Omega (\Phi'(|\nabla u_k|)|\nabla u_k| - \Phi'(|\nabla u|)|\nabla u| + V(x)(\Phi'(|u_k|)|u_k| - \Phi'(|u|)|u|))\\
&\quad \leqslant \varepsilon + \int_\Omega (f(x,u_k)u_k - f(x,u)u + \mu h(x)(u_k - u)) + \varepsilon_k \|u_k\|_{WL^\Phi(\Omega)} + \varepsilon_k \|u\|_{WL^\Phi(\Omega)}\\
&\quad \leqslant \varepsilon + \int_\Omega f(x,u_k)w_k + \int_\Omega (f(x,u_k)u - f(x,u)u) + 2\mu \|h\|_{L^\Psi(\Omega)} \|u_k - u\|_{L^\Phi(\Omega)} + 2\varepsilon\\
&\quad \leqslant C\varepsilon.
\end{aligned}
$$

That is,

$$\int_\Omega (\Phi'(|\nabla w_k|)|\nabla w_k| + V(x)\Phi'(|w_k|)|w_k|) \overset{k\to\infty}{\longrightarrow} 0.$$

Therefore, from the inequality $\Phi(t) \leqslant \Phi'(t)t$, $t \geqslant 0$ (which easily follows from the convexity of Φ and $\Phi(0) = 0$), and (2.5) we see that $\|\nabla u_k - \nabla u\|_{L^\Phi(\Omega)} \to 0$ and since we also have $\|u_k - u\|_{L^\Phi(\Omega)} \to 0$ (see (5.10)), we are done. $\square$

6. Existence results

In this section we show that the Ekeland Variational Principle (Theorem 2.8) and the Mountain Pass Theorem (Theorem 2.7) give us two different nontrivial weak solutions to (1.1).

Proposition 6.1. *There is $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$, then* (1.1) *has a nontrivial minimum-type solution $u_0 \in WL^\Phi(\Omega)$ with $J_\mu(u_0) = c_0 < 0$, where c_0 is given in Lemma* 4.2. *Moreover, there is a corresponding Palais-Smale sequence $\{u_k\} \subset WL^\Phi(\Omega)$ converging to u_0 in the sense of* (5.10) *and strongly in $WL^\Phi(\Omega)$.*

Proof. Let $\varrho_\mu > 0$ be the same as in Lemma 4.2. We can suppose that μ_0 is so small that

$$\varrho_\mu < \Big(\frac{1}{2}\Big)^{1/n} \Big(\frac{K_{l,n,\alpha}}{b}\Big)^{1/\gamma} \quad \text{for all } \mu \in (0, \mu_0).$$

Set $Y = \{v \in WL^\Phi(\Omega) \colon \|v\|_{WL^\Phi(\Omega)} \leqslant \varrho_\mu\}$. Since $WL^\Phi(\Omega)$ is a complete metric space, Y is its closed subset, the functional J_μ is a C^1-functional and bounded from below on Y (see Lemma 4.2), we can use the Ekeland Variational Principle (Theorem 2.8) to obtain a sequence $\{u_k\} \subset Y$ such that

$$J_\mu(u_k) \overset{k\to\infty}{\longrightarrow} c_0 \quad \text{and} \quad \|J'_\mu(u_k)\|_{C(WL^\Phi(\Omega),\mathbb{R})} \overset{k\to\infty}{\longrightarrow} 0. \tag{6.1}$$

Indeed, the boundedness of Y ensures that (2.7) can be used to obtain a minimizing sequence. Moreover, if $\delta > 0$ is small enough, then u_δ is an interior point of Y (see Lemma 4.2 and Lemma 4.3) and we obtain the convergence of the Fréchet derivatives in (6.1) in the standard way dealing with (2.7) and with the definition of the Fréchet derivative.

Notice that (6.1) gives us the conditions (5.1) and (5.2). Therefore we can use all our results from Section 5 for the sequence $\{u_k\}$. By Lemma 5.3, Lemma 5.4 and the continuity of J_μ we obtain that u_0 is a weak solution to (1.1) satisfying $J_\mu(u_0) = c_0$. We have $c_0 < 0$ by Lemma 4.3. Since μ and h are nontrivial, u_0 has to be nontrivial (see (1.12), (1.18) and (1.19)). $\square$

Proposition 6.2. *There is $\mu_0 > 0$ such that if $\mu \in [0, \mu_0)$, then (1.1) has a nontrivial Mountain Pass-type solution $u_M \in WL^\Phi(\Omega)$. Moreover, there is a corresponding Palais-Smale sequence $\{v_k\} \subset WL^\Phi(\Omega)$ converging to u_M in the sense of (5.10) and $J_\mu(v_k) \to c_M$, where $c_M \in (0, c_0 + 1/2(K_{l,n,\alpha}/b)^{n/\gamma})$.*

P r o o f. Since we have $J_\mu(0) = 0$, Lemmas 4.1, 4.2 and the fact that J_μ is a C^1-functional, we can apply the Mountain Pass Theorem (Theorem 2.7) which gives us a Palais-Smale sequence $\{v_k\} \subset WL^\Phi(\Omega)$ such that $J_\mu(v_k) \to c_M \geqslant \xi_\mu > 0$. Passing to a subsequence, we can further suppose that $\{v_k\}$ possesses all the properties from Section 5, except for Lemma 5.4 (since we do not have (5.37) in general). Finally, if we set $u_M = u$, where $u \in WL^\Phi(\Omega)$ is given by (5.10), then u_M is a weak solution to (1.1) by Lemma 5.3. Further, Lemma 4.5 gives us the upper estimate of the Palais-Smale level c_M.

It remains to show that u_M is nontrivial. This is plainly satisfied if $\mu > 0$, since h is nontrivial (see (1.12), (1.18) and (1.19)). In the rest of the proof let $\mu = 0$ and for the sake of contradiction suppose that $u_M = 0$. From (5.1) (with $\mu = 0$), $u_M = 0$, $F(x, 0) = 0$, (5.12) and from the upper estimate concerning the level c_M (recall that $c_0 = 0$ for $\mu = 0$ by Lemma 4.2) we obtain $\tilde{c} > c_M$ such that for k sufficiently large we have

$$\int_\Omega \Phi(|\nabla v_k|) + V(x)\Phi(|v_k|) \leqslant \tilde{c} < \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}.$$

Hence, as the second term in the above integral is nonnegative (by (1.11)), the estimates (2.1), (5.4), (3.1) and Proposition 3.2 give us $q > 1$ such that

$$\int_\Omega (\exp_{[l]}(b|v_k|^\gamma))^q \leqslant C.$$

Now, from the above estimate, (1.14), (5.10), $u_M = 0$ and Hölder's inequality we infer that

$$\Big|\int_\Omega f(x, v_k)v_k\Big| \leqslant C\|\exp_{[l]}(b|v_k|^\gamma)\|_{L^q(\Omega)}\|v_k\|_{L^{q'}(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

Therefore, (5.2) with $v = v_k$ and $\mu = 0$, (5.3) and (5.4) imply

$$\int_\Omega \Phi'(|\nabla v_k|)|\nabla v_k| + V(x)\Phi'(|v_k|)|v_k| \overset{k\to\infty}{\longrightarrow} 0.$$

Next, as Φ is a Young function, we have $\Phi(t) \leqslant t\Phi'(t)$ for every $t > 0$ and thus we obtain from the above

$$\int_\Omega \Phi(|\nabla v_k|) + V(x)\Phi(|v_k|) \overset{k\to\infty}{\longrightarrow} 0.$$

However, in view of (5.1) and (5.12) this contradicts $c_M > 0$. Hence, u_M is nontrivial and we are done. □

Proposition 6.3. *If $\mu_0 > 0$ is small enough and $\mu \in (0, \mu_0)$, then the functions u_0 and u_M given by Proposition* 6.1 *and Proposition* 6.2, *respectively, are distinct.*

P r o o f. By Proposition 6.1, Proposition 6.2 and by the properties of the Palais-Smale sequences obtained in the previous section we have $\{u_k\}, \{v_k\} \subset WL^\Phi(\Omega)$ such that

$$\begin{aligned} u_k \to u_0 \quad \text{in } WL^\Phi(\Omega) \quad &\text{and} \quad v_k \rightharpoonup u_M \quad \text{in } WL^\Phi(\Omega),\ v_k \to u_M \text{ in } L^\Phi(\Omega), \\ u_k \to u_0 \quad \text{a.e. in } \Omega \quad &\text{and} \quad v_k \to u_M \quad \text{a.e. in } \Omega, \\ \nabla u_k \to \nabla u_0 \quad \text{a.e. in } \Omega \quad &\text{and} \quad \nabla v_k \to \nabla u_M \quad \text{a.e. in } \Omega, \\ J_\mu(u_k) \to c_0 = J_\mu(u_0) \quad &\text{and} \quad J_\mu(v_k) \to c_M, \\ \langle J'_\mu(u_k), u_k\rangle \to 0 \quad &\text{and} \quad \langle J'_\mu(v_k), v_k\rangle \to 0. \end{aligned} \tag{6.2}$$

Moreover, by Propositions 6.1 and 6.2 we have

$$c_0 < 0 < c_M \quad \text{and} \quad c_M - c_0 < \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}. \tag{6.3}$$

Suppose that on the contrary, $u_0 = u_M$. As both Palais-Smale sequences converge to $u_0 = u_M$ in $L^\Phi(\Omega)$, $h \in L^\Psi(\Omega)$ and we have (5.12), we see that

$$\begin{aligned} J_\mu(u_k) &= \int_\Omega (\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) - F(x, u_0) - \mu h(x)u_0) + \mathrm{o}(1) \overset{k\to\infty}{\longrightarrow} c_0, \\ J_\mu(v_k) &= \int_\Omega (\Phi(|\nabla v_k|) + V(x)\Phi(|v_k|) - F(x, u_0) - \mu h(x)u_0) + \mathrm{o}(1) \overset{k\to\infty}{\longrightarrow} c_M \end{aligned}$$

and subtracting one from another, we obtain

$$\int_\Omega (\Phi(|\nabla u_k|) + V(x)\Phi(|u_k|)) - \int_\Omega (\Phi(|\nabla v_k|) + V(x)\Phi(|v_k|)) \overset{k\to\infty}{\longrightarrow} c_0 - c_M. \tag{6.4}$$

Next, $\langle J'_\mu(u_k), u_k\rangle \to 0$ and $\langle J'_\mu(v_k), v_k\rangle \to 0$ read by (1.18)

$$\int_\Omega (\Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k| - f(x,u_k)u_k - \mu h(x)u_k) \overset{k\to\infty}{\longrightarrow} 0,$$
$$\int_\Omega (\Phi'(|\nabla v_k|)|\nabla v_k| + V(x)\Phi'(|v_k|)|v_k| - f(x,v_k)v_k - \mu h(x)v_k) \overset{k\to\infty}{\longrightarrow} 0,$$

and thus

$$\begin{aligned}\int_\Omega (\Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k|)& \\ - \int_\Omega (\Phi'(|\nabla v_k|)|\nabla v_k| + V(x)\Phi'(|v_k|)|v_k|)& \\ - \int_\Omega (f(x,u_k)u_k - f(x,v_k)v_k) - \mu \int_\Omega h(x)(u_k - v_k) &\overset{k\to\infty}{\longrightarrow} 0.\end{aligned} \tag{6.5}$$

As both sequences converge to u_0 in $L^\Phi(\Omega)$ and $h \in L^\Psi(\Omega)$, for the last integral we have

$$\mu \int_\Omega h(x)(u_k - v_k) \overset{k\to\infty}{\longrightarrow} 0. \tag{6.6}$$

Further, since $u_k \to u_0$ in $WL^\Phi(\Omega)$ by (6.2), passing to a subsequence we can construct a common majorant $g \in WL^\Phi(\Omega)$. Hence, from (1.14) we infer that

$$|f(x,u_k)u_k| \leqslant C_b \exp_{[l]}(b|u_k|^\gamma)|u_k| \leqslant C_b \exp_{[l]}(b|g|^\gamma)|g|.$$

Since the right hand side is an $L^1(\Omega)$-function (we can use Hölder's inequality with the powers equal to 2 together with Theorem 3.1(i) and (2.1)), we can use the Lebesgue Dominated Convergence Theorem to obtain

$$\int_\Omega f(x,u_k)u_k \overset{k\to\infty}{\longrightarrow} \int_\Omega f(x,u_0)u_0. \tag{6.7}$$

Further, let us also prove that

$$\int_\Omega (f(x,v_k)v_k - f(x,u_0)u_0) \overset{k\to\infty}{\longrightarrow} 0. \tag{6.8}$$

Since $\int_\Omega \Phi(|\nabla v_k|)$ are bounded by (5.3), passing to a subsequence we can suppose that these modulars converge. Notice that by Fatou's lemma the limit is larger than or equal to $\int_\Omega \Phi(|\nabla u_0|)$. Next, we distinguish two cases.

Case 1: $\int_\Omega \Phi(|\nabla v_k|) \to \int_\Omega \Phi(|\nabla u_0|)$.

In this case we have

$$\nabla v_k \to \nabla u_0 \quad \text{in } L^\Phi(\Omega)$$

(indeed, we can use the Brézis-Lieb lemma to show that the modular of $\nabla(v_k - u_0)$ tends to zero and so does the norm of the gradient by (2.5)). Since we also have $v_k \to u_0$ in $L^\Phi(\Omega)$, we obtain $v_k \to u_0$ in $WL^\Phi(\Omega)$. Hence, we can prove (6.8) in the same way as we proved (6.7).

Case 2: $\lim\limits_{k\to\infty} \int_\Omega \Phi(|\nabla v_k|) - \int_\Omega \Phi(|\nabla u_0|) > 0$.

In this case, our first step is to prove that there is $q > 1$ such that

$$\int_\Omega (\exp_{[l]}(b|v_k|^\gamma))^q \leqslant C. \tag{6.9}$$

By the Brézis-Lieb lemma, $u_k \to u_0$ in $WL^\Phi(\Omega)$ and (6.4) we see that

$$\lim_{k\to\infty} \int_\Omega \Phi(|\nabla v_k|) + \lim_{k\to\infty} \int_\Omega V(x)\Phi(|v_k|) - \int_\Omega \Phi(|\nabla u_0|) - \int_\Omega V(x)\Phi(|u_0|) = c_M - c_0.$$

Further, from Fatou's lemma and $v_k \to u_0$ a.e. on Ω we obtain

$$\lim_{k\to\infty} \int_\Omega V(x)\Phi(|v_k|) \geqslant \int_\Omega V(x)\Phi(|u_0|).$$

Thus, (6.3) yields

$$\lim_{k\to\infty} \int_\Omega \Phi(|\nabla v_k|) - \int_\Omega \Phi(|\nabla u_0|) \leqslant c_M - c_0 < \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{b}\Big)^{n/\gamma}.$$

Therefore, there exists $q > 1$ such that

$$\lim_{k\to\infty} \int_\Omega \Phi(|\nabla v_k|) - \int_\Omega \Phi(|\nabla u_0|) < \frac{1}{2}\Big(\frac{K_{l,n,\alpha}}{bq^2}\Big)^{n/\gamma}.$$

That is,

$$bq < \frac{(\frac{1}{2})^{\gamma/n} K_{l,n,\alpha}}{q} \Big(\frac{1}{\lim\limits_{k\to\infty} \int_\Omega \Phi(|\nabla v_k|) - \int_\Omega \Phi(|\nabla u_0|)}\Big)^{\gamma/n}. \tag{6.10}$$

Next, (2.1) gives

$$\int_\Omega (\exp_{[l]}(b|v_k|^\gamma))^q \leqslant C \int_\Omega \exp_{[l]}(bq|v_k|^\gamma).$$

Now, the integral on the right hand side is uniformly bounded by Proposition 3.3 and (6.10). Thus, we have proved (6.9).

Next, we are going to estimate

$$\begin{aligned}\int_\Omega & |f(x,v_k)v_k - f(x,u_0)u_0| \\ &\leqslant \int_\Omega |(f(x,v_k) - f(x,u_0))u_0| + \int_\Omega |f(x,v_k)(v_k - u_0)| = I_1 + I_2.\end{aligned}$$

First, let us deal with I_2. Estimates (1.14), (6.9), $v_k \to u_0$ in $L^{q'}(\Omega)$ (by (5.10)) and Hölder's inequality yield

$$\begin{aligned} I_2 = \int_\Omega |f(x,v_k)(v_k - u_0)| &\leqslant C \int_\Omega \exp_{[l]}(b|v_k|^\gamma)|v_k - u_0| \\ &\leqslant C\|\exp_{[l]}(b|v_k|^\gamma)\|_{L^q(\Omega)}\|v_k - u_0\|_{L^{q'}(\Omega)} \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

It remains to deal with I_1. By (1.14), (2.1), (6.9) and Theorem 3.1(i) we know that $f(x,v_k) - f(x,u_0)$ is bounded in $L^q(\Omega)$. Further, by (6.2) and (1.12) these functions converge to zero a.e. in Ω. Hence, choosing $r \in (1,q)$ we obtain that they converge to zero in $L^r(\Omega)$ by Remark 2.5. Since $u_0 \in L^{r'}(\Omega)$ by (5.10), Hölder's inequality implies

$$I_1 = \int_\Omega |(f(x,v_k) - f(x,u_0))u_0| \leqslant \|f(x,v_k) - f(x,u_0)\|_{L^r(\Omega)}\|u_0\|_{L^{r'}(\Omega)} \overset{k\to\infty}{\longrightarrow} 0.$$

This concludes the proof of (6.8) also in the second case.

Now, by (6.6), (6.7) and (6.8) we obtain from (6.5)

$$\begin{aligned} &\int_\Omega (\Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k|) \\ &\qquad - \int_\Omega (\Phi'(|\nabla v_k|)|\nabla v_k| + V(x)\Phi'(|v_k|)|v_k|) \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Next, since $u_k \to u_0$ in $WL^\Phi(\Omega)$ (see (6.2)), by (1.10) we easily obtain the convergence of the corresponding modulars with respect to the function $t \mapsto \Phi'(t)t$ and thus

$$\begin{aligned} &\int_\Omega (\Phi'(|\nabla u_0|)|\nabla u_0| + V(x)\Phi'(|u_0|)|u_0|) \\ &\qquad - \int_\Omega (\Phi'(|\nabla v_k|)|\nabla v_k| + V(x)\Phi'(|v_k|)|v_k|) \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now, applying the Brézis-Lieb lemma for the function $t \mapsto \Phi'(t)t$ we see that

$$\int_\Omega (\Phi'(|\nabla(v_k - u_0)|)|\nabla(v_k - u_0)| + V(x)\Phi'(|v_k - u_0|)|v_k - u_0|) \overset{k\to\infty}{\longrightarrow} 0.$$

Hence, the inequality $\Phi(t) \leqslant \Phi'(t)t$, $t \in [0,\infty)$, and (2.5) give us $v_k \to u_0$ in $WL^\Phi(\Omega)$. This strong convergence together with $J_\mu \in C^1(WL^\Phi(\Omega), \mathbb{R})$ implies

$$J_\mu(v_k) \overset{k\to\infty}{\longrightarrow} J_\mu(u_0)$$

and we have a contradiction to (6.2) and (6.3). □

P r o o f of Theorem 1.1. If $\mu_0 > 0$ is sufficiently small and $\mu \in (0, \mu_0)$, then Propositions 6.1, 6.2, and 6.3 give us two nontrivial distinct weak solutions to (1.1). Moreover, since μ and h are nontrivial, there is no trivial weak solution to (1.1) in this case (see (1.12), (1.18), and (1.19)).

Finally, if $\mu = 0$, then we easily see that (1.1) admits a trivial weak solution (see (1.12), (1.18), and (1.19)) and the Mountain Pass-type solution given by Proposition 6.2 is nontrivial (hence it is distinct). $\square$

R e m a r k 6.4. Similarly as in the papers [12], [9], [11], and [8], we can use our methods to obtain the same existence results as (1.1) also in the sub-critical case. That is, we have a version of Theorem 1.1 where instead of (1.14) we have

$$\begin{aligned}&\text{for every } b > 0 \text{ there is } C_b > 0 \text{ such that}\\ &|f(x,t)| \leqslant C_b \exp_{[l]}(b|t|^{\gamma}) \quad \text{whenever } t \in \mathbb{R} \text{ and } x \in \Omega.\end{aligned}$$

In this case we do not need to assume (1.9) and (1.16) (cf. [12, Section 7]).

A c k n o w l e d g e m e n t. The author would like to thank the referee for careful reading.

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