Janusz Zieliński; Piotr Ossowski

Rings of constants of generic 4D Lotka-Volterra systems

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 2, 529–538

Persistent URL: http://dml.cz/dmlcz/143330

Czechoslovak Mathematical Journal, 63 (138) (2013), 529–538

RINGS OF CONSTANTS OF GENERIC 4D LOTKA-VOLTERRA SYSTEMS

Janusz Zieliński, Piotr Ossowski, Toruń

(Received March 6, 2012)

Abstract. We show that the rings of constants of generic four-variable Lotka-Volterra derivations are finitely generated polynomial rings. We explicitly determine these rings, and we give a description of all polynomial first integrals of their corresponding systems of differential equations. Besides, we characterize cofactors of Darboux polynomials of arbitrary four-variable Lotka-Volterra systems. These cofactors are linear forms with coefficients in the set of nonnegative integers. Lotka-Volterra systems have various applications in such branches of science as population biology and plasma physics, among many others.

Keywords: Lotka-Volterra derivation, polynomial constant, polynomial first integral, Darboux polynomial

MSC 2010: 13N15, 12H05, 92D25, 34A34

1. Introduction

Throughout this paper, k is a field of characteristic zero. By $k[X]$ we denote $k[x_1,\ldots,x_n]$, the polynomial ring in n variables. For $n \leqslant 3$ the ring of constants of any derivation of $k[X]$ is finitely generated (see [7]). For $n = 4$ the ring of constants may not be finitely generated. An example was given in [3]. There is no general procedure for determining the ring of constants, nor even deciding whether it is finitely generated. Even for a given specific derivation of $k[X]$ the problem may be difficult, see various counterexamples to Hilbert's fourteenth problem (for example [3]) and the three-variable Lotka-Volterra derivation (for example [5]). Such problems are closely linked to the invariant theory, namely for every connected algebraic group $G \subseteq \mathrm{Gl}_n(k)$ there exists a derivation d such that $k[X]^G = k[X]^d$ (see, for instance, [6]).

It is well known that Lotka-Volterra systems play a significant role in population biology. They also have many applications in other branches of science, for instance in plasma physics (for more details we refer the reader to [1] and its extensive bibliography). Moreover, they play an important part in the derivation theory itself. A derivation $d\colon k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \ldots, n$. Examples of such derivations are Lotka-Volterra derivations. How to associate a factorizable derivation with any given derivation is shown in [10]. The construction helps to establish new facts on constants of the initial derivation (see, for instance, [8]). We have thus a special interest in describing constants of factorizable derivations.

Section 3 provides some facts on Darboux polynomials of Lotka-Volterra derivations in 4 variables with arbitrary coefficients. Section 4 contains several properties of Lotka-Volterra derivations for n variables, which supply potential tools for further studies. In Section 5, we prove Theorem 5.1, which gives a full description of the ring of polynomial constants of the derivation $d\colon k[x_1, \ldots, x_4] \to k[x_1, \ldots, x_4]$ defined by

$$d = \sum_{i=1}^{4} x_i(x_{i-1} - C_i x_{i+1}) \frac{\partial}{\partial x_i},$$

for C_i not belonging to the set of positive rationals. It is the main result of the paper. As a consequence we obtain that a generic four-variable Lotka-Volterra system has a finitely generated ring of constants.

2. Notation and preliminaries

If R is a commutative k-algebra, then a k-linear map $d\colon R \to R$ is called a *derivation* of R if $d(ab) = ad(b) + d(a)b$ for all $a, b \in R$. We call $R^d = \ker d$ the *ring of constants* of the derivation d. If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d\colon k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. The set $k[X]^d \setminus k$ is equal to the set of all polynomial first integrals of the corresponding system of ordinary differential equations (see [6] for more details).

A derivation $d\colon k[X] \to k[X]$ is called *homogeneous of degree s* if the image of a homogeneous form of degree t under d is a homogeneous form of degree $s + t$ for all $t \in \mathbb{N}$. Since k is a field of characteristic zero, we have $\mathbb{Q} \subseteq k$. Let $\mathbb{Q}_+$ denote the set of positive rationals and $\mathbb{N}$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in k[X]$ and by $|\alpha|$ the sum $\alpha_1 + \ldots + \alpha_n$.

Let $n \geqslant 3$. Throughout the rest of this paper, $R = k[x_1, \ldots, x_n]$ and $d\colon R \to R$ is a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1}),$$

for $i = 1, \dots, n$, and we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$. All our considerations are in the cyclic sense; for example, $\{i, i+1\}$ admits also $\{n, 1\}$. We write a minus sign before C_i just to simplify further computations. Denote by $R_{(m)}$ the homogeneous component of R of degree m. Let $R^d_{(m)} = R_{(m)} \cap R^d$. Since d is homogeneous, we have $R^d = \bigoplus_{m=0}^{\infty} R^d_{(m)}$ and we need only to determine the homogeneous constants.

3. Darboux polynomials

A nonzero polynomial f is said to be a *Darboux polynomial* of a derivation $\delta\colon R \to R$ if $\delta(f) = \Lambda f$ for some $\Lambda \in R$. We will call Λ a *cofactor* of f. Since R is a domain, Λ is unique. The product $f_1 f_2$ of Darboux polynomials is a Darboux polynomial and its cofactor equals the sum of the cofactors of f_1 and f_2.

Proposition 3.1 is well known (see [6], Proposition 2.2.1). It is true for k being any unique factorization domain and any derivation δ of $k[x_1, \dots, x_n]$.

Proposition 3.1. *If $f \in R$ is a Darboux polynomial of δ, then all factors of f are also Darboux polynomials of δ.*

We call a polynomial $g \in R$ *strict* if it is nonzero, homogeneous and not divisible by the variables $x_1, \dots, x_n$. Every nonzero homogeneous polynomial $f \in R$ has a unique presentation $f = X^\alpha g$, where X^α is a monomial and g is strict.

If f is a Darboux polynomial of a homogeneous derivation δ with a cofactor Λ, then every homogeneous part of f is a Darboux polynomial of δ with the same cofactor Λ (see [6], Proposition 2.2.3).

If $f = X^\alpha g$ is a Darboux polynomial of the derivation d, then it is easy to compute the cofactor of the monomial X^α (see the proof of Lemma 3.4). Thus we are going to characterize cofactors of strict Darboux polynomials (Lemma 3.2 and Corollary 3.3). Such a characterization for 3 variables was done in [4]. Since d is a homogeneous derivation of degree 1, the cofactor of any homogeneous Darboux polynomial is a homogeneous form of degree 1.

Lemma 3.2. *Let $n = 4$. Let $g \in R_{(m)}$ be a Darboux polynomial of d with the cofactor $\lambda_1 x_1 + \ldots + \lambda_4 x_4$. Let $i \in \{1, 2, 3, 4\}$. If g is not divisible by x_i, then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if $g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_4) = x_{i+2}^{\beta_{i+2}} \overline{G}$ and $x_{i+2} \nmid \overline{G}$, then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2} \lambda_{i+1}$.*

P r o o f. Without loss of generality we can assume that $i=4$. Since g is a Darboux polynomial, we have

$$\sum_{i=1}^{4} x_i(x_{i-1}-C_i x_{i+1})\frac{\partial g}{\partial x_i}=(\lambda_1 x_1+\ldots+\lambda_4 x_4)g.$$

We put $x_4=0$ in the equation above and obtain

$$-x_1 C_1 x_2\frac{\partial G}{\partial x_1}+x_2(x_1-C_2x_3)\frac{\partial G}{\partial x_2}+x_3x_2\frac{\partial G}{\partial x_3}=(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)G,$$

where $G=g(x_1,x_2,x_3,0)\neq 0$, since $x_4\nmid g$.

Let $G=x_2^{\beta_2}\overline{G}$, where $x_2\nmid\overline{G}$ and $\beta_2\in\mathbb{N}$. Then

$$\begin{aligned}(3.1)\qquad -C_1x_1x_2x_2^{\beta_2}\frac{\partial\overline{G}}{\partial x_1}+x_2(x_1-C_2x_3)\Big(\beta_2x_2^{\beta_2-1}\overline{G}+x_2^{\beta_2}\frac{\partial\overline{G}}{\partial x_2}\Big)+x_3x_2x_2^{\beta_2}\frac{\partial\overline{G}}{\partial x_3}\\ =(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)x_2^{\beta_2}\overline{G}\end{aligned}$$

(if $\beta_2=0$, then we assume that expression $\beta_2x_2^{\beta_2-1}$ is equal to 0). We divide both sides of (3.1) by $x_2^{\beta_2}$, then we add $(C_2x_3-x_1)\beta_2\overline{G}$ to both sides of (3.1) and we obtain

$$\begin{aligned}(3.2)\qquad -C_1x_1x_2\frac{\partial\overline{G}}{\partial x_1}+x_2(x_1-C_2x_3)\frac{\partial\overline{G}}{\partial x_2}+x_3x_2\frac{\partial\overline{G}}{\partial x_3}\\ =((\lambda_1-\beta_2)x_1+\lambda_2x_2+(\lambda_3+C_2\beta_2)x_3)\overline{G}.\end{aligned}$$

The left-hand side of (3.2) is the divisible by x_2, so also is the right-hand side of (3.2). Since $x_2\nmid\overline{G}$, we get

$$x_2\mid(\lambda_1-\beta_2)x_1+\lambda_2x_2+(\lambda_3+C_2\beta_2)x_3.$$

Hence $\lambda_1-\beta_2=0$ and $\lambda_3+C_2\beta_2=0$. Finally, $\lambda_1=\beta_2$ and $\lambda_3=-C_2\beta_2=-C_2\lambda_1$. □

Corollary 3.3. *Let $n=4$. If $g\in R_{(m)}$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in $\mathbb{N}$.*

Lemma 3.4. *Let $n=4$. If $d(f)=0$ and $f=X^\alpha g$, where g is strict, then $d(X^\alpha)=0$ and $d(g)=0$.*

P r o o f. If $d(f) = 0$, then f is a Darboux polynomial. In view of Proposition 3.1, also X^α and g are Darboux polynomials. If $\alpha = (\alpha_1, \ldots, \alpha_4)$, then a short computation shows that the cofactor of X^α equals $(\alpha_2 - \alpha_4 C_4)x_1 + (\alpha_3 - \alpha_1 C_1)x_2 + (\alpha_4 - \alpha_2 C_2)x_3 + (\alpha_1 - \alpha_3 C_3)x_4$. The polynomial g is strict, therefore by Lemma 3.2, if $\lambda_1 x_1 + \ldots + \lambda_4 x_4$ is the cofactor of g, then $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{N}$ and $\lambda_1 = -C_4\lambda_3$, $\lambda_2 = -C_1\lambda_4$, $\lambda_3 = -C_2\lambda_1$, $\lambda_4 = -C_3\lambda_2$. The cofactor of the product $X^\alpha g$ is the sum of the cofactors of X^α and g, that is, equals

$$(\alpha_2 - \alpha_4 C_4 + \lambda_1)x_1 + (\alpha_3 - \alpha_1 C_1 + \lambda_2)x_2 + (\alpha_4 - \alpha_2 C_2 + \lambda_3)x_3 + (\alpha_1 - \alpha_3 C_3 + \lambda_4)x_4.$$

On the other hand, by assumption, this cofactor is equal to 0. Thus

$$\begin{aligned} \alpha_2 - \alpha_4 C_4 + \lambda_1 &= 0, \\ \alpha_3 - \alpha_1 C_1 + \lambda_2 &= 0, \\ \alpha_4 - \alpha_2 C_2 + \lambda_3 &= 0, \\ \alpha_1 - \alpha_3 C_3 + \lambda_4 &= 0. \end{aligned}$$

Suppose g is not a constant of d. Then $\lambda_i \neq 0$ for some $i \in \{1, \ldots, 4\}$. There is no loss of generality in assuming that $i = 1$. Then $\lambda_1 = -C_4\lambda_3$ implies that also $\lambda_3 \neq 0$. Hence $C_4 = -\lambda_1/\lambda_3 < 0$. Then $\alpha_2 \geqslant 0$, $-\alpha_4 C_4 \geqslant 0$ and $\lambda_1 > 0$. Therefore $\alpha_2 - \alpha_4 C_4 + \lambda_1 > 0$, which is a contradiction. This proves that $d(g) = 0$.

If $d(X^\alpha g) = 0$ and $d(g) = 0$, then obviously $d(X^\alpha) = 0$. □

4. Restrictions of polynomials

Let $\varphi \in R$ and $1 \leqslant q \leqslant n$. Then for every subset $\{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\}$ we denote by $\varphi^{\{i_1,\ldots,i_q\}}$ the sum of terms of φ that depend on variables $x_{i_1}, \ldots, x_{i_q}$, that is, $\varphi^{\{i_1,\ldots,i_q\}} = \varphi|_{x_j=0 \text{ for } j\notin\{i_1,\ldots,i_q\}}$. We noticed that for inductive purposes it is more convenient to deal with polynomials φ such that $d(\varphi^A)^A = 0$ for a given $A \subseteq \{1, \ldots, n\}$, than with the constants themselves.

The first three results, that is 4.1, 4.2 and 4.3, are similar to those for $C_1 = \ldots = C_n = 1$ of our paper [9]. As an obvious consequence of the fact that $x_i \mid d(x_i)$, for $i = 1, \ldots, n$, we obtain the following proposition.

Proposition 4.1. *If $A \subseteq \{1, \ldots, n\}$, then for every homogeneous polynomial $\varphi \in R_{(m)}$, we have $d(\varphi^A)^A = d(\varphi)^A$.*

Corollary 4.2. *If $A \subseteq \{1, \ldots, n\}$, then for every $\varphi \in R^d_{(m)}$ we have $d(\varphi^A)^A = 0$.*

Lemma 4.3. *If $B \subseteq A \subseteq \{1,\dots,n\}$ and $d(\varphi^A)^A = 0$, then also $d(\varphi^B)^B = 0$.*

Proof. Let $\varphi^A = \varphi^B + \psi$, where each monomial in ψ has x_j in a positive power for some $j \in A \setminus B$. Then $d(\varphi^A) = d(\varphi^B) + d(\psi)$. If $d(\varphi^A)^A = 0$, then clearly $d(\varphi^A)^B = 0$. Therefore $0 = d(\varphi^A)^B = d(\varphi^B)^B + d(\psi)^B$. Moreover $d(\psi)^B = 0$, because every monomial in $d(\psi)$ has x_j in positive a power for some $j \in A \setminus B$, by the definition of d. Finally, $d(\varphi^B)^B = 0$. □

We formulated Lemma 4.4 in [9] without a proof. Note that there is no assumption on the coefficients C_i in this lemma.

Lemma 4.4. *Let $\varphi \in R_{(m)}$ and $A = \{i, i+1\} \subset \{1,\dots,n\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_i x_{i+1})^m$, for $a \in k$.*

Proof. Let $\varphi^A = \sum\limits_{r=0}^{m} b_r x_i^{m-r} x_{i+1}^r$. Then

$$
\begin{aligned}
d(\varphi^A) &= \sum_{r=0}^{m} b_r (d(x_i^{m-r}) x_{i+1}^r + x_i^{m-r} d(x_{i+1}^r)) \\
&= \sum_{r=0}^{m} b_r x_i^{m-r} x_{i+1}^r ((m-r)(x_{i-1} - C_i x_{i+1}) + r(x_i - C_{i+1} x_{i+2})).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(\varphi^A)^A &= \sum_{r=0}^{m} b_r (r x_i^{m-r+1} x_{i+1}^r - C_i (m-r) x_i^{m-r} x_{i+1}^{r+1}) \\
&= \sum_{r=1}^{m} r b_r x_i^{m-r+1} x_{i+1}^r - C_i \sum_{r=0}^{m-1} (m-r) b_r x_i^{m-r} x_{i+1}^{r+1} \\
&= \sum_{r=1}^{m} r b_r x_i^{m-r+1} x_{i+1}^r - C_i \sum_{r=1}^{m} (m-r+1) b_{r-1} x_i^{m-r+1} x_{i+1}^r \\
&= \sum_{r=1}^{m} (r b_r - C_i (m-r+1) b_{r-1}) x_i^{m-r+1} x_{i+1}^r = 0.
\end{aligned}
$$

Hence for $r = 1,\dots,m$ we have $r b_r = C_i(m-r+1) b_{r-1}$, that is, $b_r = \frac{m-r+1}{r} C_i b_{r-1}$. Thus an easy induction on r shows that $b_r = \binom{m}{r} C_i^r b_0$ for $r = 0,\dots,m$. Consequently, $\varphi^A = b_0(x_i + C_i x_{i+1})^m$. □

Note that the above $a = b_0$ may be equal to 0. Here and throughout, by the *support* of $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ we mean the set $\operatorname{supp}(\alpha) = \{i\colon \alpha_i \neq 0\}$. Observe that there is an assumption on only one coefficient C_i in Lemma 4.5.

Lemma 4.5. *Let $n \geqslant 4$, $\varphi \in R_{(m)}$ and $A = \{i, i+1, i+2\} \subset \{1,\dots,n\}$. If $d(\varphi^A)^A = 0$ and $C_i \notin \mathbb{Q}_+$, then $\varphi^A \in k[x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2}]$.*

P r o o f. Let $m = 1$. By assumption and Lemma 4.3, $d(\varphi^{\{i,i+1\}})^{\{i,i+1\}} = 0$. In view of Lemma 4.4, we have $\varphi^{\{i,i+1\}} = a_1(x_i + C_i x_{i+1})$. Similarly, we obtain $\varphi^{\{i+1,i+2\}} = a_2(x_{i+1} + C_{i+1}x_{i+2})$. Thus $a_2 = a_1 C_i$ and $\varphi^A = a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})$. Now let $m = 2$. Since $d(\varphi^{\{i,i+1\}})^{\{i,i+1\}} = 0$, it follows that $\varphi^{\{i,i+1\}} = a_1(x_i + C_i x_{i+1})^2$. Analogously $\varphi^{\{i+1,i+2\}} = a_2(x_{i+1} + C_{i+1}x_{i+2})^2$. Hence $a_2 = a_1 C_i^2$ and $\varphi^{\{i+1,i+2\}} = a_1(C_i x_{i+1} + C_i C_{i+1} x_{i+2})^2$. Therefore, $\varphi^A = a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^2 + b x_i x_{i+2}$ for some $b \in k$. Applying first $d(\cdot)$ and then $(\cdot)^A$ to both sides of the last equation we get $0 = b(1 - C_i)x_i x_{i+1} x_{i+2}$. Since $C_i \neq 1$, we have $b = 0$.

Assume $m \geqslant 3$. Then φ^A is a linear combination of monomials X^α such that $|\alpha| = m$ and $\operatorname{supp}(\alpha) \subseteq \{i, i+1, i+2\}$. We have $\varphi^{\{i,i+1\}} = a_1(x_i + C_i x_{i+1})^m$ and $\varphi^{\{i+1,i+2\}} = a_2(x_{i+1} + C_{i+1}x_{i+2})^m$, for $a_1, a_2 \in k$. Thus $a_2 = a_1 C_i^m$ and $\varphi^{\{i+1,i+2\}} = a_1(C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m$. The terms of the form $x_i^r x_{i+1}^{m-r}$ and $x_{i+1}^r x_{i+2}^{m-r}$ for $r = 0, \ldots, m$ have the same coefficients in φ^A and in $a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m$. Therefore

$$\varphi^A = a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m + \sum_{\operatorname{supp}(\alpha)=\{i,i+2\}} b_\alpha X^\alpha + \sum_{\operatorname{supp}(\alpha)=\{i,i+1,i+2\}} b_\alpha X^\alpha,$$

that is, $\varphi^A = a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m + x_i x_{i+2}\psi$, where $\psi \in R_{(m-2)}$ and ψ depends on the variables x_i, x_{i+1}, x_{i+2} only. We show that $\psi = 0$. First,

$$\begin{aligned} d(\varphi^A) &= a_1 d((x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m) + d(x_i x_{i+2})\psi + x_i x_{i+2} d(\psi) \\ &= a_1 m (x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^{m-1}(x_i x_{i-1} - C_i C_{i+1} C_{i+2} x_{i+2} x_{i+3}) \\ &\quad + (x_{i-1} + (1 - C_i)x_{i+1} - C_{i+2}x_{i+3}) x_i x_{i+2}\psi + x_i x_{i+2} d(\psi). \end{aligned}$$

Obviously, $\psi^A = \psi$. Therefore,

$$0 = d(\varphi^A)^A = (1 - C_i)x_i x_{i+1} x_{i+2}\psi + x_i x_{i+2} d(\psi)^A.$$

Hence $d(\psi)^A = (C_i - 1)x_{i+1}\psi$.

Suppose $\psi \neq 0$. Let $s = \deg_{x_{i+1}} \psi$. Let $b x_i^r x_{i+1}^s x_{i+2}^t$ be a term of ψ with $b \in k \setminus \{0\}$ (we fix one of the terms of ψ that are divisible by x_{i+1}^s). Then the coefficient of the monomial $x_i^r x_{i+1}^{s+1} x_{i+2}^t$ in the expansion of $(C_i - 1)x_{i+1}\psi$ equals $(C_i - 1)b$. The coefficient of $x_i^r x_{i+1}^{s+1} x_{i+2}^t$ in the expansion of $d(\psi)^A$ is equal to $b(t - rC_i)$ (because in all terms of the d-image of any term the exponent of only one variable may be increased). Therefore $C_i = (t + 1)/(r + 1) \in \mathbb{Q}_+$. The contradiction obtained proves that $\psi = 0$.

Thus $\varphi^A = a_1(x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2})^m \in k[x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2}]$. □

Theorem 5.1. *Let $R = k[x_1, \dots, x_4]$ and $C_1, \dots, C_4 \notin \mathbb{Q}_+$. Let $d\colon R \to R$ be a derivation of the form*

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1}),$$

for $i = 1, \dots, 4$. If $C_1C_2C_3C_4 = 1$, then

$$R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4].$$

If $C_1C_2C_3C_4 \neq 1$, then $R^d = k$.

P r o o f. First we show that $R^d_{(m)} \subseteq k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4]$, for all $m \geqslant 0$. Let $A_1 = \{2,3,4\}, A_2 = \{1,3,4\}, A_3 = \{1,2,4\}, A_4 = \{1,2,3\}$ and let $\varphi \in R^d_{(m)}$. By Corollary 4.2 and Lemma 4.5, $\varphi^{A_i} = a_{i+1}(x_{i+1} + C_{i+1}x_{i+2} + C_{i+1}C_{i+2}x_{i+3})^m$, for $i = 1, \dots, 4$. Comparison of the coefficients of x_2^m in φ^{A_1} and φ^{A_4} gives $a_2 = a_1C_1^m$. Analogously, $a_3 = a_2C_2^m = a_1C_1^mC_2^m$ and $a_4 = a_3C_3^m = a_1C_1^mC_2^mC_3^m$. Let $\psi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m$. Then $\varphi^{A_i} = \psi^{A_i}$, for $i = 1, \dots, 4$. This means that the polynomials φ and ψ have the same terms that depend on at most three variables. Therefore

$$\varphi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m + \eta,$$

where each term of the polynomial η has all four variables in positive powers, that is, η is divisible by $x_1x_2x_3x_4$.

We show that η is a constant of the derivation d. If $m < 4$, then $\eta = 0$, since $x_1x_2x_3x_4 \mid \eta$. Assume, then, that $m \geqslant 4$. If $C_1C_2C_3C_4 = 1$, then φ and $x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4$ are constants of d, so also is η. If $C_1C_2C_3C_4 \neq 1$, then

$$0 = d(\varphi) = a_1m(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^{m-1}x_1x_4(1 - C_1C_2C_3C_4) + d(\eta).$$

The derivation d is factorizable, hence $x_1x_2x_3x_4 \mid \eta$ implies $x_1x_2x_3x_4 \mid d(\eta)$. Therefore, the coefficient of $x_1^mx_4$ in $d(\varphi)$ equals 0, on the one hand, and is equal to $a_1m(1 - C_1C_2C_3C_4)$, on the other hand. Thus $a_1 = 0$ and $\varphi = \eta$. In particular, η is a constant of d.

We show that $\eta = 0$. Suppose that η is a monomial. Let $\eta = cx_1^rx_2^sx_3^tx_4^u$, where $r, s, t, u \geqslant 1$. Then

$$0 = d(\eta) = cx_1^rx_2^sx_3^tx_4^u((s - uC_4)x_1 + (t - rC_1)x_2 + (u - sC_2)x_3 + (r - tC_3)x_4).$$

If $c \neq 0$, then $C_4 = s/u \in \mathbb{Q}_+$, which is a contradiction. Then $c = 0$ and $\eta = 0$.

Suppose that η is not a term. Then $\eta = X^\alpha g$, where X^α is a monomial and g is strict. Since η is divisible by $x_1x_2x_3x_4$, the monomial X^α has positive exponents. Since η is a constant, by Lemma 3.4 also X^α and g are. However, the considerations above prove that no monomial of positive exponents is a constant of d.

Thus $\eta = 0$ and $\varphi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m \in k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$. Consequently, $R^d \subseteq k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$.

Case $C_1C_2C_3C_4 = 1$. Since $d(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4) = x_1x_4 - C_1C_2C_3C_4x_1x_4 = 0$, we have $k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4] \subseteq R^d$.

Case $C_1C_2C_3C_4 \neq 1$. Let $a \in k \setminus \{0\}$ and $m \in \{1, 2, \ldots\}$. Then

$$\begin{aligned} &d(a(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m) \\ &\quad = am(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^{m-1}(x_1x_4 - C_1C_2C_3C_4x_1x_4) \neq 0. \end{aligned}$$

Thus $a = 0$ or $m = 0$. Hence, $R^d = k$. □

Corollary 5.2. *If $k = \mathbb{R}$ or $k = \mathbb{C}$, then in the generic case a four-variable Lotka-Volterra derivation has a finitely generated (even trivial) ring of constants.*

Lotka-Volterra derivations with positive rational coefficients are investigated for instance in [4], [5], [9], [11].

Note that if we consider a field k of a positive characteristic p, then all elements of the form x_i^p are constants of any polynomial derivation. For more information on this case we refer the reader to [2] and its bibliography.

References

[1] *O. I. Bogoyavlenskij*: Algebraic constructions of integrable dynamical systems: extension of the Volterra system. Russ. Math. Surv. *46* (1991), 1–64; translation from Usp. Mat. Nauk *46* (1991), 3–48. (In Russian.)

[2] *P. Jędrzejewicz*: Positive characteristic analogs of closed polynomials. Cent. Eur. J. Math. *9* (2011), 50–56.

[3] *S. Kuroda*: Fields defined by locally nilpotent derivations and monomials. J. Algebra *293* (2005), 395–406.

[4] *J. Moulin Ollagnier*: Polynomial first integrals of the Lotka-Volterra system. Bull. Sci. Math. *121* (1997), 463–476.

[5] *J. Moulin-Ollagnier, A. Nowicki*: Polynomial algebra of constants of the Lotka-Volterra system. Colloq. Math. *81* (1999), 263–270.

[6] *A. Nowicki*: Polynomial Derivations and Their Rings of Constants. Uniwersytet Mikołaja Kopernika, Toruń, 1994.

[7] *A. Nowicki, M. Nagata*: Rings of constants for k-derivations in $k[x_1, \ldots, x_n]$. J. Math. Kyoto Univ. *28* (1988), 111–118.

[8] *A. Nowicki, J. Zieliński*: Rational constants of monomial derivations. J. Algebra *302* (2006), 387–418.

[9] *P. Ossowski, J. Zieliński*: Polynomial algebra of constants of the four variable Lotka-Volterra system. Colloq. Math. *120* (2010), 299–309.

[10] *J. Zieliński*: Factorizable derivations and ideals of relations. Commun. Algebra *35* (2007), 983–997.
[11] *J. Zieliński*: The five-variable Volterra system. Cent. Eur. J. Math. *9* (2011), 888–896.

Authors' address: Janusz Zieliński (corresponding author), Piotr Ossowski, Faculty of Mathematics and Computer Science, N. Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland, e-mails: `ubukrool@mat.uni.torun.pl`, `ossowski@mat.uni.torun.pl`.