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ESTIMATES IN THE HARDY-SOBOLEV SPACE OF THE ANNULUS AND STABILITY RESULT

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Abstract. The main purpose of this work is to establish some logarithmic estimates of optimal type in the Hardy-Sobolev space $H^{k,\infty}$; $k \in \mathbb{N}^*$ of an annular domain. These results are considered as a continuation of a previous study in the setting of the unit disk by L. Baratchart and M. Zerner, On the recovery of functions from pointwise boundary values in a Hardy-Sobolev class of the disk, J. Comput. Appl. Math. 46 (1993), 255–269 and by S. Chaabane and I. Feki, Optimal logarithmic estimates in Hardy-Sobolev spaces $H^{k,\infty}$, C. R., Math., Acad. Sci. Paris 347 (2009), 1001–1006.

As an application, we prove a logarithmic stability result for the inverse problem of identifying a Robin parameter on a part of the boundary of an annular domain starting from its behavior on the complementary boundary part.

Keywords: annular domain, Poisson kernel, Hardy-Sobolev space, logarithmic estimate, Robin parameter

MSC 2010: 30H10, 30C40, 35R30

1. Introduction

The purpose of this paper is to establish logarithmic estimates of optimal type in the Hardy-Sobolev space $H^{1,\infty}(G_s)$ where $s \in]0,1[$ and G_s is the annulus of radius $(s,1)$. More precisely, we study the behavior on the whole boundary of the annulus G_s with respect to the uniform norm of any function f in the unit ball of the Hardy-Sobolev space $H^{1,\infty}(G_s)$ starting from its behavior on any open connected subset $I \subset \partial G_s$ with respect to the L^1-norm. Our result can be viewed as an extension of those established in [7], [14], [15].

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The particular case where $I = \mathbb{T}$ has been considered by L. Leblond, M. Mahjoub and J. R. Partington in [14]. The authors proved in this case that the L^2-norm of any function f in the unit ball of the Hardy-Sobolev space $H^{1,2}(G_s)$ on the inner boundary $s\mathbb{T}$ is controlled by the corresponding norm taken on the outer boundary $\mathbb{T}$. In the same context, H. Meftahi and F. Wieolonsky gave recently in [15] an explicit logarithmic inequality exhibiting the dependence with respect to the inner radius s of the above control. The first estimate of this kind remounts to L. Baratchart and M. Zerner in [2] where the authors proved, a $\log\log / \log$ control with L^2-norm in the Hardy-Sobolev space $H^{1,2}$ of the unit disk $\mathbb{D}$. In [1], Alessandrini et al. have proved by a quite different method an estimate of $1/\log^{\alpha}$-type, $0 < \alpha < 1$. Recently, the author of this paper together with S. Chaabane [5] proved in the uniform norm some optimal logarithmic estimates in the Hardy-Sobolev space $H^{k,\infty}(\mathbb{D})$; $k \in \mathbb{N}^*$.

For more regular functions, we improve inequality (3.6) amid the class of bounded $H^{k,\infty}(G_s)$ functions.

These logarithmic estimates allow us to prove a stability result for the inverse problem of recovering a Robin coefficient on a part of the boundary of an annular domain starting from its behavior on the complementary boundary part. The particular case where the inaccessible part of the boundary is the inner circle has been proved in [14]. We can also refer the reader to [1], [5], [7], [14], [15] for stability estimates in the case of simply or doubly connected domains.

2. Notation and preliminary results

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ with boundary $\mathbb{T}$ and let G_s denote the annulus:

$$G_s = \{z \in \mathbb{C};\ s < |z| < 1\}; \quad 0 < s < 1.$$

The boundary of the annular domain G_s consists of two pieces $s\mathbb{T}$ and $\mathbb{T}$: $\partial G_s = s\mathbb{T} \cup \mathbb{T}$. Let I be any connected open subset of the boundary of G_s and let $J = \partial G_s \backslash I$. We also equip the boundary ∂G_s with the usual Lebesgue measure μ normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$ each have unit measure. Furthermore, we denote $\lambda = \mu(I)/(2\pi)$, assume that $\lambda \in\ ]0,1[$ and define

$$\|f\|_{L^1(I)} = \frac{1}{2\pi\lambda}\int_I |f(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta,$$

for the L^1-norm of f on I, where $r = s$ if $I \subset s\mathbb{T}$ and $r = 1$ if $I \subset \mathbb{T}$.

In the sequel, the Hardy space $H^\infty(G_s)$ is defined as the space of bounded analytic functions on G_s. According to ([11], Theorem 7.1), the Hardy space $H^\infty(G_s)$ can be identified with the direct sum

$$H^\infty(G_s) = H^\infty(\mathbb{D}) \oplus H_0^\infty(\mathbb{C} \setminus s\overline{\mathbb{D}}),$$

where the Hardy space $H_0^\infty(\mathbb{C} \setminus s\overline{\mathbb{D}})$ is defined as the set of analytic functions in $\mathbb{C} \setminus s\overline{\mathbb{D}}$, with a zero limit at infinity. Hence we can regard it as a closed subspace $H^\infty(\partial G_s)$ of $L^\infty(\partial G_s)$. Equivalent definitions of Hardy spaces on annular domains are discussed by several authors ([3], [10], [11], [17], [18]). We can also refer the reader to [12] for more comprehensive details on Hardy spaces.

For $k \in \mathbb{N}^*$, we designate by $H^{k,\infty}(G_s)$ the Hardy-Sobolev space of order k of the annulus

$$H^{k,\infty}(G_s) = \{f \in H^\infty(G_s)\colon\ f^{(j)} \in H^\infty(G_s),\ j = 0, \dots, k\},$$

where $f^{(j)}$ denotes the j^{th} complex derivative of f. We endow $H^{k,\infty}(G_s)$ with the norm inherited from the space $L^\infty(\partial G_s)$:

$$\|f\|_{H^{k,\infty}(G_s)} = \max_{0 \leqslant j \leqslant k} (\|f^{(j)}\|_{L^\infty(s\mathbb{T})} \ + \ \|f^{(j)}\|_{L^\infty(\mathbb{T})}).$$

Let $\mathscr{B}_{k,\infty} = \{f \in H^{k,\infty}(G_s);\ \|f\|_{H^{k,\infty}(G_s)} \leqslant 1\}$ be the closed unit ball of $H^{k,\infty}$.

Next, we introduce the Poisson kernel p for the annulus G_s. Following Sarason [18] and Hwai [19], we consider the holomorphic function

$$F(t, r) = \frac{1}{2q_0}\ \tanh\Big(-\frac{\pi t}{2q_0} + \mathrm{i}\Big(\frac{\pi}{4} + \frac{\pi}{2q_0}\log\frac{r}{\sqrt{s}}\Big)\Big),$$

where $q_0 = -\log s,\ 0 < s < r < 1$ and $t \in \mathbb{R}$.

The imaginary part $P(t, r)$ of $F(t, r)$ is the harmonic function given by:

$$P(t, r) = \frac{1}{2q_0}\,\frac{\cos(\pi {q_0}^{-1}\log(r/\sqrt{s}))}{\cosh({q_0}^{-1}\pi t) - \sin(\pi {q_0}^{-1}\log(r/\sqrt{s}))}.$$

Referring to [19, p. 92], we recall the following lemma.

Lemma 2.1. *The harmonic function P posseses the following properties:*

(i) $P(t, r) > 0$ *for* $s < r < 1$ *and* $t \in \mathbb{R}$.
(ii) $\int_{-\infty}^{+\infty} P(t, r)\,\mathrm{d}t + \int_{-\infty}^{+\infty} P(t, s/r)\,\mathrm{d}t = 1$ *for* $s < r < 1$.
(iii) *There exists a non negative constant C such that for every $|t| \leqslant \pi$ and j large enough, we have:*

$$|P(t + 2\pi j, r)| \leqslant \min\Big(\frac{C}{j^4},\ \frac{C\cos(\pi/q_0)\log(r/\sqrt{s})}{t^4}\Big).$$

This lemma allows us to define the Poisson kernel p for the annular domain G_s:

$$p(t,r)=\sum_{j=-\infty}^{+\infty} P(t+2\pi j,r) \quad \text{for } |t|\leqslant \pi \text{ and } s<r<1.$$

We also have, from [19], the following lemma.

Lemma 2.2.

(i) *$p(t,r)$ is a harmonic function on the annulus G_s.*

(ii) *$p(t,r)>0$ for $s<r<1$ and $|t|\leqslant\pi$.*

(iii) *$(2\pi)^{-1}\int_0^{2\pi} p(t,r)\,\mathrm{d}t+(2\pi)^{-1}\int_0^{2\pi} p(t,s/r)\,\mathrm{d}t=1$ for $s<r<1$.*

In the next lemma, we recall the Poisson-Jensen formula for the annulus, see ([18, p. 25]). This will be of interest later.

Lemma 2.3. *Let $f\not\equiv 0$ be a function in $H^q(G_s)$ for $1\leqslant q\leqslant\infty$. Then for all $r\mathrm{e}^{\mathrm{i}t}\in G_s$ we have*

$$\log|f(r\mathrm{e}^{\mathrm{i}t})|\leqslant \frac{1}{2\pi}\int_0^{2\pi} p(t,r)\log|f(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t+\frac{1}{2\pi}\int_0^{2\pi} p\Big(t,\frac{s}{r}\Big)\log|f(s\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t.$$

3. Optimal logarithmic estimates in $H^{k,\infty}$

Our objective in this section is to establish some logarithmic estimates in the Hardy-Sobolev space $H^{k,\infty}(G_s)$; $k\in\mathbb{N}^*$ that can be viewed as a continuation of the results already established by [5], [14], [15]. We start by recording a variant of the Hardy-Landau-Littlewood inequality which will be used crucially in the proofs of Theorem 3.5 and Theorem 3.7, see [4, chapter VIII p. 147] and [16].

Lemma 3.1. *Let $\mathscr{I}$ be a bounded interval and let $j\in\mathbb{N}$ satisfy $j\geqslant 2$. Then, there exists a non negative constant $C_\infty(\mathscr{I},j)$ such that*

$$\|g'\|_{L^\infty(\mathscr{I})}\leqslant C_\infty(\mathscr{I},j)\|g\|^{1/j}_{W^{j,\infty}(\mathscr{I})}\|g\|^{1-1/j}_{L^\infty(\mathscr{I})} \quad \text{for all } g\in W^{j,\infty}(\mathscr{I}). \tag{3.1}$$

Next, we give a lower bound for the Poisson kernel p which will be useful for the proof of Lemma 3.3.

Lemma 3.2. *There exists a non negative constant C_s depending only on $s \in]0,1[$ such that for every $|t| \leqslant \pi$ we have*

$$p(t,r) \geqslant \frac{2C_s}{\log s}(\log s - \log r) \quad \textit{if } s < r \leqslant \sqrt{s},$$
$$p(t,r) \geqslant \frac{2C_s}{\log s}\log r \quad \textit{if } \sqrt{s} \leqslant r < 1.$$

Proof. Let us recall that $q_0 = -\log s$. Let $r \in]s,1[$, then $\pi q_0^{-1}\log(r/\sqrt{s}) \in]-\frac{1}{2}\pi, \frac{1}{2}\pi[$ and therefore

$$P(t+2\pi j, r) \geqslant \frac{1}{2q_0(1+\cosh \pi {q_0}^{-1}(t+2\pi j))}\cos\Big(\frac{\pi}{q_0}\log\Big(\frac{r}{\sqrt{s}}\Big)\Big).$$

Since

$$C_s(t) = \frac{1}{2q_0}\sum_{j=-\infty}^{+\infty}\frac{1}{1+\cosh\pi {q_0}^{-1}(t+2\pi j)} < \infty \quad \text{for every } |t| \leqslant \pi,$$

we deduce that

$$p(t,r) \geqslant C_s \cos\Big(\frac{\pi}{q_0}\log\Big(\frac{r}{\sqrt{s}}\Big)\Big), \quad C_s = \inf_{|t|\leqslant\pi} C_s(t). \tag{3.2}$$

In the case where $r \in]s,\sqrt{s}]$, we have $\pi {q_0}^{-1}\log\big(r/\sqrt{s}\big) \in \big]-\frac{1}{2}\pi, 0\big]$. Using the inequality $\cos x \geqslant 2\pi^{-1}x + 1$ for $x \in \big]-\frac{1}{2}\pi, 0\big]$, we obtain

$$p(t,r) \geqslant \frac{2C_s}{\log s}(\log s - \log r).$$

Otherwise, $r \in [\sqrt{s}, 1[$ and $\pi {q_0}^{-1}\log(r/\sqrt{s}) \in [0, \frac{1}{2}\pi[$. Using the inequality $\cos x \geqslant -2\pi^{-1}x + 1$ for $x \in [0, \frac{1}{2}\pi[$, we obtain

$$p(t,r) \geqslant \frac{2C_s}{\log s}\log r, \tag{3.3}$$

which completes the proof of the Lemma. □

We adapt the same arguments as those developed in ([2], Lemma 4.1) with some slight shifts to prove

Lemma 3.3. *Let $g \in H^\infty(G_s)$ and $m \geqslant \|g\|_{L^\infty(\partial G_s)}$. Then, for every $z \in \overline{G}_s$, we have*

$$|g(z)| \leqslant m\Big\|\frac{g}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s(\log s - \log|z|)} \quad \textit{if } s < |z| \leqslant \sqrt{s},$$

$$|g(z)| \leqslant m\Big\|\frac{g}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s \log|z|} \quad \textit{if } \sqrt{s} \leqslant |z| < 1.$$

Proof. Let $h = g/m$ and let $z = re^{it} \in G_s$. From Lemma 2.3 and the fact that $\log|h|$ is a non positive subharmonic function, we get

$$\log(|h(re^{it})|) \leqslant \frac{1}{2\pi}\int_0^{2\pi} p(t-\theta, r)\log(|h(e^{i\theta})|)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} p\Big(t-\theta, \frac{s}{r}\Big)\log(|h(se^{i\theta})|)\,d\theta.$$

If we suppose that $I \subset \mathbb{T}$, then by using the facts that $p(t,r) > 0$ and $\log|h| \leqslant 0$ we deduce that

$$\log(|h(re^{it})|) \leqslant \lambda \int_I p(t-\theta, r)\log(|h(e^{i\theta})|)\frac{d\theta}{2\pi\lambda};$$

consequently, from Lemma 3.2 we obtain

$$\log(|h(z)|) \leqslant \frac{2\lambda C_s}{\log s}(\log s - \log r)\int_I \log(|h(e^{i\theta})|)\frac{d\theta}{2\pi\lambda} \quad \text{if } s < |z| \leqslant \sqrt{s},$$

$$\log(|h(z)|) \leqslant \frac{2\lambda C_s}{\log s}\log r \int_I \log(|h(e^{i\theta})|)\frac{d\theta}{2\pi\lambda} \quad \text{if } \sqrt{s} < |z| < 1.$$

By using Jensen's inequality we deduce that

$$|g(z)| \leqslant m\Big\|\frac{g}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s(\log s - \log|z|)} \quad \text{if } s < |z| \leqslant \sqrt{s},$$

$$|g(z)| \leqslant m\Big\|\frac{g}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s \log|z|} \quad \text{if } \sqrt{s} < |z| < 1.$$

If we suppose that $I \subset s\mathbb{T}$, then by using again the facts that $p(t,r) > 0$ and $\log|h| \leqslant 0$ we get

$$\log(|h(re^{it})|) \leqslant \lambda \int_I p\Big(t-\theta, \frac{s}{r}\Big)\log(|h(se^{i\theta})|)\frac{d\theta}{2\pi\lambda}$$

and the proof can be completed in a way similar to the first case. $\square$

Let $f \in H^\infty(G_s)$ and let t be a real number such that $|t| \leqslant \pi$. We designate by F_t the radial primitive of f that vanishes at s and is defined by

$$F_t(r) = \int_s^r f(xe^{it})\,dx \quad \text{for all } r \in \mathscr{I} = \,]s, 1[. \tag{3.4}$$

From Lemma 3.3 we obtain

Lemma 3.4. *Let $f \in H^{\infty}(G_s)$ and $m \geqslant \|f\|_{L^{\infty}(\partial G_s)}$. We suppose that f is not identically zero and that $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. Then for all $|t| \leqslant \pi$ and $r \in\,]s,1[$ we get*

$$|F_t(r)| \leqslant \frac{(2s+1)q_0 m}{\big|2\lambda C_s \log \|f/m\|_{L^1(I)}\big|}. \tag{3.5}$$

P r o o f. Let $|t| \leqslant \pi$ and $r \in\,]s,1[$. From (3.4) and the monotonicity of the function $\eta(y) = \int_s^y |f(x\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}x$ we have

$$|F_t(r)| \leqslant \int_s^{\sqrt{s}} |f(x\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}x + \int_{\sqrt{s}}^1 |f(x\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}x,$$

and according to Lemma 3.3 we get

$$\begin{aligned}|F_t(r)| &\leqslant m\int_s^{\sqrt{s}} \Big\|\frac{f}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s(\log s - \log x)}\,\mathrm{d}x + m\int_{\sqrt{s}}^1 \Big\|\frac{f}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log^{-1} s \log x}\,\mathrm{d}x\\ &\leqslant \frac{ms}{|1 + 2\lambda C_s q_0{}^{-1}\log\|f/m\|_{L^1(I)}|} + \frac{m}{|1 - 2\lambda C_s q_0{}^{-1}\log\|f/m\|_{L^1(I)}|}.\end{aligned}$$

From the assumption that $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$ we have

$$\frac{1}{|1 + 2\lambda C_s q_0{}^{-1}\log\|f/m\|_{L^1(I)}|} \leqslant \frac{2}{|2\lambda C_s q_0{}^{-1}\log\|f/m\|_{L^1(I)}|},$$

and therefore, we conclude the desired inequality

$$|F_t(r)| \leqslant \frac{(2s+1)q_0 m}{|2\lambda C_s \log \|f/m\|_{L^1(I)}|}.$$

$\square$

We are now in a position to establish the main control theorem in the Hardy-Sobolev space $H^{1,\infty}(G_s)$.

Theorem 3.5. *Let $f \in \mathscr{B}_{1,\infty}$ and $m \geqslant \|f\|_{L^{\infty}(\partial G_s)}$. We suppose that f is not identically zero and that $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. Then*

$$\|f\|_{L^{\infty}(\partial G_s)} \leqslant \frac{C_{\infty}^2(\mathscr{I},2)/(1-1/2\mathrm{e})}{|\lambda_0 \log\|f\|_{L^1(I)}|} \quad \textit{where} \quad \lambda_0 = \min\Big(1, \frac{2\lambda C_s}{(1+2s)q_0}\Big). \tag{3.6}$$

Moreover, for $I = \mathbb{T}$ there exists a sequence of functions $f_n \in \mathscr{B}_{1,\infty}$ such that

$$\lim_{n\to+\infty} \|f_n\|_{L^{\infty}(\partial G_s)} |\log\|f_n\|_{L^1(\mathbb{T})}| \geqslant s|\log s|. \tag{3.7}$$

P r o o f. For every $|t| \leqslant \pi$ let F_t be the radial primitive of f defined by equation (3.4) and let $m \geqslant \max(\|f\|_{L^\infty(\partial G_s)}, 1)$. According to Lemma 3.4, we have

$$|F_t(r)| \leqslant \frac{m}{|\lambda_0 \log \|f/m\|_{L^1(I)}|}, \quad \text{where} \quad \lambda_0 = \min\Big(1, \frac{2\lambda C_s}{(1+2s)q_0}\Big). \tag{3.8}$$

Since $f \in \mathscr{B}_{1,\infty}$, by virtue of the Hardy-Landau-Littlewood inequality (3.1) there exists a non negative constant $C = C_\infty(\mathscr{I}, 2)$ such that

$$\|f\|_{L^\infty(\partial G_s)} \leqslant C\|F\|^{1/2}_{L^\infty(G_s)},$$

and consequently,

$$\|f\|_{L^\infty(\partial G_s)} \leqslant m_1 := C\bigg(\frac{m}{|\lambda_0 \log \|f/m\|_{L^1(I)}|}\bigg)^{1/2}. \tag{3.9}$$

Making use of (3.8) and (3.9) for the new estimate m_1 of $\|f\|_{L^\infty(\partial G_s)}$, one obtains

$$\|f\|_{L^\infty(\partial G_s)} \leqslant C\bigg(\frac{m_1}{|\lambda_0 \log \|f/m_1\|_{L^1(I)}|}\bigg)^{1/2}. \tag{3.10}$$

Let $\eta(x) = x|\log x|^{1/2}$ and $\alpha = 1 - 1/(2\mathrm{e})$. Since $m \geqslant 1$, $\lambda_0 \leqslant 1$ and $g(x) \leqslant x^\alpha$ in $]0,1]$, we get

$$\Big\|\frac{f}{m_1}\Big\|_{L^1(I)} = \frac{(m\lambda_0)^{1/2}}{C}\eta\Big(\Big\|\frac{f}{m}\Big\|_{L^1(I)}\Big) \leqslant \|f\|^\alpha_{L^1(I)}.$$

From (3.10) and the monotonicity of the mapping $\varepsilon(x) = 1/|\mathrm{Log} x|$ we obtain

$$\|f\|_{L^\infty(\partial G_s)} \leqslant C^{1+1/2}\frac{m^{(1/2)^2}(1/\alpha)^{1/2}}{|\lambda_0 \log\|f\|_{L^1(I)}|^{1/2(1+1/2)}}.$$

Proceeding thus repeatedly, we obtain for every $k \in \mathbb{N}^*$

$$\|f\|_{L^\infty(\partial G_s)} \leqslant C^{b_k}\frac{m^{(1/2)^{k+1}}(1/\alpha)^{c_k}}{|\lambda_0 \log\|f\|_{L^1(I)}|^{a_k}},$$

where a_k, b_k and c_k are three recurrent sequences satisfying

$$a_1 = \frac{1}{2}\Big(1 + \frac{1}{2}\Big),\ b_1 = 1 + \frac{1}{2},\ c_1 = \frac{1}{2},\ a_{k+1} = \frac{1 + a_k}{2},\ b_{k+1} = 1 + \frac{b_k}{2},\ c_{k+1} = \frac{1 + c_k}{2}.$$

The proof of inequality (3.6) is completed by letting $k \to +\infty$. $\square$

To prove equation (3.7), we consider the sequence of functions $u_n(z) = 1/z^n$; $n \in \mathbb{N}^*$.

Let $I = \mathbb{T}$ and let $f_n = u_n/\|u_n\|_{H^{1,\infty}(G_s)}$ be the $H^{1,\infty}(G_s)$ normalized function of u_n. Then

$$\|f_n\|_{L^\infty(s\mathbb{T})} = \frac{1/s^n}{n(1+1/s^{n+1})}, \quad \|f_n\|_{L^\infty(\mathbb{T})} = \frac{1}{n(1+1/s^{n+1})}$$
$$\text{and} \quad \|f_n\|_{L^\infty(\partial G_s)} = \frac{1+1/s^n}{n(1+1/s^{n+1})}.$$

Let $A_n = \|f_n\|_{L^\infty(\partial G_s)} |\log \|f_n\|_{L^\infty(\mathbb{T})}|$, then we have

$$A_n = s\frac{1+s^n}{n(1+s^{n+1})}|\log n + \log(1+s^{n+1}) - (n+1)\log s|.$$

Hence, $\lim\limits_{n\to\infty} A_n = s|\log s|$ and this completes the proof. $\square$

Remark 1. The estimate (3.6) still holds in a more general situation of a smooth doubly-connected domain $G \subset \mathbb{R}^2$ (see [13] for more details on conformal mapping).

Remark 2. The estimate (3.6) of Theorem 3.5 is of optimal type: it is impossible to find a function ε which tends to zero at zero such that for all $f \in \mathscr{B}_{1,\infty}$,

$$\|f\|_{L^\infty(\partial G_s)} \leqslant \frac{1}{|\log\|f\|_{L^1(I)}|}\, \varepsilon(\|f\|_{L^1(I)}).$$

Remark 3. The estimate (3.6) of Theorem 3.5 is false in the general setting of bounded function $f \in H^\infty(G_s)$ (we consider the H^∞-normalized function of u_n).

Remark 4. The problem under investigation is to give the optimal constant C in equation (3.6):

$$C = \max_{f\in\mathscr{B}_{1,\infty}} \|f\|_{L^\infty(\partial G_s)}|\log\|f\|_{L^1(I)}|.$$

The following corollary is a direct consequence of Theorem 3.5.

Corollary 3.6. *Let $K > 0$ and $f \in H^{1,\infty}(G_s)$ be such that $\|f\|_{H^{1,\infty}(\partial G_s)} \leqslant K$ and $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. Then we have*

$$\|f\|_{L^\infty(\partial G_s)} \leqslant \frac{C_\infty^2(\mathscr{I},2)\max(1,K)/(1-1/(2\mathrm{e}))}{|\lambda_0 \log\|f\|_{L^1(I)}|}.$$

If we suppose that f is more regular, then we can improve inequality (3.6) in the same way as in the proof of Theorem 3.5.

Theorem 3.7. *Let $k \in \mathbb{N}^*$. There exists a non negative constant C depending only on k, s and λ such that every $f \in \mathscr{B}_{k,\infty}$ satisfying $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$ also satisfies*

$$\|f\|_{L^\infty(\partial G_s)} \leqslant \frac{C_k(s)}{|\log\|f\|_{L^1(I)}|^k}.$$

Moreover, for $I = \mathbb{T}$, there exists a sequence f_n in $\mathscr{B}_{k,\infty}$ such that

$$\lim_{n\to+\infty} \|f_n\|_{L^\infty(\mathbb{T})} |\log \|f_n\|_{L^1(I)}|^k \geqslant s|\log s|^k. \tag{3.11}$$

P r o o f. For every $|t| \leqslant \pi$ we have according to the proof of the previous theorem that the radial primitive F_t of f satisfies inequality (3.8). Since $f \in \mathscr{B}_{k,\infty}$, then from the Hardy-Landau-Littlewood inequality (3.1) applied to $j = k+1$ we prove that there exists a non negative constant $C = C_\infty(\mathscr{I}, k+1)$ such that

$$\|f\|_{L^\infty(\partial G_s)} \leqslant m_1 := C\Big(\frac{m}{|\lambda_0 \log \|f/m\|_{L^1(I)}|}\Big)^{k/(k+1)}. \tag{3.12}$$

Similarly to the proof of Theorem 3.5, consider $\varrho = k/(k+1)$, $g_\varrho(x) = x|\log x|^\varrho$ and $\sigma = 1 - \varrho/\mathrm{e}$. Then we have $g_\varrho(x) \leqslant x^\sigma$ in $]0,1]$ and consequently we establish for every $j \in \mathbb{N}^*$ the inequality

$$\|f\|_{L^\infty(\partial G_s)} \leqslant C^{b_j} \frac{m^{(\varrho)^{j+1}}(1/\sigma)^{c_j}}{|\lambda_0 \log\|f\|_{L^1(I)}|^{a_j}},$$

where a_j, b_j and c_j are three recurrent sequences satisfying

$a_1 = \varrho(1+\varrho)$; $b_1 = 1+\varrho$; $c_1 = \varrho$; $a_{j+1} = \varrho(1+a_j)$; $b_{j+1} = 1+\varrho b_j$ and $c_{j+1} = \varrho(1+c_j)$.

Then by letting $j \to +\infty$ we obtain

$$\|f\|_{L^\infty(\partial G_s)} \leqslant \frac{C_k(s)}{|\log \|f\|_{L^1(I)}|^k}.$$

To prove equation (3.11), we consider the same sequence as in the proof of equation (3.7), with the suitable $H^{k,\infty}(G_s)$ normalization norm. □

Corollary 3.8. *Let $K > 0$, let j and k be integers with $0 \leqslant j < k$. Let $f \in H^{k,\infty}$ be such that $\|f\|_{H^{k,\infty}(G_s)} \leqslant K$ and $\|f\|_{H^{j,\infty}(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. Then there exist non negative constants C, ε depending only on K, k, j, s and λ such that*

$$\|f\|_{H^{j,\infty}(\partial G_s)} \leqslant \frac{C}{|\log\|f\|_{L^1(I)}|^{k-j}}$$

provided that $\|f\|_{L^1(I)} < \varepsilon$.

Proof. Let $K_1 = \max(K, 1)$ and $g = f/K_1$, then the derivative $g^{(i)}$ of order $i \in \{0, \ldots, j\}$ belongs to $\mathscr{B}_{k-i,\infty}$ and satisfies the assumptions of Theorem 3.7. Hence, there exists a non-negative constant C_1 depending only on K, k, i, s and λ such that

$$\|g^{(i)}\|_{L^\infty(\partial G_s)} \leqslant \frac{C_1}{\left|\log \|g^{(i)}\|_{L^1(I)}\right|^{k-i}}. \tag{3.13}$$

According to [16, Theorem 1] and the assumption that $g \in \mathscr{B}_{k,\infty}$, there exists a non-negative constant C_2 such that

$$\|g^{(i)}\|_{L^1(I)} \leqslant C_2\|g\|_{L^1(I)}^{1-i/k}.$$

We derive from (3.13) and the monotonicity of the mapping $\eta_i(x) = 1/\log^{k-i}(1/x)$ that

$$\|g^{(i)}\|_{L^\infty(\partial G_s)} \leqslant C_1\eta_i\big(C_2\|g\|_{L^1(I)}^{1-i/k}\big). \tag{3.14}$$

Let us choose $\varepsilon > 0$ small enough such that

$$\eta_i(C_2\|g\|_{L^1(I)}^{1-i/k}) \leqslant 2\eta_i(\|g\|_{L^1(I)}), \tag{3.15}$$

then from (3.14) and (3.15) we obtain

$$\|g^{(i)}\|_{L^\infty(\partial G_s)} \leqslant \frac{2C_1}{|\log \|g\|_{L^1(I)}|^{k-i}}.$$

Taking the maximum over all $i = 0, \ldots, j$ we complete the proof of the corollary. $\square$

As an immediate consequence, we prove that if the L^1-norm of a bounded $H^{k,\infty}(\partial G_s)$ function is known to be small on a connected open subset I of ∂G_s, it remains also small (with uniform norm) on the whole boundary ∂G_s. The same result with the L^2-norm has been established by Leblond et al. in [14].

Corollary 3.9. *Let j and k be integers with $0 \leqslant j < k$, and let $I \subset \partial G_s$ be any connected open subset. Let (f_p) be a sequence of functions in the unit ball of the Hardy-Sobolev spaces $H^{k,\infty}(\partial G_s)$ such that $\|f_p\|_{L^1(I)} \longrightarrow 0$. Then $\|f_p\|_{H^{j,\infty}(\partial G_s)} \longrightarrow 0$.*

In the particular case where $I = \mathbb{T}$, the following corollary provides logarithmic estimates with respect to the L^∞-norm similar to those proved with the L^2-norm by Leblond and al. in [14].

Corollary 3.10. *Let $I = \mathbb{T}$, let k and j be integers with $0 \leqslant j < k$. Then there exist non negative constants C, ε depending only on K, k, j and I such that whenever $f \in \mathscr{B}_{k,\infty}$ satisfies $\|f\|_{H^{j,\infty}(I)} < \mathrm{e}^{-q_0/\lambda C_s}$, we have*

$$\|f\|_{H^{j,\infty}(s\mathbb{T})} \leqslant \frac{C}{|\log\|f\|_{L^1(\mathbb{T})}|^{k-j}}$$

provided that $\|f\|_{L^1(I)} < \varepsilon$.

4. Application

In this section we prove a logarithmic stability result for the inverse problem of identification of a Robin parameter in two dimensional annular domain. Let I be any connected open subset of the boundary of the annulus G_s and let $J = \partial G_s \setminus I$. We consider the following inverse problem (I.P).

Given a function φ and a prescribed flux ϕ on I, find a function $q \in \mathbf{Q}^n_{\mathrm{ad}}$ such that the solution u to the problem

$$(\mathrm{N.R}) \begin{cases} \triangle u = 0 & \text{in } G_s, \\ \partial_n u = \phi & \text{on } I, \\ \partial_n u + qu = 0 & \text{on } J \end{cases}$$

also satisfies $u|_I = \varphi$, where ∂_n stands for the partial derivative with respect to the outer normal unit vector to ∂G_s and the admissible set $\mathbf{Q}^n_{\mathrm{ad}}$ of smooth Robin coefficient is defined by

$$\mathbf{Q}^n_{\mathrm{ad}} = \{q \in \mathscr{C}^n_0(\overline{J})\colon\ \ |q^{(k)}| \leqslant c',\ 0 \leqslant k \leqslant n,\ \text{and } q \geqslant c\},$$

where c, c' are non negative constants and K is a nonempty connected subset of J far from the boundary of J. For $q \in \mathbf{Q}^n_{\mathrm{ad}}$ we denote by u_q the solution of the Neumann-Robin problem (N.R).

Referring to [6], [8], [9] we have the following

Lemma 4.1 ([6], [8], [9]). *Let $n \in \mathbb{N}$, $\phi \in W^{n,2}(I)$ with non-negative value such that $\phi \not\equiv 0$ and assume that $q \in \mathbf{Q}^n_{\rm ad}$ for some constants $c, c' > 0$. Then the solution u_q of the inverse problem* (I. P) *belongs to $W^{n+3/2,2}(G_s)$.*

Furthermore, there exist non negative constants α, β such that for every $q \in \mathbf{Q}^n_{\rm ad}$ and every $\phi \in W^{n,2}(I)$ we have

$$u_q \geqslant \alpha > 0 \quad \text{and} \quad \|u\|_{W^{n+1,2}(\partial G_s)} \leqslant \beta.$$

The following identifiability result proves the uniqueness of the solution q of the inverse problem (I. P).

Lemma 4.2 ([6], [9]). *The mapping*

$$F\colon\ \mathbf{Q}^n_{\rm ad} \longrightarrow L^2(\Gamma_d),$$
$$q \longmapsto u_{q/_{\Gamma_d}}$$

is well defined, continuous and injective.

Applying to Theorem 3.7, we establish the following stability result.

Theorem 4.3. *Let $n \geqslant 2$ and let $\phi \in W^{n,2}_0(I)$ be such that $\phi \not\equiv 0$ and $\phi \geqslant 0$. Then there exists a non negative constant C such that for any $q_1, q_2 \in \mathbf{Q}^n_{\rm ad}$ we have*

$$\|q_1 - q_2\|_{L^\infty(J)} \leqslant \frac{C}{|\log\|u_{q_1} - u_{q_2}\|_{L^1(I)}|^{n-1}}$$

provided that $\|u_{q_1} - u_{q_2}\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$.

P r o o f. Referring to ([14], Lemma 12), we introduce for every $i = 1, 2$ the analytic function f_i in G_s satisfying $u_{q_i} = \mathrm{Re} f_i$ and $f_i \in H^{n+1,2}(\partial G_s)$. Moreover, Lemma 4.1 together with the Gagliardo-Nirenberg inequalities proves that there exists non negative constants M, K depending only on s and the class $\mathbf{Q}^n_{\rm ad}$ such that

$$\|f_i\|_{H^{n,\infty}(G_s)} \leqslant M\|f_i\|_{H^{n+1,2}(G_s)} \leqslant K \quad \text{for } i = 1, 2. \tag{4.1}$$

Using the equation $\partial_n u + qu = 0$ on J, we get for $f = f_1 - f_2$ that

$$q_1 - q_2 = -\frac{1}{\mathrm{Re} f_1}\frac{\partial\,\mathrm{Im} f_1}{\partial\theta} + \frac{1}{\mathrm{Re} f_2}\frac{\partial\,\mathrm{Im} f_2}{\partial\theta} = -\frac{1}{\mathrm{Re} f_1}\frac{\partial\,\mathrm{Im} f}{\partial\theta} + \frac{\partial\,\mathrm{Im} f_2}{\partial\theta}\,\frac{\mathrm{Re} f}{\mathrm{Re} f_1 \mathrm{Re} f_2}.$$

It follows from Lemma 4.1 that

$$\|q_1 - q_2\|_{L^\infty(J)} \leqslant \frac{1}{\alpha}\|f\|_{W^{1,\infty}(J)} + \frac{\beta}{\alpha^2}\|f\|_{L^\infty(J)} \leqslant \Big(\frac{1}{\alpha} + \frac{\beta}{\alpha^2}\Big)\|f\|_{W^{1,\infty}(J)}.$$

Hence, from (4.1) and Corollary 3.8 we get

$$\|q_1 - q_2\|_{L^\infty(J)} \leqslant \frac{C}{|\log\|u_{q_1} - u_{q_2}\|_{L^1(I)}|^{n-1}}$$

provided that $\|u_{q_1} - u_{q_2}\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. □

The particular case where $I = \mathbb{T}$ has been recently established by Leblond et al. in [14].

Corollary 4.4. *Let $n \geqslant 2$, let $\phi \in W_0^{n,2}(\mathbb{T})$ be such that $\phi \not\equiv 0$ and $\phi \geqslant 0$. Then there exists a non negative constant C such that for any $q_1, q_2 \in \mathbf{Q}_{\mathrm{ad}}^n$ we have*

$$\|q_1 - q_2\|_{L^\infty(s\mathbb{T})} \leqslant \frac{C}{|\log\|u_{q_1} - u_{q_2}\|_{L^1(\mathbb{T})}|^{n-1}},$$

provided that $\|u_{q_1} - u_{q_2}\|_{L^1(\mathbb{T})} < \mathrm{e}^{-q_0/\lambda C_s}$.

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