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Wojciech Młocek; Marek Ptak

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ON THE REFLEXIVITY OF SUBSPACES OF TOEPLITZ OPERATORS ON THE HARDY SPACE ON THE UPPER HALF-PLANE

Wojciech Młocek, Marek Ptak, Kraków

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Abstract. The reflexivity and transitivity of subspaces of Toeplitz operators on the Hardy space on the upper half-plane are investigated. The dichotomic behavior (transitive or reflexive) of these subspaces is shown. It refers to the similar dichotomic behavior for subspaces of Toeplitz operators on the Hardy space on the unit disc. The isomorphism between the Hardy spaces on the unit disc and the upper half-plane is used. To keep weak* homeomorphism between L^∞ spaces on the unit circle and the real line we redefine the classical isomorphism between L^1 spaces.

Keywords: reflexive subspace, transitive subspace, Toeplitz operator, Hardy space, upper half-plane

MSC 2010: 47L80, 47L05, 47L45

1. Introduction

If $\mathscr{H}$ is a Hilbert space, then $\mathscr{B}(\mathscr{H})$ stands for the Banach algebra of all bounded linear operators on $\mathscr{H}$. The *reflexive closure* of a subspace $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is given by

$$\operatorname{ref} \mathscr{S} = \{B \in \mathscr{B}(\mathscr{H})\colon\ Bh \in \overline{\mathscr{S}h} \quad \text{for all} \quad h \in \mathscr{H}\}.$$

The subspace $\mathscr{S}$ is said to be *reflexive*, if $\operatorname{ref}\mathscr{S} = \mathscr{S}$ and *transitive*, if $\operatorname{ref}\mathscr{S} = \mathscr{B}(\mathscr{H})$. The theory of Toeplitz operators on the Hardy space on the unit disc gave exact examples of natural spaces having reflexivity or transitivity property. In [11] the reflexivity of the algebra of all analytic Toeplitz operators on this space was proved. Transitivity of the whole space of Toeplitz operators was shown in [1]. In fact, in [1] there was proved the dichotomic behavior (transitive or reflexive) of subspaces of Toeplitz operators on the Hardy space on the unit disc. The precise condition verifying dichotomy between transitivity and reflexivity was given. It completely characterized subspaces of Toeplitz operators from the reflexive-transitive point of view. It is also natural to consider Toeplitz operators on the Hardy space on the upper half-plane.

The aim of the paper is to investigate reflexivity and transitivity of subspaces of Toeplitz operators on the upper half-plane. There is an isomorphism between L^p spaces and the Hardy spaces on the unit disc and L^p spaces and the Hardy spaces on the upper half-plane (see (3.2), [10, p. 143]). Investigating reflexivity-transitivity it is convenient to assume weak* closedness of subspaces. Thus it is necessary to redefine (see (3.4)) the classical isomorphism between L^1 spaces to obtain a weak* homeomorphism between L^∞ spaces. Theorem 3.4 shows weak* properties of this isomorphism. Section 4 gives a relation between Toeplitz operators on the Hardy spaces on the unit disc and the upper half-plane.

Theorem 5.4, which can be regarded as the main result of the paper, shows the dichotomic behavior (transitive or reflexive) of subspaces of Toeplitz operators on the Hardy space on the upper half-plane. In Section 6 several examples are given.

2. Preliminaries

2.1. Duality. If X_* is a Banach space, by X we denote the dual of X_* and the dual action is given by $\langle\cdot,\cdot\rangle$. Similarly we have a Banach space Y_* and its dual Y. Recall the relation between an operator on spaces X and Y and on the preduals Y_* and X_*. If $T\colon X \to Y$ is a weak* continuous, bounded linear transformation, then there exists a bounded linear transformation $T_*\colon Y_* \to X_*$ satisfying the following formula

$$\langle x, T_* y_* \rangle = \langle Tx, y_* \rangle, \quad \text{for all } x \in X,\ y_* \in Y_*. \tag{2.1}$$

If $\mathscr{S} \subset X$ then by $\mathscr{S}_\perp$ we denote the *preannihilator* of $\mathscr{S}$.

The dual pair considered in the paper will be the algebra $\mathscr{B}(\mathscr{H})$ and the space of trace class operators $\mathscr{B}_1(\mathscr{H})$. Recall also that the bilinear functional given by

$$\langle A, t \rangle := \operatorname{tr}(At), \quad A \in \mathscr{B}(\mathscr{H}),\ t \in \mathscr{B}_1(\mathscr{H}),$$

allows us to identify $\mathscr{B}_1(\mathscr{H})^*$ with $\mathscr{B}(\mathscr{H})$ i.e. $\mathscr{B}(\mathscr{H})_* = \mathscr{B}_1(\mathscr{H})$.

2.2. Reflexivity. For the sake of completeness we establish the following technical lemma. It will be useful in Section 5.

Lemma 2.1. *Let $\mathscr{H}$, $\mathscr{K}$ Hilbert spaces, $U\colon \mathscr{H} \to \mathscr{K}$ be a unitary operator. If the operator $\widetilde{U}\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ is given by $\widetilde{U}(A) = UAU^{-1}$, then*

(a) *$\widetilde{U}$ is an isometric isomorphism,*

(b) *$\operatorname{ref}(\widetilde{U}(\mathscr{S})) = \widetilde{U}(\operatorname{ref}\mathscr{S})$ for $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$,*

(c) *$\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is reflexive (respectively transitive) if and only if $\widetilde{U}(\mathscr{S})$ is reflexive (respectively transitive).*

For the proof of (a) see [4, Exercise 2, p. 61], and (c) is a consequence of (b), which can be proved similarly to [1, Lemma 4.5].

2.3. Hardy spaces. Let $\mathbb{D} = \{w \in \mathbb{C}\colon |w| < 1\}$ denote the open unit disc, $\mathbb{T} = \{\omega \in \mathbb{C}\colon |\omega| = 1\}$ the unit circle and $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ the upper half-plane. The Hardy space $H^p(\mathbb{D})$ $(1 \leqslant p \leqslant \infty)$ is the space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that $\|f\|^p_{H^p(\mathbb{D})} := \sup\limits_{0\leqslant r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \,d\theta < \infty$ for $1 \leqslant p < \infty$ and $\|f\|_{H^\infty(\mathbb{D})} := \sup\limits_{z\in\mathbb{D}} |f(z)| < \infty$ for $p = \infty$. By [10, Theorem 3.4.1], each function from $H^p(\mathbb{D})$ has radial and also non-tangential limits on the unit circle $\mathbb{T}$ and moreover the space $H^p(\mathbb{D})$ can be identified with a corresponding subspace of $L^p(\mathbb{T})$.

Definition 2.2. The Hardy space $H^p(\mathbb{C}_+)$ $(1 \leqslant p < \infty)$ on $\mathbb{C}_+$ is the space of all analytic functions $F\colon \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|F\|_{H^p(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{\mathbb{R}} |F(x+iy)|^p \,dx \right)^{1/p} < \infty.$$

For $p = \infty$, by $H^\infty(\mathbb{C}_+)$ we denote the space of all bounded analytic functions on $\mathbb{C}_+$ with $\|F\|_{H^\infty(\mathbb{C}_+)} := \sup\limits_{y>0} |F(x+iy)|$.

The spaces $H^p(\mathbb{C}_+)$ $(1 \leqslant p \leqslant \infty)$ are Banach spaces and $H^2(\mathbb{C}_+)$ is a Hilbert space. By [9, Theorem p. 153], each function from $H^p(\mathbb{C}_+)$ has non-tangential limits on the real line $\{z \in \mathbb{C}\colon z = 0\}$ and moreover the space $H^p(\mathbb{C}_+)$ can be identified with a corresponding subspace of $L^p(\mathbb{R})$. For more information about the Hardy spaces $H^p(\mathbb{C}_+)$ see [7], [8], [9], [10].

Let $\gamma\colon \mathbb{C}_+ \to \mathbb{D}$, $\gamma(z) = (z-i)/(z+i)$, be the usual conformal mapping of the upper half-plane onto the unit disc. The function $\gamma(t) = (t-i)/(t+i)$, then considered as $\gamma\colon \mathbb{R} \to \mathbb{T}$, gives a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{T} \setminus \{1\}$. The function γ will be often used in the whole paper in both contexts.

3. Isomorphisms between spaces on the unit disc and the upper half-plane

For $1 \leqslant p \leqslant \infty$, let $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$ denote L^p spaces of complex functions with the normalized Lebesgue measure m on $\mathbb{T}$ and the usual Lebesgue measure on $\mathbb{R}$, respectively. Firstly let us recall the well-known isomorphism between the spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{R})$.

Lemma 3.1. *The operator $U_2\colon\ L^2(\mathbb{T}) \to L^2(\mathbb{R})$ defined by*

$$(U_2 f)(t) = \frac{1}{\sqrt{\pi}} \frac{1}{t+\mathrm{i}} f(\gamma(t)) \tag{3.1}$$

is unitary.

Remark. Note that the result above can be extended to L^p spaces. The mapping

$$(U_p f)(t) = \Bigl(\frac{1}{\pi(t+\mathrm{i})^2}\Bigr)^{1/p} f(\gamma(t)), \quad t \in \mathbb{R} \tag{3.2}$$

is an isometric isomorphism of the space $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$, for $1 \leqslant p < \infty$ (see [10, p. 143]).

The following is well known and easy to prove.

Lemma 3.2. *The operator $U_\infty\colon\ L^\infty(\mathbb{T}) \to L^\infty(\mathbb{R})$ defined by*

$$U_\infty \varphi = \varphi \circ \gamma \tag{3.3}$$

is an isometric isomorphism.

Let $(X, \mathscr{B}, \mu)$ be a (positive) measure space. Recall that $L^\infty(X,\mu)$ is the dual space to $L^1(X,\mu)$ and this duality is given by $\langle \varphi, f\rangle = \int_X \varphi f \,\mathrm{d}\mu$, where $\varphi \in L^\infty(X,\mu)$, $f \in L^1(X,\mu)$. We will especially use the duality between $L^1(\mathbb{T})$ and $L^\infty(\mathbb{T})$ and also between $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. Hence we have to define an isomorphism between $L^1(\mathbb{T})$ and $L^1(\mathbb{R})$ differently than (3.2).

Lemma 3.3. *The operator $U_1\colon L^1(\mathbb{T}) \to L^1(\mathbb{R})$ defined by*

$$(U_1 f)(t) = \frac{1}{\pi}\frac{1}{1+t^2} f(\gamma(t)) \tag{3.4}$$

is an isometric isomorphism.

Proof. Let $f \in L^1(\mathbb{T})$. To see that U_1 is well defined and that it is, in fact an isometry, note that

$$\begin{aligned}\|U_1 f\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |(U_1 f)(t)|\,\mathrm{d}t = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2}|f(\gamma(t))|\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\tau})|\,\mathrm{d}\tau = \|f\|_{L^1(\mathbb{T})},\end{aligned}$$

where $(t-\mathrm{i})/(t+\mathrm{i}) = \mathrm{e}^{\mathrm{i}\tau}$. For the surjectivity of U_1 let us take $F \in L^1(\mathbb{R})$. Now put $f(\mathrm{e}^{\mathrm{i}\tau}) := \pi(1+t^2)F(t)$, where $t = \gamma^{-1}(\mathrm{e}^{\mathrm{i}\tau})$. Then $(U_1 f)(t) = F(t)$ and

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\tau})|\,\mathrm{d}\tau = \int_{\mathbb{R}} |F(t)|\,\mathrm{d}t < \infty.$$

Thus $f \in L^1(\mathbb{T})$. Therefore U_1 is surjective and isometric. □

The definition (3.4) of U_1 enables to see U_∞ given by (3.3) as a dual action to $(U_1)^{-1}\colon L^1(\mathbb{R}) \to L^1(\mathbb{T})$. Namely

Theorem 3.4. *Let $U_\infty\colon L^\infty(\mathbb{T}) \to L^\infty(\mathbb{R})$ be given by $U_\infty\varphi = \varphi\circ\gamma$, where $\gamma\colon \mathbb{C}_+ \to \mathbb{D}$ with $\gamma(z) = (z-\mathrm{i})/(z+\mathrm{i})$, and let $U_1\colon L^1(\mathbb{T}) \to L^1(\mathbb{R})$ be given by $(U_1 f)(t) = (\pi^{-1}/(1+t^2))f(\gamma(t))$, then*

(a) $\langle\varphi, f\rangle = \langle U_\infty\varphi, U_1 f\rangle$, *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$,
(b) $U_\infty(H^\infty(\mathbb{D})) = H^\infty(\mathbb{C}_+)$,
(c) $U_1(H^\infty(\mathbb{D})_\perp) = H^\infty(\mathbb{C}_+)_\perp$,
(d) $U_\infty = (U_1^{-1})^*$,
(e) U_∞ *is a weak* homeomorphism.*

Proof. To see (a) we make the direct computation

$$\begin{aligned}\langle U_\infty\varphi, U_1 f\rangle &= \int_{\mathbb{R}} (U_\infty\varphi)(t)(U_1 f)(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{\mathbb{R}} \varphi(\gamma(t))f(\gamma(t))\frac{\mathrm{d}t}{1+t^2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}\tau})f(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{d}\tau = \int_{\mathbb{T}} \varphi f\,\mathrm{d}m = \langle\varphi, f\rangle.\end{aligned}$$

(b) is an easy consequence of Lemma 3.2 and the definition of γ. Combining (a) with (b) we get (c). Condition (d) follows from (a) by [3, Proposition 2.5]. It also gives weak* continuity of U_∞. Finally, by [3, Theorem 2.7] we get that U_∞ is a weak* homeomorphism. □

By [10, Theorem 6.3.4] we have $U_2(H^2(\mathbb{D})) = H^2(\mathbb{C}_+)$ and

$$\|f\|_{H^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{T})} = \|U_2 f\|_{L^2(\mathbb{R})} = \|U_2 f\|_{H^2(\mathbb{C}_+)}.$$

Thus by Lemma 3.1 we have the following.

Lemma 3.5. *If* $f \in H^2(\mathbb{D})$, $z \in \mathbb{C}_+$ *and*

$$(U_2 f)(z) := \frac{1}{\sqrt{\pi}} \frac{1}{z + \mathrm{i}} f(\gamma(z)), \tag{3.5}$$

then $U_2\colon\ H^2(\mathbb{D}) \to H^2(\mathbb{C}_+)$ *is a unitary operator.*

It is known (see [10, Theorem 3.4.1]) that the spaces $H^\infty(\mathbb{D})$ and $L^\infty(\mathbb{T}) \cap H^2(\mathbb{D})$ are isomorphic. By Theorem 3.4 and Lemma 3.5 we obtain the isomorphism between $H^\infty(\mathbb{C}_+)$ and $L^\infty(\mathbb{R}) \cap H^2(\mathbb{C}_+)$.

4. Toeplitz operators

The following lemma gives relation between multiplication operators on $L^2(\mathbb{T})$ and on $L^2(\mathbb{R})$.

Lemma 4.1. *If* $\varphi \in L^\infty(\mathbb{T})$ *and* M_φ *is the multiplication operator by* φ *on the space* $L^2(\mathbb{T})$ *then* $U_2 M_\varphi U_2^{-1} = M_{\varphi\circ\gamma}$ *is the multiplication operator by* $\varphi \circ \gamma$ *on the space* $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{T})$. Then for $t \in \mathbb{R}$ we have

$$\begin{aligned} ((M_{\varphi\circ\gamma} U_2)(f))(t) &= (M_{\varphi\circ\gamma} U_2 f)(t) = (\varphi \circ \gamma)(t)(U_2 f)(t) \\ &= \varphi(\gamma(t)) \frac{1}{\sqrt{\pi}} \frac{1}{t + \mathrm{i}} f(\gamma(t)) = \frac{1}{\sqrt{\pi}} \frac{1}{t + \mathrm{i}} (\varphi f)(\gamma(t)) \\ &= (U_2 \varphi f)(t) = ((U_2 M_\varphi)(f))(t). \end{aligned}$$

□

Recall that the operator T_φ with symbol $\varphi \in L^\infty(\mathbb{T})$ given by

$$T_\varphi f = P_{H^2(\mathbb{D})}(\varphi f), \quad f \in H^2(\mathbb{D}),$$

where $P_{H^2(\mathbb{D})}$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$, is called the *Toeplitz operator* on $H^2(\mathbb{D})$. If $\varphi \in H^\infty(\mathbb{D})$ then T_φ is called an *analytic Toeplitz operator.* By $\mathscr{T}(\mathbb{D})$ we denote the space of all Toeplitz operators and by $\mathscr{A}(\mathbb{D})$ the algebra of all analytic Toeplitz operators on $H^2(\mathbb{D})$. Let us now introduce the Toeplitz operators on the Hardy space on the upper half-plane.

Definition 4.2. For each $\Phi \in L^\infty(\mathbb{R})$, the *Toeplitz operator* on $H^2(\mathbb{C}_+)$ with symbol Φ is the operator T_Φ defined by

$$T_\Phi F = P_{H^2(\mathbb{C}_+)}(\Phi F), \quad F \in H^2(\mathbb{C}_+),$$

where $P_{H^2(\mathbb{C}_+)}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. If $\Phi \in H^\infty(\mathbb{C}_+)$, then T_Φ is called an *analytic Toeplitz operator*.

Similarly as above $\mathscr{T}(\mathbb{C}_+)$ denotes the space of all Toeplitz operators and $\mathscr{A}(\mathbb{C}_+)$ the algebra of all analytic Toeplitz operators on $H^2(\mathbb{C}_+)$.

Definition 4.3. *Symbol maps* of Toeplitz operators are the functions $\xi\colon L^\infty(\mathbb{T}) \to \mathscr{T}(\mathbb{D}) \subset \mathscr{B}(H^2(\mathbb{D}))$ defined by $\xi(\varphi) = T_\varphi$ and $\eta\colon L^\infty(\mathbb{R}) \to \mathscr{T}(\mathbb{C}_+) \subset \mathscr{B}(H^2(\mathbb{C}_+))$ defined by $\eta(\Phi) = T_\Phi$.

Considering the relation between Toeplitz operators on $H^2(\mathbb{D})$ and Toeplitz operators on $H^2(\mathbb{C}_+)$ let us first observe that the equality $P_{H^2(\mathbb{C}_+)}U_2 = U_2 P_{H^2(\mathbb{D})}$ and Lemma 4.1 give us the following relation for all $\varphi \in L^\infty(\mathbb{T})$:

$$T_{\varphi\circ\gamma}U_2 = P_{H^2(\mathbb{C}_+)}M_{\varphi\circ\gamma}U_2 = P_{H^2(\mathbb{C}_+)}U_2 M_\varphi = U_2 P_{H^2(\mathbb{D})}M_\varphi = U_2 T_\varphi. \tag{4.1}$$

By the observation above the relationship between Toeplitz operators on the Hardy space on the unit disc and Toeplitz operators on the Hardy space on the upper half-plane can be characterized as follows.

Theorem 4.4. *Let $\xi\colon L^\infty(\mathbb{T}) \to \mathscr{T}(\mathbb{D}) \subset \mathscr{B}(H^2(\mathbb{D}))$, $\xi(\varphi) = T_\varphi$ and $\eta\colon L^\infty(\mathbb{R}) \to \mathscr{T}(\mathbb{C}_+) \subset \mathscr{B}(H^2(\mathbb{C}_+))$, $\eta(\Phi) = T_\Phi$ be the symbol maps of the Toeplitz operators on $H^2(\mathbb{D})$ and on $H^2(\mathbb{C}_+)$. If $\widetilde{U}_2\colon \mathscr{B}(H^2(\mathbb{D})) \to \mathscr{B}(H^2(\mathbb{C}_+))$ is given by*

$$\widetilde{U}_2(A) = U_2 A U_2^{-1}, \quad A \in \mathscr{B}(H^2(\mathbb{D})), \tag{4.2}$$

where U_2 is defined by (3.5), then
(a) *$U_2 T_\varphi U_2^{-1} = T_{\varphi\circ\gamma}$, for all $\varphi \in L^\infty(\mathbb{T})$,*
(b) *$U_2(\mathscr{T}(\mathbb{D}))U_2^{-1} = \mathscr{T}(\mathbb{C}_+)$ and $U_2(\mathscr{A}(\mathbb{D}))U_2^{-1} = \mathscr{A}(\mathbb{C}_+)$,*
(c) *$\widetilde{U}_2$ is a weak* homeomorphism,*
(d) *the following diagram commutes*

$$\begin{array}{ccc} L^\infty(\mathbb{T}) & \xrightarrow{\ \xi\ } & \mathscr{T}(\mathbb{D}) \\ {\scriptstyle U_\infty}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{U}_2} \\ L^\infty(\mathbb{R}) & \xrightarrow[\ \eta\ ]{} & \mathscr{T}(\mathbb{C}_+), \end{array}$$

(e) *η is a weak* homeomorphism.*

P r o o f. Condition (a) is just the equality (4.1) and (b) follows directly from this equality and Theorem 3.4. To see (c) note that by Lemma 2.1 we have that $\widetilde{U}_2$ is an isomorphism. First observe that $\widetilde{U}_2(\mathscr{B}_1(H^2(\mathbb{D})) = \mathscr{B}_1(H^2(\mathbb{C}_+))$ and this yields

$$(4.3)\qquad \langle A, t\rangle = \langle \widetilde{U}_2(A), \widetilde{U}_2(t)\rangle, \quad \text{for all} \quad A \in \mathscr{B}(H^2(\mathbb{D})),\ t \in \mathscr{B}_1(H^2(\mathbb{D})).$$

Note that $\widetilde{U}_2^{-1}(B) = U_2^{-1}BU_2$ for $B \in H^2(\mathbb{C}_+)$, thus by (4.3) we have

$$(4.4)\qquad \langle B, T\rangle = \langle \widetilde{U}_2^{-1}(B), \widetilde{U}_2^{-1}(T)\rangle, \quad B \in \mathscr{B}(H^2(\mathbb{C}_+)),\ T \in \mathscr{B}_1(H^2(\mathbb{C}_+)).$$

From the equalities (4.3) and (4.4) it follows that $\widetilde{U}_2$ and $\widetilde{U}_2^{-1}$ are weak* continuous, so the proof of the condition (c) is complete.

Let $\varphi \in L^\infty(\mathbb{T})$. Then (d) follows from the equality

$$(\widetilde{U}_2 \circ \xi)(\varphi) = \widetilde{U}_2(T_\varphi) = T_{\varphi\circ\gamma} = \eta(\varphi \circ \gamma) = (\eta \circ U_\infty)(\varphi).$$

Since $\xi\colon L^\infty(\mathbb{T}) \to \mathscr{T}(\mathbb{D})$ is a weak* homeomorphism, see [1, Corollary 2.3 (2)], the condition (e) follows from the conditions (c), (d) and Theorem 3.4. □

The next lemma follows immediately from the similar facts concerning Toeplitz operators on $H^2(\mathbb{D})$ (see [6, Proposition 7.5]) and the condition (a) of Theorem 4.4.

Corollary 4.5. *If $\Phi \in L^\infty(\mathbb{R})$ and $G \in H^\infty(\mathbb{C}_+)$, then*

(a) $T_\Phi^* = T_{\overline{\Phi}}$,

(b) $T_\Phi T_G = T_{\Phi G}$ *and* $T_{\overline{G}} T_\Phi = T_{\overline{G}\Phi}$.

Since $\mathscr{B}_1(H^2(\mathbb{D})) = \mathscr{B}(H^2(\mathbb{D}))_*$, $\mathscr{T}(\mathbb{D})$ is a weak* closed subspace of $\mathscr{B}(H^2(\mathbb{D}))$. Similarly $\mathscr{B}_1(H^2(\mathbb{C}_+)) = \mathscr{B}(H^2(\mathbb{C}_+))_*$, so $\mathscr{T}(\mathbb{C}_+)$ is also a weak* closed subspace of $\mathscr{B}(H^2(\mathbb{C}_+))$. Hence Corollary 2.2 of [3] implies that

$$\mathscr{T}(\mathbb{D})_* = \mathscr{B}_1(H^2(\mathbb{D}))/\mathscr{T}(\mathbb{D})_\perp \quad \text{and} \quad \mathscr{T}(\mathbb{C}_+)_* = \mathscr{B}_1(H^2(\mathbb{C}_+))/\mathscr{T}(\mathbb{C}_+)_\perp.$$

Moreover, since $\xi\colon L^\infty(\mathbb{T}) \to \mathscr{T}(\mathbb{D})$ and $\eta\colon L^\infty(\mathbb{R}) \to \mathscr{T}(\mathbb{C}_+)$ are weak* homeomorphisms, by [3, Proposition 2.5], there are weak homeomorphisms $\xi_*\colon \mathscr{T}(\mathbb{D})_* \to L^1(\mathbb{T})$ and $\eta_*\colon \mathscr{T}(\mathbb{C}_+)_* \to L^1(\mathbb{R})$ such that $\langle T_\varphi, \xi_*^{-1}(f)\rangle = \langle \varphi, f\rangle$ for $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$ and $\langle T_\Phi, \eta_*^{-1}(F)\rangle = \langle \Phi, F\rangle$ for $\Phi \in L^\infty(\mathbb{R})$, $F \in L^1(\mathbb{R})$.

The relationship between these spaces is given by the following.

Theorem 4.6. *Let* $\xi\colon L^\infty(\mathbb{T}) \to \mathscr{T}(\mathbb{D}) \subset \mathscr{B}(H^2(\mathbb{D}))$, $\xi(\varphi) = T_\varphi$ *and* $\eta\colon L^\infty(\mathbb{R}) \to \mathscr{T}(\mathbb{C}_+) \subset \mathscr{B}(H^2(\mathbb{C}_+))$, $\eta(\Phi) = T_\Phi$ *be the symbol maps of the Toeplitz operators on* $H^2(\mathbb{D})$ *and on* $H^2(\mathbb{C}_+)$. *If the operator* $\widetilde{U}_2$ *is given by* (4.2) *and the operator* U_1 *is given by* (3.4) *then*

(a) $\langle T_\varphi, \xi_*^{-1}(f)\rangle = \langle T_{U_\infty\varphi}, \eta_*^{-1}(U_1 f)\rangle$ *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$,

(b) *the following diagram commutes*

$$\begin{array}{ccc} \mathscr{T}(\mathbb{C}_+)_* & \xrightarrow{\ \eta_*\ } & L^1(\mathbb{R}) \\ {\scriptstyle \widetilde{U}_{2*}}\big\downarrow & & \big\downarrow{\scriptstyle U_1^{-1}} \\ \mathscr{T}(\mathbb{D})_* & \xrightarrow[\ \xi_*\ ]{} & L^1(\mathbb{T}). \end{array}$$

P r o o f. By Theorem 3.4 we have

$$\langle T_\varphi, \xi_*^{-1}(f)\rangle = \langle \varphi, f\rangle = \langle U_\infty\varphi, U_1 f\rangle = \langle T_{U_\infty\varphi}, \eta_*^{-1}(U_1 f)\rangle,$$

which proves (a). To see (b) by Theorem 4.4(d) we only need to show that $U_{\infty *} = U_1^{-1}$. Let $\varphi \in L^\infty(\mathbb{T})$ and $F \in L^1(\mathbb{R})$. Using Theorem 3.4 we get

$$\langle \varphi, U_{\infty *}F\rangle = \langle U_\infty\varphi, F\rangle = \langle U_\infty\varphi, U_1U_1^{-1}F\rangle = \langle \varphi, U_1^{-1}F\rangle.$$

□

Remark. The theorem above holds since we have defined the operator U_1 by the formula (3.4) instead of (3.2).

It is known (see [6, Exercise 7.3, p. 203] and [10, Theorem 4.1.4]) that $A \in \mathscr{B}(H^2(\mathbb{D}))$ is a Toeplitz operator if and only if $A = T_z^* A T_z$. Considering Toeplitz operators on the Hardy space on $H^2(\mathbb{C}_+)$ it was pointed in [10, p. 273] that

Theorem 4.7. *Let* $B \in \mathscr{B}(H^2(\mathbb{C}_+))$. *The operator* B *is a Toeplitz operator on* $H^2(\mathbb{C}_+)$ *if and only if* $B = T^*_{\mathrm{e}^{\mathrm{i}\lambda t}} B T_{\mathrm{e}^{\mathrm{i}\lambda t}}$ *for all* $\lambda > 0$.

A proof of the above is an imitation of the proof of the characterization of the Toeplitz operators on $H^2(\mathbb{D})$ (see [10, Theorem 4.1.4]), changing the relation between the groups $\mathbb{T}$ and $\mathbb{Z}$ to the relation between $\mathbb{R}$ and $\mathbb{R}$.

The following is a useful characterization of the Toeplitz operators on $H^2(\mathbb{D})$.

Theorem 4.8. *Let $A \in \mathscr{B}(H^2(\mathbb{D}))$ and $\varphi_\lambda(\omega) := \exp(\lambda(\omega+1)/(\omega-1))$ where $\omega \in \mathbb{T} \setminus \{1\}$, $\lambda > 0$. Then the following conditions are equivalent.*
(a) *A is a Toeplitz operator on $H^2(\mathbb{D})$.*
(b) *$A = T_z^* A T_z$.*
(c) *$A = T_{\varphi_\lambda}^* A T_{\varphi_\lambda}$ for all $\lambda > 0$.*

P r o o f. Note that φ_λ is an inner function for all $\lambda > 0$. Hence for any $\varphi \in L^\infty(\mathbb{T})$ by [6, Proposition 7.5] we get

$$T_{\varphi_\lambda}^* T_\varphi T_{\varphi_\lambda} = T_{\overline{\varphi}_\lambda} T_{\varphi\varphi_\lambda} = T_{\overline{\varphi}_\lambda \varphi\varphi_\lambda} = T_\varphi,$$

which proves (a) $\Rightarrow$ (c).

For the proof of (c) $\Rightarrow$ (a) put $\Phi_\lambda := \varphi_\lambda \circ \gamma$ and $B := U_2 A U_2^{-1}$, where U_2 is given by (3.5). Then $\Phi_\lambda(t) = \mathrm{e}^{\mathrm{i}\lambda t}$ and $T_{\Phi_\lambda} = U_2 T_{\varphi_\lambda} U_2^{-1}$ by Theorem 4.4. An easy computation shows that

$$B = U_2 A U_2^{-1} = U_2 T_{\varphi_\lambda}^* A T_{\varphi_\lambda} U_2^{-1} = U_2 T_{\varphi_\lambda}^* U_2^{-1} B U_2 T_{\varphi_\lambda} U_2^{-1} = T_{\Phi_\lambda}^* B T_{\Phi_\lambda}.$$

Therefore, $B \in \mathscr{T}(\mathbb{C}_+)$ by Theorem 4.7 and finally, $A \in \mathscr{T}(\mathbb{D})$ by Theorem 4.4. So the proof is complete. $\square$

5. Reflexivity and transitivity results

In [11] Sarason proved that $\mathscr{A}(\mathbb{D})$ is reflexive and in [1] it was pointed out that $\mathscr{T}(\mathbb{D})$ is transitive. By Theorem 4.4 we have $\mathscr{A}(\mathbb{C}_+) = U_2 \mathscr{A}(\mathbb{D}) U_2^{-1}$ and $\mathscr{T}(\mathbb{C}_+) = U_2 \mathscr{T}(\mathbb{D}) U_2^{-1}$, thus by Lemma 2.1 we obtain the following.

Theorem 5.1. *The algebra $\mathscr{A}(\mathbb{C}_+)$ is reflexive and the subspace $\mathscr{T}(\mathbb{C}_+)$ is transitive.*

If $\mathscr{F} \subsetneq \mathscr{T}(\mathbb{C}_+)$ is a weak* closed subspace and $\mathscr{A}(\mathbb{C}_+) \subset \mathscr{F}$, then $\mathscr{A}(\mathbb{D}) \subset U_2^{-1} \mathscr{F} U_2 \subsetneq \mathscr{T}(\mathbb{D})$. Thus, by [1, Theorem 1.2] we get

Theorem 5.2. *If $\mathscr{A}(\mathbb{C}_+) \subset \mathscr{F} \subsetneq \mathscr{T}(\mathbb{C}_+)$ and $\mathscr{F}$ is a weak* closed subspace, then $\mathscr{F}$ is reflexive.*

A dichotomy between reflexivity and transitivity of subspaces of Toeplitz operators on the Hardy space on the unit disc was given in [1, Theorem 1.1′]. Namely:

Theorem 5.3. *Suppose that $\mathscr{B} \subset \mathscr{T}(\mathbb{D})$ is a weak* closed subspace. Then the following statements are equivalent.*

(1) *$\mathscr{B}$ is not transitive.*

(2) *There is a function $f\colon \mathbb{T} \to \mathbb{C}$ such that $f \in L^1(\mathbb{T})$, $\log|f| \in L^1(\mathbb{T})$ and $\int_{\mathbb{T}} \varphi f \, \mathrm{d}m = 0$ for all $T_\varphi \in \mathscr{B}$.*

(3) *$\mathscr{B}$ is reflexive.*

The condition (2) of the above clearly characterizes the dichotomy. Now we will prove a corresponding dichotomy for subspaces of Toeplitz operators on the Hardy space on the upper half-plane and we will also give an appropriate condition, which verifies this dichotomy.

Theorem 5.4. *Suppose that $\mathscr{F} \subset \mathscr{T}(\mathbb{C}_+)$ is a weak* closed subspace. Then the following statements are equivalent.*

(1) *$\mathscr{F}$ is not transitive.*

(2) *There is a function $F\colon \mathbb{R} \to \mathbb{C}$ such that $F \in L^1(\mathbb{R})$, $\log|F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$ and $\int_{\mathbb{R}} \Phi F \, \mathrm{d}t = 0$ for all $T_\Phi \in \mathscr{F}$.*

(3) *$\mathscr{F}$ is reflexive.*

P r o o f. At the beginning let us note that there is a positive constant C such that

$$\int_{\mathbb{R}} \frac{\log(1+t^2)}{1+t^2} \, \mathrm{d}t \leqslant C < \infty. \tag{5.1}$$

Put $\mathscr{B} := \widetilde{U}_2^{-1}(\mathscr{F})$ ($\widetilde{U}_2$ is given by (4.2)). Then $\mathscr{B} \subset \mathscr{T}(\mathbb{D})$ and $\mathscr{B}$ is weak* closed by Theorem 4.4. To see that (1) $\Rightarrow$ (2) observe that if $\mathscr{F}$ is not transitive we have that $\mathscr{B}$ is not transitive by Lemma 2.1. Therefore there is a function f such that the condition (2) of Theorem 5.3 holds. Let us denote $F := U_1 f$. Then $F \in L^1(\mathbb{R})$. Since $\log|f| \in L^1(\mathbb{T})$ and the inequality (5.1) holds, it follows that

$$\begin{aligned}\int_{\mathbb{R}} \bigl|\log|F(t)|\bigr| \frac{\mathrm{d}t}{1+t^2} &= \int_{\mathbb{R}} \Bigl|\log\Bigl|\frac{1}{\pi}\frac{1}{1+t^2} f(\gamma(t))\Bigr|\Bigr| \frac{\mathrm{d}t}{1+t^2} \\ &< \pi\log\pi + C + \int_{\mathbb{R}} \bigl|\log|f(\gamma(t))|\bigr| \frac{\mathrm{d}t}{1+t^2} \\ &= \pi\log\pi + C + \pi\|\log|f|\|_{L^1(\mathbb{T})} < \infty.\end{aligned}$$

Therefore $\log|F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$. To see (2) let us take $\Phi \in \eta^{-1}(\mathscr{F})$ and put $\varphi := U_\infty^{-1}\Phi$. From the condition (d) of Theorem 4.4 we have that

$$T_\varphi = \xi(\varphi) = (\xi \circ U_\infty^{-1})(\Phi) = (\xi \circ U_\infty^{-1} \circ \eta^{-1})(T_\Phi) = \widetilde{U}_2^{-1}(T_\Phi) \in \mathscr{B}.$$

Now by Theorem 3.4

$$\int_{\mathbb{R}} \Phi F \, \mathrm{d}t = \langle \Phi, F \rangle = \langle \varphi, f \rangle = \int_{\mathbb{T}} \varphi f \, \mathrm{d}m = 0.$$

Hence (2) is shown.

Assume (2) and put $f := U_1^{-1} F$. Since $\log |F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$, thus $\log |\pi(1+t^2)F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$ by (5.1). The equality

$$\int_{\mathbb{R}} \big|\log |\pi(1+t^2)F(t)|\big| \frac{\mathrm{d}t}{1+t^2} = \int_{\mathbb{R}} \big|\log |f(\gamma(t))|\big| \frac{\mathrm{d}t}{1+t^2}$$

shows that $\log |f| \in L^1(\mathbb{T})$ and the condition (2) from Theorem 5.3 holds for the function f. Thus $\mathscr{B}$ is reflexive, hence $\mathscr{F}$ is reflexive by Lemma 2.1. Finally the implication (3) $\Rightarrow$ (1) follows from Lemma 2.1 and Theorem 5.3. □

6. Examples

By Theorem 5.4 we have the following examples of reflexive and transitive subspaces consisting of Toeplitz operators on the Hardy space on the upper half-plane.

Example 6.1. If $G \in L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \big|\log |G(t)|\big| \frac{\mathrm{d}t}{1+t^2} = \infty$, then $T_G \mathscr{A}(\mathbb{C}_+)$ is transitive. Indeed, assuming that $T_G \mathscr{A}(\mathbb{C}_+)$ is reflexive, then, by Theorem 5.4, there is a function $F \in L^1(\mathbb{R})$ such that $\log |F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$ and $\int_{\mathbb{R}} \Phi G F \, \mathrm{d}t = 0$ for all $\Phi \in H^\infty(\mathbb{C}_+)$. Hence $GF \in H^\infty(\mathbb{C}_+)_\perp$ and by Theorem 3.4 we have that $GF = U_1 f$, where $f \in H^1(\mathbb{D})$ and $f(0) = 0$, see [2]. Thus

$$\int_{\mathbb{R}} \big|\log |G(t)|\big| \frac{\mathrm{d}t}{1+t^2} = \int_{\mathbb{R}} \big|\log |U_1 f(t)|\big| \frac{\mathrm{d}t}{1+t^2} - \int_{\mathbb{R}} \big|\log |F(t)|\big| \frac{\mathrm{d}t}{1+t^2}.$$

But this leads to the contradiction, since $\int_{\mathbb{R}} \big|\log |U_1 f(t)|\big| \frac{\mathrm{d}t}{1+t^2} < \infty$ by (5.1) and $\log |f| \in L^1(\mathbb{T})$, see [10, Corollary 3.6.1].

Taking an appropriate function G we get in particular;

(a) if $G(t) = \exp(-|t|)$ or $G(t) = \exp(-t^2/2)$, then $T_G \mathscr{A}(\mathbb{C}_+)$ is transitive,

(b) if G is the characteristic function of $E \subset \mathbb{R}$ with E having finite non-zero Lebesgue measure, then $T_G \mathscr{A}(\mathbb{C}_+)$ is transitive.

Example 6.2. If $G \in L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \big|\log |G(t)|\big| \frac{\mathrm{d}t}{1+t^2} < \infty$ then $T_G \mathscr{A}(\mathbb{C}_+)$ is reflexive. Note first that $T_G \mathscr{A}(\mathbb{C}_+) \subsetneq \mathscr{T}(\mathbb{C}_+)$. Suppose now that $F \in L^1(\mathbb{R})$ is such that $\int_{\mathbb{R}} \Phi G F \, \mathrm{d}t = 0$ for all $\Phi \in H^\infty(\mathbb{C}_+)$. Then $GF \in H^\infty(\mathbb{C}_+)_\perp$, thus $GF = U_1 f$ by Theorem 3.4. As above $f \in H^1(\mathbb{D})$, $f(0) = 0$ and $\log |f| \in L^1(\mathbb{T})$, therefore

$\log|U_1 f| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$. Since $\log|G| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$ then $\log|F| \in L^1(\mathbb{R}, \frac{\mathrm{d}t}{1+t^2})$. So, $T_G\mathscr{A}(\mathbb{C}_+)$ is reflexive by Theorem 5.4.

Taking an appropriate function G we get in particular;

(a) the subspace $T_{\mathrm{e}^{\mathrm{i}\lambda t}}\mathscr{A}(\mathbb{C}_+)$ is reflexive for any $\lambda < 0$,

(b) if $\overline{G}$ is an inner function on $\mathbb{C}_+$ (i.e. $\overline{G} \in H^\infty(\mathbb{C}_+)$ and $|\overline{G}(t)| = 1$ a.e.), then $T_G\mathscr{A}(\mathbb{C}_+)$ is reflexive,

(c) if $G(t) = (1+t^2)^{-1}$, then $T_G\mathscr{A}(\mathbb{C}_+)$ is reflexive.

Example 6.3. Let $G \in L^1(\mathbb{R})$ and $\mathscr{B}_G := \{T_\Phi \in \mathscr{T}(\mathbb{C}_+)\colon \int_{\mathbb{R}} G\Phi\,\mathrm{d}t = 0\}$. Let $F \in L^1(\mathbb{R})$ then $\int_{\mathbb{R}} F\Phi\,\mathrm{d}t = 0$ for all Φ such that $T_\Phi \in \mathscr{B}_G$ iff $F \in \operatorname{span}\{G\}$. Hence the following holds:

(a) if G is the characteristic function of $E \subset \mathbb{R}$ with E having finite non-zero Lebesgue measure, then $\mathscr{B}_G$ is transitive,

(b) if $G(t) = \exp(-|t|^\alpha)$ and $0 \leqslant \alpha < 1$ ($\alpha \geqslant 1$, respectively), then the subspace $\mathscr{B}_G$ is reflexive (transitive, respectively),

(c) if $G(t) = (1+t^2)^{-1}$ (or more generally $G(t) = (1+t^2)^\alpha, \alpha < -\frac{1}{2}$), then $\mathscr{B}_G$ is reflexive.

Example 6.4. If $\mathscr{F}$ is a weak* closed subspace (subalgebra) of $\mathscr{A}(\mathbb{C}_+)$, then $\mathscr{F}$ is reflexive. Indeed, recall that $\mathscr{A}(\mathbb{D})$ has the property $\mathbb{A}_1(1)$, see [5, Definition 59.1, Proposition 60.5]. Thus $\mathscr{A}(\mathbb{C}_+) = \widetilde{U}_2(\mathscr{A}(\mathbb{D}))$ has this property, since $\widetilde{U}_2$ is a weak* homeomorphism. Since $\mathscr{A}(\mathbb{C}_+)$ is reflexive, it is hereditarily reflexive by [1, Proposition 1.7].

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Authors' addresses: Wojciech Młocek, Institute of Mathematics, University of Agriculture, ul. Balicka 253c, 30-198 Kraków, Poland, e-mail: `wojciech.mlocek@ur.krakow.pl`; Marek Ptak, Institute of Mathematics, University of Agriculture, ul. Balicka 253c, 30-198 Kraków, Poland and Institute of Mathematics, Pedagogical University, ul. Podchorążych 2, 30-084 Kraków, Poland, e-mail: `rmptak@cyf-kr.edu.pl`.