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The conformal change of the metric of an almost Hermitian manifold applied to the antiholomorphic curvature tensor

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Abstract. By using the technique of decomposition of a Hermitian vector space under the action of a unitary group, Ganchev [2] obtained a tensor which he named *the Weyl component of the antiholomorphic curvature tensor*. We show that the same tensor can be obtained by direct application of the conformal change of the metric to the antiholomorphic curvature tensor. Also, we find some other conformally curvature tensors and examine some relations between them.

1 Introduction

Let (M, g, J) be an almost Hermitian manifold, $\dim M = 2n \geq 4$, with the complex structure J and the Hermitian metric g, i.e., $J^2 = -\operatorname{Id}$, $g(JX, JY) = g(X, Y)$ for all $X, Y \in T_p(M)$, where $T_p(M)$ is the tangent vector space of M at $p \in M$. Then the fundamental 2-form is $F(X, Y) = g(JX, Y) = -F(Y, X)$.

We consider the tensors having the standard symmetries of the curvature tensors of a Riemannian manifold. In [7] the linear space $\mathcal{R}(V)$ of such tensors over a $2n$-dimensional Hermitian vector space V was decomposed into irreducible components under the action of the unitary group. Furthermore, all conformally invariant subspaces of $\mathcal{R}(V)$ were found.

In [1] and [2] the holomorphic and antiholomorphic curvature tensors for an almost Hermitian manifold are introduced, and, using the same technique as in the paper [7], the *generalized Bochner curvature tensor* and the *Weyl component of the antiholomorphic curvature tensor* are obtained.

We ask the question: Is it possible to get these tensors in the classical way, i.e., by direct application of the conformal change of this metric?

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In [6], we examined conformally invariant tensors associated with holomorphic curvature tensor, and we found, among others, the generalized Bochner curvature tensor.

In the present paper, we deal with the antiholomorphic curvature tensor and, in Section 3, we find the Weyl component of the antiholomorphic curvature tensor. In Sections 2 and 4, we determine some other conformally invariant tensors. In Section 5 we find some relations between obtained conformally invariant tensors, and in Section 6 we discuss the case of Kähler manifold.

We denote by R the Riemannian curvature tensor. Then the first and the second Ricci tensors are defined as follows: $\rho(X,Y) = \sum_{i=1}^{2n} R(e_i, X, Y, e_i)$, and $\overset{*}{\rho}(X,Y) = \sum_{i=1}^{2n} R(e_i, X, JY, Je_i)$, where $\{e_i\}$ is an orthonormal basis of $T_p(M)$. Finally, the first and the second scalar curvatures are

$$\tau = \sum_{i=1}^{2n} \rho(e_i, e_i), \quad \overset{*}{\tau} = \sum_{i=1}^{2n} \overset{*}{\rho}(e_i, e_i)\,.$$

In general, the second Ricci tensor is not symmetric, but it satisfies the condition

$$\overset{*}{\rho}(JX, JY) = \overset{*}{\rho}(Y, X)\,. \tag{1}$$

To make some formulas clearer we use the following notations:

$$\begin{aligned}
\pi_1(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\,,\\
\pi_2(X,Y,Z,W) &= F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\,,\\
\varphi(X,Y,Z,W) &= g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W)\\
&\quad - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z),\\
(J\varphi)(X,Y,Z,W) &= \varphi(JX,JY,JZ,JW) = g(X,W)\rho(JY,JZ) + g(Y,Z)\rho(JX,JW)\\
&\qquad - g(X,Z)\rho(JY,JW) - g(Y,W)\rho(JX,JZ)\,,\\
\psi(X,Y,Z,W) &= F(X,W)\big[\rho(Y,JZ) - \rho(JY,Z)\big] + F(Y,Z)\big[\rho(X,JW)\\
&\qquad - \rho(JX,W)\big] - F(X,Z)\big[\rho(Y,JW) - \rho(JY,W)\big]\\
&\quad - F(Y,W)\big[\rho(X,JZ) - \rho(JX,Z)\big] - 2F(X,Y)\big[\rho(Z,JW)\\
&\qquad - \rho(JZ,W)\big] - 2F(Z,W)\big[\rho(X,JY) - \rho(JX,Y)\big]\,,\\
\overset{*}{\varphi}(X,Y,Z,W) &= g(X,W)\big[\overset{*}{\rho}(Y,Z) + \overset{*}{\rho}(Z,Y)\big] + g(Y,Z)\big[\overset{*}{\rho}(X,W) + \overset{*}{\rho}(W,X)\big]\\
&\quad - g(X,Z)\big[\overset{*}{\rho}(Y,W) + \overset{*}{\rho}(W,Y)\big] - g(Y,W)\big[\overset{*}{\rho}(X,Z) + \overset{*}{\rho}(Z,X)\big]\,,\\
\overset{*}{\psi}(X,Y,Z,W) &= F(X,W)\big[\overset{*}{\rho}(Y,JZ) - \overset{*}{\rho}(JY,Z)\big] + F(Y,Z)\big[\overset{*}{\rho}(X,JW)\\
&\qquad - \overset{*}{\rho}(JX,W)\big] - F(X,Z)\big[\overset{*}{\rho}(Y,JW) - \overset{*}{\rho}(JY,W)\big]\\
&\quad - F(Y,W)\big[\overset{*}{\rho}(X,JZ) - \overset{*}{\rho}(JX,Z)\big]\\
&\quad - 2F(X,Y)\big[\overset{*}{\rho}(Z,JW) - \overset{*}{\rho}(JZ,W)\big]\\
&\quad - 2F(Z,W)\big[\overset{*}{\rho}(X,JY) - \overset{*}{\rho}(JX,Y)\big]\,.
\end{aligned} \tag{2}$$

A $(0,4)$ tensor $T(X,Y,Z,W)$ is said to be generalized curvature tensor if it satisfies all algebraic properties of the Riemannian curvature tensor, i.e.,

$$\begin{gathered} T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y)\,, \\ T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W) = 0\,. \end{gathered}$$

All tensors in the relations (2) are generalized curvature tensors. Hence we can consider the corresponding first Ricci tensor and the second Ricci tensor of each of them, and denote them as $\rho(T)$ and $\overset{*}{\rho}(T)$, respectively. In particular, we have $\rho(R) = \rho$ and $\overset{*}{\rho}(R) = \overset{*}{\rho}$.

For the latter use, we note that

$$\begin{aligned} \rho(\pi_1)(X,Y) &= (2n-1)g(X,Y), \quad \rho(\pi_2)(X,Y) = 3g(X,Y)\,, \\ \rho(\varphi)(X,Y) &= 2(n-1)\rho(X,Y) + \tau g(X,Y)\,, \\ \rho(J\varphi)(X,Y) &= 2(n-1)\rho(JX,JY) + \tau g(X,Y)\,, \\ \rho(\psi)(X,Y) &= -6[\,\rho(X,Y) + \rho(JX,JY)]\,, \\ \rho(\overset{*}{\varphi})(X,Y) &= 2(n-1)\big[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big] + 2\overset{*}{\tau} g(X,Y)\,, \\ \rho(\overset{*}{\psi})(X,Y) &= -6\big[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big]\,, \\ \overset{*}{\rho}(\pi_1)(X,Y) &= g(X,Y), \quad \overset{*}{\rho}(\pi_2)(X,Y) = (2n+1)g(X,Y)\,, \\ \overset{*}{\rho}(\varphi)(X,Y) &= \rho(X,Y) + \rho(JX,JY)\,, \\ \overset{*}{\rho}(J\varphi)(X,Y) &= \rho(X,Y) + \rho(JX,JY)\,, \\ \overset{*}{\rho}(\psi)(X,Y) &= -2(n-1)[\,\rho(X,Y) + \rho(JX,JY)] - 2\tau g(X,Y)\,, \\ \overset{*}{\rho}(\overset{*}{\varphi})(X,Y) &= 2[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)]\,, \\ \overset{*}{\rho}(\overset{*}{\psi})(X,Y) &= -2(n-1)[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)] - 2\overset{*}{\tau}(X,Y)\,. \end{aligned} \tag{3}$$

2 Some conformally invariant tensors

For an almost Hermitian manifold (M,g,J), we consider the conformal change of metric $\bar{g} = e^{2f}g$, where f is a scalar function. If ∇ and $\bar{\nabla}$ are the Levi-Civita connections with respect to the metrics g and $\bar{g}$ respectively, we have

$$(\bar{\nabla} - \nabla)(X,Y) = \theta(X)(Y) + \theta(Y)(X) - g(X,Y)U\,,$$

for any vector fields $X, Y \in T(M)$, where $\theta = df$, and U is the vector field such that $g(U,X) = \theta(X)$.

From now on, all geometric objects in $(M,\bar{g},J)$ will be denoted by analogous letters as in (M,g,J), but with “bar”.

It is well known (see e.g., [3]) that the Riemannian curvature tensors $\bar{R}$ and R are related as follows

$$\begin{aligned} e^{-2f}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(X,W)\sigma(Y,Z) + g(Y,Z)\sigma(X,W) \\ &- g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(X,Z) \end{aligned} \tag{4}$$

where

$$\sigma(X,Y) = (\nabla_X\theta)(Y) - \theta(X)\theta(Y) + \frac{1}{2}g(X,Y)\theta(U)\,.$$

We note that $\sigma(X,Y)=\sigma(Y,X)$.

The relation (4) implies

$$\bar\rho(Y,Z)=\rho(Y,Z)+2(n-1)\sigma(Y,Z)+g(Y,Z)\sigma\,,$$

where $\sigma=\sum_{i=1}^{2n}\sigma(e_i,e_i)$. Therefore

$$\sigma=\frac{e^{2f}\bar\tau-\tau}{2(2n-1)}\,, \tag{5}$$

and

$$\begin{aligned}\sigma(Y,Z)=&\frac{1}{2(n-1)}\Big[\bar\rho(Y,Z)-\frac{\bar\tau}{2(2n-1)}\bar g(Y,Z)\Big]\\&-\frac{1}{2(n-1)}\Big[\rho(Y,Z)-\frac{\tau}{2(2n-1)}g(Y,Z)\Big]\,.\end{aligned} \tag{6}$$

Thus

$$\begin{aligned}\sigma(Y,Z)+\sigma(JY,JZ)=&\frac{1}{2(n-1)}\big[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\big]-\frac{\bar\tau}{2(n-1)(2n-1)}\bar g(Y,Z)\\&-\frac{1}{2(n-1)}\big[\rho(Y,Z)+\rho(JY,JZ)\big]\\&+\frac{\tau}{2(n-1)(2n-1)}g(Y,Z)\,.\end{aligned} \tag{7}$$

On the other hand, the relation (4) implies

$$\begin{aligned}e^{-2f}\bar R(X,Y,JZ,JW)=&R(X,Y,JZ,JW)-F(X,W)\sigma(Y,JZ)\\&-F(Y,Z)\sigma(X,JW)+F(X,Z)\sigma(Y,JW)\\&+F(Y,W)\sigma(X,JZ)\,,\end{aligned}$$

wherefrom it follows

$$\sigma(Y,Z)+\sigma(JY,JZ)=\bar{\overset{*}{\rho}}(Y,Z)-\overset{*}{\rho}(Y,Z)\,, \tag{8}$$

or

$$\sigma(Y,Z)+\sigma(JY,JZ)=\frac12\big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\big]-\frac12\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big] \tag{9}$$

and

$$\sigma=\frac12\left(e^{2f}\bar{\overset{*}{\tau}}-\overset{*}{\tau}\right)\,. \tag{10}$$

Comparing (7) and (9), as well as (5) and (10), we find

$$\begin{aligned}&\frac{1}{n-1}\big[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\big]-\big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\big]-\frac{\bar\tau}{(n-1)(2n-1)}\bar g(Y,Z)\\&=\frac{1}{n-1}\big[\rho(Y,Z)+\rho(JY,JZ)\big]-\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]-\frac{\tau}{(n-1)(2n-1)}g(Y,Z),\end{aligned} \tag{11}$$

and

$$e^{2f}\left(\bar{\overset{*}{\tau}} - \frac{\bar{\tau}}{2n-1}\right) = \overset{*}{\tau} - \frac{\tau}{2n-1}\,. \tag{12}$$

The relation (11) shows that the tensor

$$\begin{aligned} V(Y,Z) = {} & \frac{1}{n-1}[\,\rho(Y,Z)+\rho(JY,JZ)] - [\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)] \\ & - \frac{\tau}{(n-1)(2n-1)}g(Y,Z) \end{aligned} \tag{13}$$

is conformally invariant, i.e., $\bar{V}(Y,Z) = V(Y,Z)$. Thus

$$e^{-2f}\bar{g}(X,W)\bar{V}(Y,Z) = g(X,W)V(Y,Z)$$

and therefore

$$\begin{aligned} & e^{-2f}\left\{\bar{g}(X,W)\bar{V}(Y,Z)+\bar{g}(Y,Z)\bar{V}(X,W)-\bar{g}(X,Z)\bar{V}(Y,W)-\bar{g}(Y,W)\bar{V}(X,Z)\right\} \\ & \quad = g(X,W)V(Y,Z)+g(Y,Z)V(X,W)-g(X,Z)V(Y,W)-g(Y,W)V(X,Z)\,. \end{aligned}$$

This relation, in view of (13) and (2), can be rewritten in the form

$$\begin{aligned} & e^{-2f}\left[\frac{1}{n-1}(\bar{\varphi}+J\bar{\varphi}) - \bar{\overset{*}{\varphi}} - \frac{2\bar{\tau}}{(n-1)(2n-1)}\bar{\pi}_1\right] \\ & \qquad\qquad = \frac{1}{n-1}(\varphi+J\varphi) - \overset{*}{\varphi} - \frac{2\tau}{(n-1)(2n-1)}\pi_1\,. \end{aligned}$$

This means that the tensor

$$W_3 = \frac{1}{n-1}(\varphi+J\varphi) - \overset{*}{\varphi} - \frac{2\tau}{(n-1)(2n-1)}\pi_1 \tag{14}$$

satisfies the condition

$$e^{-2f}\bar{W}_3 = W_3\,. \tag{15}$$

Since all the tensors $\varphi, J\varphi, \overset{*}{\varphi}$ and π_1 are generalized curvature tensors, the tensor (14) is also a generalized curvature tensor.

In a similar way, we find that the tensors

$$\begin{gathered} W_4 = \frac{1}{n-1}\psi - \overset{*}{\psi} + \frac{2\tau}{(n-1)(2n-1)}\pi_2 \\ W_5 = \left(\frac{\tau}{2n-1} - \overset{*}{\tau}\right)\pi_1 \qquad \text{and} \qquad W_6 = \left(\frac{\tau}{2n-1} - \overset{*}{\tau}\right)\pi_2 \end{gathered} \tag{16}$$

satisfy the conditions

$$e^{-2f}\bar{W}_4 = W_4, \quad e^{-2f}\bar{W}_5 = W_5 \quad \text{and} \quad e^{-2f}\bar{W}_6 = W_6 \tag{17}$$

respectively.

Thus, we can state the following theorem.

Theorem 1. *For an almost Hermitian manifold we have*

1. *the tensor* (13) *is conformally invariant;*
2. *the generalized curvature tensors* (16) *satisfy the conditions* (17) *respectively.*

3 The Weyl component of the antiholomorphic curvature tensor

A 2-plane $\alpha \in T_p(M)$ is said to be *holomorphic* if $J\alpha = \alpha$ and *antiholomorphic* if $J\alpha \perp \alpha$. The sectional curvatures with respect to such 2-planes are *holomorphic* and *antiholomorphic* respectively. The holomorphic sectional curvatures can be examined using the *holomorphic curvature tensor* (see e.g., [2], [4], [5]). In [6] we found some conformally invariant tensors associated with holomorphic curvature tensor.

To examine the antiholomorphic sectional curvatures, G. Ganchev [1], [2] introduced *antiholomorphic curvature tensor* which is

$$\begin{aligned}(AR)(X,Y,Z,W) = R(X,Y,Z,W) &\\ + \frac{1}{2n+2}\Big\{&F(X,W)\overset{*}{\rho}(Y,JZ) + F(Y,Z)\overset{*}{\rho}(X,JW) \\ &- F(X,Z)\overset{*}{\rho}(Y,JW) - F(Y,W)\overset{*}{\rho}(X,JZ) \\ &- 2F(X,Y)\overset{*}{\rho}(Z,JW) - 2F(Z,W)\overset{*}{\rho}(X,JY)\Big\} \\ + \frac{\overset{*}{\tau}}{(2n+2)(2n+1)}&\pi_2(X,Y,Z,W)\,.\end{aligned} \tag{18}$$

The corresponding Ricci tensor is

$$\begin{aligned}\rho(AR)(X,Y) &\equiv \sum_{i=1}^{2n}(AR)(e_i,X,Y,e_i) \\ &= \rho(X,Y) - \frac{3}{2n+2}\big[\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big] + \frac{3\overset{*}{\tau}}{(2n+2)(2n+1)}g(X,Y)\,,\end{aligned} \tag{19}$$

and therefore the corresponding scalar curvature is

$$\tau(AR) = \sum_{i=1}^{2n}\rho(AR)(e_i,e_i) = \tau - \frac{3\overset{*}{\tau}}{2n+1}\,, \tag{20}$$

while the second Ricci tensor vanishes.

In this section we apply the conformal change of the metric to the tensor (18). Let $A\bar{R}$ be the antiholomorphic curvature tensor with respect to the metric $\bar{g}$. Then, using (4), (8), and (10), we get

$$\begin{aligned}e^{-2f}(A\bar{R})(X,Y,Z,W) &= (AR)(X,Y,Z,W) + g(X,W)\sigma(Y,Z) \\ &+ g(Y,Z)\sigma(X,W) - g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(X,Z) \\ &+ \frac{1}{2n+2}\Big\{F(X,W)[\sigma(Y,JZ) - \sigma(JY,Z)] \\ &\quad + F(Y,Z)[\sigma(X,JW) - \sigma(JX,W)] - F(X,Z)[\sigma(Y,JW) - \sigma(JY,W)] \\ &\quad - F(Y,W)[\sigma(X,JZ) - \sigma(JX,Z)] - 2F(X,Y)[\sigma(Z,JW) - \sigma(JZ,W)] \\ &\quad - 2F(Z,W)[\sigma(X,JY) - \sigma(JX,Y)]\Big\} \\ &+ \frac{\sigma}{(n-1)(2n+1)}\pi_2(X,Y,Z,W)\,.\end{aligned} \tag{21}$$

To determine $\sigma(X,Y)$, we put into (21) $X=W=e_i$, sum up and obtain

$$\begin{aligned}\rho(A\bar{R})(Y,Z)-\rho(AR)(Y,Z)=\frac{2n^2-5}{n+1}\sigma(Y,Z)-\frac{3}{n+1}\sigma(JY,JZ)\\ +\frac{2n^2+3n+4}{(n+1)(2n+1)}g(Y,Z)\sigma\,.\end{aligned}\tag{22}$$

It follows from (22) that

$$e^{2f}\tau(A\bar{R})-\tau(AR)=\frac{8(n^2-1)}{2n+1}\sigma\,,$$

or, in view of (20),

$$\sigma=\frac{2n+1}{8(n^2-1)}\Big\{e^{2f}\Big(\bar\tau-\frac{3\bar{\overset{*}{\tau}}}{2n+1}\Big)-\Big(\tau-\frac{3\overset{*}{\tau}}{2n+1}\Big)\Big\}\,.\tag{23}$$

On the other hand, the relation (22) yields

$$\begin{aligned}4(n-1)(n^2-4)\sigma(Y,Z)&=(2n^2-5)[\,\rho(A\bar{R})(Y,Z)-\rho(AR)(Y,Z)]\\&\quad+3[\,\rho(A\bar{R})(JY,JZ)-\rho(AR)(JY,JZ)]\\&\quad-\frac{2(n^2-1)(2n^2+3n+4)}{(n+1)(2n+1)}g(Y,Z)\sigma\end{aligned}$$

wherefrom, using (19) and (23) and supposing $n>2$, we find

$$\begin{aligned}\sigma(Y,Z)=&\frac{1}{4(n-1)(n^2-4)}\big[(2n^2-5)\bar\rho(Y,Z)+3\bar\rho(JY,JZ)\big]\\&-\frac{3}{4(n^2-4)}\Big[\,\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\Big]\\&-\frac{1}{16(n^2-1)(n^2-4)}\Big[(2n^2+3n+4)\bar\tau-9n\bar{\overset{*}{\tau}}\,\Big]\bar g(Y,Z)\\&-\Big\{\frac{1}{4(n-1)(n^2-4)}\big[(2n^2-5)\rho(Y,Z)+3\rho(JY,JZ)\big]\\&\quad-\frac{3}{4(n^2-4)}\big[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]\\&\quad-\frac{1}{16(n^2-1)(n^2-4)}\big[(2n^2+3n+4)\tau-9n\overset{*}{\tau}\big]g(Y,Z)\Big\}\,.\end{aligned}\tag{24}$$

Finally, substituting (23) and (24) into (21) and using the notations (2), we get

$$\begin{aligned}e^{-2f}\Big\{A\bar{R}+&\frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\bar\varphi-\frac{3}{n-1}J\bar\varphi-\bar\psi+3\Big(\bar{\overset{*}{\varphi}}+\frac{1}{n+1}\bar{\overset{*}{\psi}}\Big)\\&+\frac{(2n^2+3n+4)\bar\tau-9n\bar{\overset{*}{\tau}}}{2(n^2-1)}\bar\pi_1-\frac{3}{2(n^2-1)}\Big(n\bar\tau-\frac{7n-4}{2n+1}\bar{\overset{*}{\tau}}\Big)\bar\pi_2\Big]\Big\}\\=AR+&\frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\varphi-\frac{3}{n-1}J\varphi-\psi+3\Big(\overset{*}{\varphi}+\frac{1}{n+1}\overset{*}{\psi}\Big)\\&+\frac{(2n^2+3n+4)\tau-9n\overset{*}{\tau}}{2(n^2-1)}\pi_1-\frac{3}{2(n^2-1)}\Big(n\tau-\frac{7n-4}{2n+1}\overset{*}{\tau}\Big)\pi_2\Big]\,.\end{aligned}\tag{25}$$

Thus, putting

$$W_1 = AR + \frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\varphi - \frac{3}{n-1}J\varphi - \psi + 3\Big(\overset{*}{\varphi} + \frac{1}{n+1}\overset{*}{\psi}\Big) + \frac{(2n^2+3n+4)\tau - 9n\overset{*}{\tau}}{2(n^2-1)}\pi_1 - \frac{3}{2(n^2-1)}\Big(n\tau - \frac{7n-4}{2n+1}\overset{*}{\tau}\Big)\pi_2\Big] \tag{26}$$

we have

$$e^{-2f}\bar{W}_1 = W_1 \,.$$

In [2] the tensor (26) is obtained applying the technique of a decomposition of the Hermitian vector space under an action of a unitary group, is denoted by $(AR)_W$ and is called the *Weyl component of the antiholomorphic curvature tensor*. It is here obtained as a result of a direct application of the conformal change of the metric to the antiholomorphic curvature tensor.

It can be proved, using the relation (3) that $\rho(W_1) = \overset{*}{\rho}(W_1) = 0$, and this explains the name “Weyl component...” for the tensor (26)

4 The second conformally invariant tensor associated with the antiholomorphic curvature tensor

Putting into (7) JZ instead of Z, we find

$$\begin{aligned}\sigma(Y,JZ) - \sigma(JY,Z) = {} & \frac{1}{2(n-1)}\big[\bar{\rho}(Y,JZ) - \bar{\rho}(JY,Z)\big] + \frac{\bar{\tau}}{2(n-1)(2n-1)}\bar{F}(Y,Z) \\ & - \frac{1}{2(n-1)}\big[\rho(Y,JZ) - \rho(JY,Z)\big] \\ & - \frac{\tau}{2(n-1)(2n-1)}F(Y,Z)\,.\end{aligned}$$

Substituting this relation as well as the relations (5) and (6) into (21), and using the notation (2), we obtain

$$\begin{aligned} e^{-2f}\Big\{ & A\bar{R} - \frac{1}{2(n-1)}\bar{\varphi} - \frac{1}{4(n^2-1)}\bar{\psi} \\ & + \frac{\bar{\tau}}{2(n-1)(2n-1)}\Big(\bar{\pi}_1 - \frac{3n}{(n+1)(2n+1)}\bar{\pi}_2\Big)\Big\} \\ = {} & AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi \\ & + \frac{\tau}{2(n-1)(2n-1)}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big)\end{aligned} \tag{27}$$

or

$$e^{-2f}\bar{W}_2 = W_2\,, \tag{28}$$

where

$$\begin{aligned} W_2 = {} & AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi \\ & + \frac{\tau}{2(n-1)(2n-1)}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big)\,.\end{aligned} \tag{29}$$

We say that (29) is the *second conformally invariant* (in the sense of (28)) *curvature tensor* of an *almost Hermitian manifold associated with the antiholomorphic curvature tensor*.

In the case $n > 2$, the relation (27) can also be obtained from (25), using (11) and (12). Namely, we have, according to (11)

$$\begin{aligned}\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y) &= \frac{1}{(n-1)}\left[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\right]-\frac{\bar\tau}{(n-1)(2n-1)}\bar g(Y,Z)\\ &\quad+\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)-\frac{1}{(n-1)}\left[\rho(Y,Z)+\rho(JY,JZ)\right]\\ &\quad+\frac{\tau}{(n-1)(2n-1)}g(Y,Z)\end{aligned} .$$

Thus

$$\begin{aligned}e^{-2f}\bar g(X,W)\Big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\Big] &= e^{-2f}\Big\{\frac{1}{(n-1)}\bar g(X,W)\left[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\right]\\ &\quad-\frac{\bar\tau}{(n-1)(2n-1)}\bar g(X,W)\bar g(Y,Z)\Big\}\\ &\quad+g(X,W)[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)]\\ &\quad-\frac{1}{(n-1)}g(X,W)\left[\rho(Y,Z)+\rho(JY,JZ)\right]\\ &\quad+\frac{\tau}{(n-1)(2n-1)}g(X,W)g(Y,Z)\,,\end{aligned}$$

such that

$$\begin{aligned}e^{-2f}\bar{\overset{*}{\varphi}} &= e^{-2f}\Big\{\frac{1}{n-1}(\bar\varphi+J\bar\varphi)-\frac{2\bar\tau}{(n-1)(2n-1)}\bar\pi_1\Big\}\\ &\quad+\overset{*}{\varphi}-\frac{1}{n-1}(\varphi+J\varphi)+\frac{2\tau}{(n-1)(2n-1)}\bar\pi_1.\end{aligned}$$

We get, in a similar way,

$$\begin{aligned}e^{-2f}\bar{\overset{*}{\psi}} &= e^{-2f}\Big\{\frac{1}{n-1}\bar\psi+\frac{2\bar\tau}{(n-1)(2n-1)}\bar\pi_2\Big\}\\ &\quad+\overset{*}{\psi}-\frac{1}{n-1}\psi-\frac{2\tau}{(n-1)(2n-1)}\pi_2\,.\end{aligned}$$

Therefore

$$\begin{aligned}3e^{-2f}\Big(\bar{\overset{*}{\varphi}}+\frac{\bar{\overset{*}{\psi}}}{n+1}\Big) &= e^{-2f}\Big\{\frac{3}{n-1}(\bar\varphi+J\bar\varphi)+\frac{3\bar\psi}{n^2-1}\\ &\quad-\frac{6\bar\tau}{(n-1)(2n-1)}\Big(\pi_1-\frac{1}{n+1}\pi_2\Big)\Big\}\\ &\quad+3\Big(\overset{*}{\varphi}+\frac{\overset{*}{\psi}}{n+1}\Big)-\frac{3}{n-1}(\varphi+J\varphi)-\frac{3\psi}{n^2-1}\\ &\quad+\frac{6\tau}{(n-1)(2n-1)}\Big(\pi_1-\frac{1}{n+1}\pi_2\Big)\,.\end{aligned}$$

Substituting this into (25), and using (12), we get just (27).

Thus we can state a theorem.

Theorem 2. *For $n > 2$ the relation* (27) *is a consequence of* (25), (11) *and* (12).

The first Ricci tensor of the tensor W_2 is

$$\rho(W_2)(Y,Z) = \frac{3}{2(n+1)}\Big\{\frac{1}{n-1}[\rho(Y,Z)+\rho(JY,JZ)] - [\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)] - \frac{1}{2n+1}\Big[\frac{3n}{(n-1)(2n-1)}\tau - \overset{*}{\tau}\Big]g(Y,Z)\Big\},$$

while for the second Ricci tensor we have $\overset{*}{\rho}(W_2) = 0$.

5 The relations between some conformally invariant tensors

5.1

According to (14) and (16), we have

$$\overset{*}{\varphi} = -W_3 + \frac{1}{n-1}(\varphi + J\varphi) - \frac{2\tau}{(n-1)(2n-1)}\pi_1\,,$$
$$\overset{*}{\psi} = -W_4 + \frac{1}{n-1}\psi + \frac{2\tau}{(n-1)(2n-1)}\pi_2\,.$$

Substituting this into (26), we get

$$\begin{aligned}4(n^2-4)W_1 = {}& 4(n^2-4)\Big[AR - \frac{\varphi}{2(n-1)} - \frac{\psi}{4(n^2-1)}\Big] - 3W_3 - \frac{3}{n+1}W_4\\ &+ \Big[\Big(\frac{2n^2+3n+4}{2(n^2-1)} - \frac{6}{(n-1)(2n-1)}\Big)\tau - \frac{9n}{2(n^2-1)}\overset{*}{\tau}\Big]\pi_1\\ &+ \Big[\Big(-\frac{3n}{2(n^2-1)} + \frac{6}{(n^2-1)(2n-1)}\Big)\tau + \frac{3(7n-4)}{2(n^2-1)(2n+1)}\overset{*}{\tau}\Big]\pi_2\,.\end{aligned}$$

But, in view of (29), we have

$$AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi = W_2 - \frac{\tau}{2(n-1)(2n-1)}\pi_1 + \frac{3n\tau}{2(n^2-1)(4n^2-1)}\pi_2,$$

because of which and using (16) the preceding relation can be rewritten in the form

$$\begin{aligned}W_1 \equiv (AR)_W = W_2 &- \frac{3}{4(n^2-4)}\Big(W_3 + \frac{1}{n+1}W_4\Big)\\ &+ \frac{3}{8(n^2-1)(n^2-4)}\Big[3nW_5 - \frac{7n-4}{2n+1}W_6\Big]\,. \end{aligned} \tag{30}$$

Thus, we can state the following theorem.

Theorem 3. *The Weyl component of the antiholomorphic curvature tensor can be expressed as a linear combination of the tensors W_2, W_3, W_4, W_5 and W_6 such that* (30) *holds. Each of the tensors W is a generalized curvature tensor and each satisfies the condition of the type $e^{-2f}\bar{W} = W$.*

5.2

It is well known (see e.g., [3]) that the Weyl conformal curvature tensor for the Riemannian manifold (M,g), $\dim M = 2n$, is

$$\begin{aligned} C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2(n-1)}\big[g(X,W)\rho(Y,Z) \\ + g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z)\big] \\ + \frac{\tau}{2(n-1)(2n-1)}\,\pi_1(X,Y,Z,W) \end{aligned} \tag{31}$$

and that satisfies the condition

$$e^{-2f}\bar{C}(X,Y,Z,W) = C(X,Y,Z,W)\,.$$

Now, let us apply (18) to the tensor (31), instead to the tensor R. In such a way we obtain

$$\begin{aligned} (AC)(X,Y,Z,W) = C(X,Y,Z,W) + \frac{1}{2(n+1)}\big[F(X,W)\overset{*}{\rho}(C)(Y,JZ) \\ + F(Y,Z)\overset{*}{\rho}(C)(X,JW) - F(X,Z)\overset{*}{\rho}(C)(Y,JW) \\ - F(Y,W)\overset{*}{\rho}(C)(X,JZ) - 2F(X,Y)\overset{*}{\rho}(C)(Z,JW) \\ - 2F(Z,W)\overset{*}{\rho}(C)(X,JY)\big] \\ + \frac{\overset{*}{\tau}(C)}{(2n+2)(2n+1)}\pi_2(X,Y,Z,W)\,. \end{aligned} \tag{32}$$

The first Ricci tensor of the tensor (31) vanishes. But, for the second Ricci tensor we have

$$\begin{aligned} \overset{*}{\rho}(C)(Y,Z) &= \sum_{i=1}^{2n} C(e_i,Y,JZ,Je_i) \\ &= \overset{*}{\rho}(Y,Z) - \frac{1}{2(n-1)}[\,\rho(Y,Z)+\rho(JY,JZ)] \\ &\quad + \frac{\tau}{2(n-1)(2n-1)}g(Y,Z)\,, \end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(C) = \overset{*}{\tau} - \frac{\tau}{2n-1}\,.$$

Substituting this and (31) into (32) and using (18), we get

$$A(C) = AR - \frac{1}{2(n-1)}\Big[\varphi + \frac{\psi}{2(n+1)} - \frac{\tau}{2n-1}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big)\Big]\,,$$

such that, and in view of (29), we can state the following theorem.

Theorem 4. *The second conformally invariant curvature tensor, associated with the antiholomorphic curvature tensor, satisfies the relation* $W_2 = A(C)$, *while* (11) *can be expressed in the form*

$$\overset{*}{\rho}(\bar{C})(Y,Z) + \overset{*}{\rho}(\bar{C})(Z,Y) = \overset{*}{\rho}(C)(Y,Z) + \overset{*}{\rho}(C)(Z,Y)\,.$$

5.3

We note that, starting from (14), and using (3), we have

$$\begin{aligned}\overset{*}{\rho}(W_3)(Y,Z) &= \sum_{i=1}^{2n} W_3(e_i, Y, JZ, Je_i)\\ &= 2\Big\{\frac{1}{n-1}\big[\,\rho(Y,Z)+\rho(JY,JZ)\big] - \big[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]\\ &\quad - \frac{\tau}{(n-1)(2n-1)}g(Y,Z)\Big\}\\ &= 2V(Y,Z)\,,\end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(W_3) = 4\Big(\frac{\tau}{2n-1} - \overset{*}{\tau}\Big)\,.$$

Substituting this into

$$\begin{aligned}A(W_3)(X,Y,Z,W) = W_3(X,Y,Z,W) &+ \frac{1}{2(n+1)}\big\{F(X,W)\overset{*}{\rho}(W_3)(Y,JZ)\\ &+ F(Y,Z)\overset{*}{\rho}(W_3)(X,JW) - F(X,Z)\overset{*}{\rho}(W_3)(Y,JW)\\ &- F(Y,W)\overset{*}{\rho}(W_3)(Y,JZ) - 2F(X,Y)\overset{*}{\rho}(W_3)(Z,JW)\\ &- 2F(Z,W)\overset{*}{\rho}(W_3)(X,JY)\big\}\\ &+ \frac{\overset{*}{\tau}(W_3)}{2(n+1)(2n+1)}\,\pi_2(X,Y,Z,W)\,,\end{aligned}$$

we get

$$A(W_3) = W_3 + \frac{1}{n+1}W_4 + \frac{2}{(n+1)(2n+1)}W_6\,. \tag{33}$$

As for the tensor W_4, we have

$$\begin{aligned}\overset{*}{\rho}(W_4)(Y,Z) = -2(n+1)&\Big\{\frac{1}{n-1}[\,\rho(Y,Z)+\rho(JY,JZ)] - [\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)]\Big\}\\ &+ 2\Big[\frac{2\tau}{(n-1)(2n-1)} + \overset{*}{\tau}\Big]g(Y,Z)\,,\end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(W_4) = -4(2n+1)\Big[\frac{\tau}{2n-1} - \overset{*}{\tau}\Big]\,.$$

Thus

$$A(W_4) = W_4 - \frac{1}{n-1}\psi + \overset{*}{\psi} - \frac{2\tau}{(n-1)(2n-1)}\pi_2 = 0\,.$$

In a similar way we find

$$A(W_5) = \Big(\frac{\tau}{2n-1} - \overset{*}{\tau}\Big)\Big(\pi_1 - \frac{\pi_2}{2n+1}\Big)\,, \qquad A(W_6) = 0\,.$$

Summing up the preceding results, we can state the following theorem.

Theorem 5. *Applying* (18) *to the tensors* W, *we get*

$$A(W_1) \equiv A(AR_W) = (AR)_W\,, \qquad A(W_2) = W_2\,,$$
$$A(W_3) = W_3 + \frac{1}{n+1}W_4 + \frac{2}{(n+1)(2n+1)}W_6\,, \qquad A(W_4) = 0\,,$$
$$A(W_5) = W_5 - \frac{1}{2n+1}W_6\,, \qquad A(W_6) = 0\,.$$

6 Kähler spaces

In the case of Kähler spaces, we have

$$\overset{*}{\rho} = \rho, \quad \overset{*}{\tau} = \tau, \quad \rho(JX, JY) = \rho(X, Y)\,, \tag{34}$$

and therefore, the conformally invariant tensor (13) has the form

$$V(Y, Z) = \frac{2(n-2)}{n-1}\rho(Y, Z) - \frac{\tau}{(n-1)(2n-1)}g(Y, Z)\,.$$

Also, putting

$$\begin{aligned}\psi_0(X, Y, Z, W) = {} & F(X, W)\rho(Y, JZ) + F(Y, Z)\rho(X, JW) \\ & - F(X, Z)\rho(Y, JW) - F(Y, W)\rho(X, JZ) \\ & - 2F(X, Y)\rho(Z, JW) - 2F(Z, W)\rho(X, JY)\,,\end{aligned}$$

we get

$$\psi = \overset{*}{\psi} = 2\psi_0\,. \tag{35}$$

Thus, for Kähler manifolds, the antiholomorphic curvature tensor is

$$AR = R + \frac{1}{2(n+1)}\psi_0 + \frac{\tau}{2(n+1)(2n+1)}\pi_2\,. \tag{36}$$

Using (34), (35) and (36), we find that, for Kähler manifolds, the relation (26) becomes

$$W_1 \equiv (AR)_W = R - \frac{1}{2(n+2)}(\varphi - \psi_0) + \frac{\tau}{4(n+1)(n+2)}(\pi_1 + \pi_2)\,,$$

or, explicitly,

$$\begin{aligned}(W_1)(X, Y, Z, W) \equiv {} & (AR)_W(X, Y, Z, W) = R(X, Y, Z, W) \\ & - \frac{1}{2(n+1)}\Big\{g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) - g(X, Z)\rho(Y, W) \\ & - g(Y, W)\rho(X, Z) - F(X, W)\rho(Y, JZ) - F(Y, Z)\rho(X, JW) + F(X, Z)\rho(Y, JW) \\ & + F(Y, W)\rho(X, JZ) + 2F(X, Y)\rho(Z, JW) + 2F(Z, W)\rho(X, JY)\Big\} \\ & + \frac{\tau}{4(n+1)(n+2)}[\pi_1(X, Y, Z, W) + \pi_2(X, Y, Z, W)]\,.\end{aligned}$$

But this is the Bochner curvature tensor of a Kähler manifold. Thus, we can state the following theorem.

Theorem 6. *For a Kähler manifold, the Weyl component of the antiholomorphic curvature tensor, $(AR)_W$, is Bochner curvature tensor.*

In a similar way we find that, in the case of Kähler manifolds,

$$
\begin{aligned}
W_2 &= R - \frac{1}{2(n-1)}\varphi + \frac{n-2}{2(n^2-1)}\psi_0 \\
&\quad + \tau\Big[\frac{\pi_1}{2(n-1)(2n-1)} + \frac{2n^2-6n+1}{2(n^2-1)(4n^2-1)}\pi_2\Big], \\
W_3 &= -\frac{2(n-2)}{n-1}\varphi - \frac{2\tau}{(n-1)(2n-1)}\pi_1, \qquad W_5 = -\frac{2(n-1)}{2n-1}\pi_1, \\
W_4 &= -\frac{2(n-2)}{n-1}\psi_0 + \frac{2\tau}{(n-1)(2n-1)}\pi_2, \qquad W_6 = -\frac{2(n-1)}{2n-1}\pi_2.
\end{aligned}
$$

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