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Abstract. We prove among other theorems that it is consistent with ZFC that there exists a set $X \subseteq 2^\omega$ which is not meager additive, yet it satisfies the following property: for each F_σ measure zero set F, $X + F$ belongs to the intersection ideal $\mathcal{M} \cap \mathcal{N}$.

Keywords: F_σ measure zero sets; intersection ideal $\mathcal{M} \cap \mathcal{N}$; meager additive sets; sets perfectly meager in the transitive sense; γ-sets

Classification: 03E05, 03E17

0. In the first part of this paper we show that in the Cohen real model there is a set $X \subseteq 2^\omega$ which is not meager additive, but it satisfies the following condition: for every F_σ measure zero set F, $X + F$ is meager and has measure zero. This contrasts with the recent result of O. Zindulka (see [12]) which states that for $X \subseteq 2^\omega$, being meager additive is equivalent to the property: $X + F$ is contained in an F_σ measure zero set for every F_σ measure zero set F. Next we give a "new example" of an AFC' set, and in the second part we consider relations between various ideals of subsets of 2^ω defined in terms of translations of sets that belong to the intersection ideal $\mathcal{M} \cap \mathcal{N}$.

All the arguments that appear in this paper are quite usual and can be found in the previous literature. Throughout the paper, we assume that the reader is familiar with standard definitions and terminology of special sets of real numbers, and we recall below notions that may be less common. By $\mathcal{M}$ we denote the σ-ideal of meager subsets of 2^ω, $\mathcal{N}$ is the σ-ideal of measure zero subsets of 2^ω, and $\mathcal{E}$ stands for the σ-ideal of F_σ measure zero subsets of 2^ω. It is well-known (see [1, p. 73]) that $\mathcal{M} \cap \mathcal{N}$ is a strictly larger σ-ideal than $\mathcal{E}$.

Let $+$ be the standard modulo 2 coordinatewise addition in 2^ω, and suppose that I and J are σ-ideals of subsets of 2^ω with $I \subseteq J$.

Definition 1. We shall say that $X \subseteq 2^\omega$ is I additive, or $X \in I^*$, if $X + A = \{x + a : x \in X, a \in A\} \in I$, for any set $A \in I$, and $X \in (I, J)^*$ if for every set $A \in I$, $X + A \in J$.

Some authors use this very notation for the sets which can be "translated away" from each set in I, i.e., $X \in I^*$ if $X + A \neq 2^\omega$ for any set $A \in I$ (see [11]).

Definition 2. $X \subseteq 2^\omega$ is called an SMZ (strongly measure zero) set if $X + F \neq 2^\omega$, for every meager set F, and $Y \subseteq 2^\omega$ is an SFC (strongly first category or strongly meager) set if $Y + H \neq 2^\omega$, for every measure zero set H.

Definition 3. $X \subseteq 2^\omega$ is an AFC (always first category or perfectly meager) set if for any perfect $P \subseteq 2^\omega$, $X \cap P$ is meager in the relative topology of P.

Definition 4. $X \subseteq 2^\omega$ is said to be an AFC' (perfectly meager in the transitive sense) set if for any perfect $P \subseteq 2^\omega$, one can find an F_σ set F, with $X \subseteq F$, so that for each $t \in 2^\omega$, $(F+t) \cap P$ is meager in the relative topology of P.

Evidently, Definition 3 and 4 imply that $AFC' \subseteq AFC$.

We call a family $\mathcal{F}$ of subsets of $X \subseteq 2^\omega$ an ω-cover of X if each finite subset of X can be covered by an element of $\mathcal{F}$.

Definition 5. $X \subseteq 2^\omega$ is a γ-set if for every $\mathcal{F}$, an open ω-cover of X, we can choose a sequence $\{D_n\}_{n\in\omega} \in \mathcal{F}$ such that $X \subseteq \bigcup_{m\in\omega} \bigcap_{n\geq m} D_n$.

1. Let $\mathbb{P}_{\aleph_1}$ be an $\aleph_1$-iteration of Cohen forcing (with finite supports) over a model V of $ZFC+GCH$. Assume that G is a generic filter in $\mathbb{P}_{\aleph_1}$ over V.

Lemma 6. *There exists a set X in V which is SFC in V, and such that X is not meager additive in $V[G]$.*

Proof: We apply a reasoning similar to those in Lemma 8.5.3 of [1] and Theorem 8.5 of [6]. Working in V, we construct sets $X = \{x_\alpha\}_{\alpha<\mathfrak{c}=\omega_1}$, $Y = \{y_\alpha\}_{\alpha<\mathfrak{c}}$ and $R = \{r_\alpha\}_{\alpha<\mathfrak{c}}$ by induction. Let $\{H_\alpha\}_{\alpha<\mathfrak{c}}$ be a list of all measure zero Borel subsets of 2^ω, and let $\{z_\alpha\}_{\alpha<\mathfrak{c}}$ denote a bijective enumeration of 2^ω. Suppose now that $\{x_\alpha\}_{\alpha<\lambda<\mathfrak{c}}$, $\{y_\alpha\}_{\alpha<\lambda<\mathfrak{c}}$ and $\{r_\alpha\}_{\alpha<\lambda<\mathfrak{c}}$ are already defined, and for every $\alpha < \lambda < \mathfrak{c}$, $x_\alpha + y_\alpha = z_\alpha$. Let r_λ be a real number that does not belong to any set of the form $x_\alpha + H_\lambda$, $y_\alpha + H_\lambda$, for $\alpha < \lambda$. Pick x_λ, y_λ which do not belong to $\bigcup_{\alpha\leq\lambda}(r_\alpha + H_\alpha)$ and such that $x_\lambda + y_\lambda = z_\lambda$. It is easy to verify that for every $\lambda < \mathfrak{c}$, we have $(X + r_\lambda) \cap H_\lambda = \emptyset$, and $(Y + r_\lambda) \cap H_\lambda = \emptyset$, thus X and Y are strongly meager sets satisfying $X + Y = 2^\omega$. Since $V \cap 2^\omega$ is a non-meager set in $V[G]$, both X and Y are not meager additive in $V[G]$. □

Lemma 7. *Suppose that $F \in V[G]$, $F = \bigcup_{n\in\omega} F_n$, where for every $n \in \omega$, F_n is a closed measure zero set, and $F_n \subseteq F_{n+1}$. Then there exists H, a G_δ measure zero set coded in V, such that $F \subseteq H$.*

Proof: For $n \in \omega$, let $S_n = \left\{s : s \text{ is a clopen subset of } 2^\omega \text{ and } \mu(s) < \frac{1}{2^n}\right\}$, where μ denotes the Lebesgue measure on 2^ω. Since each S_n is countable, we can identify it with ω. Assume that $\dot{F}$, $\{\dot{F}_n\}_{n\in\omega}$, and $\dot{F}'$ are $\mathbb{P}_{\aleph_1}$-names such that

$$\mathbb{1}_{\mathbb{P}_{\aleph_1}} \Vdash \dot{F} = \bigcup_{n\in\omega} \dot{F}_n,\ \forall_{n\in\omega} \dot{F}_n \text{ is a closed measure zero set},\ \dot{F}_n \subseteq \dot{F}_{n+1},$$

and (by compactness)

$$\mathbb{1}_{\mathbb{P}_{\aleph_1}} \Vdash \dot{F}' \in \omega^\omega,\ \forall_{n\in\omega} \dot{F}_n \subseteq \dot{F}'(n), \quad \text{and} \quad \mu(\dot{F}'(n)) < \frac{1}{2^n}.$$

Then there exists $\widetilde{H} \in \omega^\omega \cap V$ (compare it to Lemma 3.1.2 in [1]) such that

$$\mathbb{1}_{\mathbb{P}_{\aleph_1}} \Vdash \exists^\infty_n \dot{F}'(n) = \widetilde{H}(n).$$

Clearly,

$$F = \bigcup_{n\in\omega} F_n \subseteq \bigcup_{m\in\omega} \bigcap_{n\geq m} F'(n) \subseteq \bigcap_{m\in\omega} \bigcup_{n\geq m} \widetilde{H}(n) = H. \qquad \square$$

Remark 8. It was pointed out to us by the referee that the following argument proves the crucial step in Lemma 7 as well: Assume for all $H \in \omega^\omega \cap V$, $\Vdash \forall^\infty_n\ F'(n) \neq H(n)$; then in the extension, the *meager* set $\bigcup_{k\in\omega}\{H \in \omega^\omega : \forall n \geq k\ F'(n) \neq H(n)\}$ would cover the ground model reals $\omega^\omega \cap V$, which is a contradiction.

Let $X \in V$ be the SFC set in V defined in Lemma 6 above.

Lemma 9. *For every set $F \in \mathcal{E} \cap V[G]$, we have that $X + F \in \mathcal{M}$.*

Proof: Let $F \subseteq \bigcup_{n\in\omega} F_n$, where for each $n \in \omega$, F_n is a closed measure zero set, and $F_n \subseteq F_{n+1}$. Suppose that $\bigcup_{n\in\omega} F_n \subseteq H$, where H is a G_δ measure zero set coded in V (see Lemma 7). We may assume without loss of generality that there are $f \in \omega^{\omega\uparrow}$ and a sequence $\{H_n\}_{n\in\omega}$ such that $H = \bigcap_{m\in\omega} \bigcup_{n\geq m} [H_n]$, every $H_n \subseteq 2^{[f(n),f(n+1))}$, $[H_n] = \{x \in 2^\omega : x \restriction [f(n), f(n+1)) \in H_n\}$, and

$$\sum_{n\in\omega} \frac{|H_n|}{|2^{[f(n),f(n+1))}|} < +\infty.$$

For each $n \in \omega$, define a finite subsequence $\{H_{k_n}, \dots, H_{k_n+m_n}\}$ of the sequence $\{H_n\}_{n\in\omega}$ satisfying $k_n = k_{n-1}+m_{n-1}+1$, and so that $F_n \subseteq [H_{k_n}]\cup\dots\cup[H_{k_n+m_n}]$.

For $n \in \omega$, put

$$M_n = \{x \in 2^\omega : x \restriction [f(k_n), f(k_{n+1})) \equiv 0\}.$$

Also, let $H'_n \subseteq 2^{[f(k_n),f(k_{n+1}))}$ be chosen in such a way that $[H'_n] = [H_{k_n}] \cup \dots \cup [H_{k_n+m_n}]$. We have that

$$\begin{aligned}\bigcup_{n\in\omega} F_n &\subseteq \bigcup_{m\in\omega} \bigcap_{n\geq m} [H'_n] \subseteq \bigcup_{m\in\omega} \bigcap_{n\geq m} [H'_n] + \bigcap_{m\in\omega} \bigcup_{n\geq m} M_n \\ &= \bigcap_{m\in\omega} \bigcup_{n\geq m} [H'_n] = \bigcap_{m\in\omega} \bigcup_{n\geq m} [H_n].\end{aligned}$$

Since X is an SFC set in V, and $\bigcap_{m\in\omega} \bigcup_{n\geq m} [H_n]$ is coded in V, we have (see Theorem 8.5.21 in [1] and (5), page 182 in [10] for a similar argument)

$$X + \bigcup_{n\in\omega} F_n \subseteq X + \bigcup_{m\in\omega} \bigcap_{n\geq m} [H'_n] + \bigcap_{m\in\omega} \bigcup_{n\geq m} M_n \neq 2^\omega.$$

This implies that for some $t \in 2^\omega$, the set

$$X + \bigcup_{n\in\omega} F_n \subseteq X + \bigcup_{m\in\omega} \bigcap_{n\geq m} [H'_n]$$

is disjoint with

$$t + \bigcap_{m\in\omega} \bigcup_{n\geq m} M_n,$$

and since the latter is a dense G_δ set, we are done. □

Theorem 10. *In $V[G]$ there exists a set X which is not meager additive and such that for every F_σ measure zero set F, $X + F \in \mathcal{M} \cap \mathcal{N}$.*

Proof: Suppose that $X \in V$ is as in Lemma 6. By Lemma 9, for every F_σ measure zero set F, $X + F \in \mathcal{M}$. Since we add iteratively $\aleph_1$ Cohen reals over V, X becomes strongly measure zero in $V[G]$. This implies that $X + F \in \mathcal{N}$, for each $F \in \mathcal{E}$ (see Theorem on page 172 in [10] or Theorem 8.1.18 in [1]). □

Remark 11. Notice that the conclusion of Theorem 10 holds also in $V[G]$, where G is a generic filter in $\mathbb{P}_{\aleph_2}$, the $\aleph_2$-iteration of Cohen forcing (with finite supports) over a model V of $ZFC + GCH$. By Carlson's argument, we have that in $V[G]$ all SFC sets are countable (see Theorem 8.5.22 in [1]).

One can prove that Theorem 10 holds as well in a single Cohen real extension, $V[c]$. This is a consequence of a non-trivial but well-known fact in the theory of forcing which we state below. Let us add that the author of this paper could not find the explicit proof of the below Lemma 12 in the literature.

Lemma 12 (Folklore)**.** *The set $2^\omega \cap V$ (old reals) is strongly measure zero in the extension $V[c]$, where c is a Cohen real over V.*

Proof: The proofs, due to M. Goldstern and J. Steprāns, can be found through the Internet (see [13]). □

Next we show that X defined in Lemma 6 is a "new example" of an AFC' set, that is, it neither belongs to the σ-ideal generated by meager additive sets and SFC sets, nor it is a carefully constructed scale $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^{\omega\uparrow}$, identified with a subset of 2^ω by characteristic functions (see [7], [8] and [9]).

Remark 13. To see that a scale, treated as a subset of 2^ω, does not need to be meager additive, consider a set X of cardinality $\mathfrak{c}$ in the iterated Laver model obtained by a successive adding of Laver reals. Since in this model Borel Conjecture holds, X cannot be strongly measure zero (see Section 8.3 in [1]). On the other hand, X, as a subset of 2^ω, is not strongly meager. This follows from the result of Bartoszyński and Shelah which states that in the iterated Laver model we have $SFC \subseteq [2^\omega]^{<\mathfrak{c}}$ (see Theorem 23, 24 and 26 in [3]).

Theorem 14. *Let $X \in V$ be the set constructed in Lemma 6, and let c be a Cohen real over V. Then X is an AFC' set in $V[c]$.*

Proof: Suppose that P is a perfect set in 2^ω, coded in $V[c]$. We follow the argument from the proof of Theorem 9 in [8]. Let $\{S_n\}_{n\in\omega}$ be a bijective enumeration of all basic clopen sets in 2^ω. For each $n \in \omega$, we put $P_n = S_n \cap P$. If P_n is a perfect set, let A_n be a perfect measure zero set such that $P_n + A_n = 2^\omega$. Assume that $\bigcup_{n\in\omega} A_n \subseteq F$, where F is an F_σ measure zero set. Let H with $F \subseteq H$, be a G_δ measure zero set coded in V as in Lemma 7. Then $X + H \neq 2^\omega \cap V$. Hence $X \cap (H + r) = \emptyset$, for some $r \in 2^\omega$. To finish the proof of Theorem 14 we proceed as in Theorem 9 from [8] to show that $2^\omega \setminus (H + r)$ is an F_σ set containing X and chosen for P as in Definition 4. □

By Lemma 6 above, X is not meager additive, and strongly meagerness of X is destroyed by adding a single Cohen real. Also, X cannot be equal to the set of characteristic functions of a scale as it would contradict the fact that no real in $\omega^{\omega\uparrow} \cap V$ dominates a Cohen real seen as a member of $\omega^{\omega\uparrow}$.

Remark 15. It is quite obvious that Lemma 9 and Theorem 14 are true in every extension $V[G]$, where G is a generic filter in a forcing notion (not necessarily Cohen) $\mathbb{P}$ such that each F_σ measure zero set in $V[G]$ can be covered by a G_δ measure zero set coded in V (which is in particular true if $\mathbb{P}$ preserves non-meagerness).

The following notion appeared in a recent paper by J. Kraszewski (see [5]).

Definition 16. An $X \subseteq 2^\omega$ is said to be an $\mathcal{EM}$ (or everywhere meager) set if for any infinite $a \subseteq \omega$, the set $\{x \restriction a : x \in X\} \subseteq 2^a$ is a meager subset of 2^a.

In [5] the author investigates the relationship between some well-known special subsets of 2^ω and the σ-ideal $\mathcal{EM}$, and he proves that $X \subseteq 2^\omega$ is an $\mathcal{EM}$ set if and only if for every set A of the form $\{x \in 2^\omega : x \restriction a = \mathbb{O}\}$, where a is an infinite subset of ω, $X + A$ is meager. So, in particular, $(\mathcal{E}, \mathcal{M})^* \subseteq \mathcal{EM}$. To prove that a scale viewed as a subset of 2^ω is in $\mathcal{EM}$, we argue as follows. First we use the easy lemma below, and then we apply Rothberger's theorem (see Theorem 5.6 in [6]) which states that every scale in $\omega^{\omega\uparrow}$ and in 2^ω is a perfectly meager set.

Lemma 17. *Suppose that $X \subseteq \omega^{\omega\uparrow}$ is a scale. Then for every infinite $a \subseteq \omega$, the set $\{x \restriction a : x \in X\}$ is a scale in $a^{\omega\uparrow}$.*

Proof: Obvious. □

Since SFC sets are in $(\mathcal{E}, \mathcal{M})^*$ (see Theorem 8.5.21 in [1] or (5), page 182 in [10]), and $\mathcal{M}^* = (\mathcal{M}, \mathcal{M})^* \subseteq (\mathcal{E}, \mathcal{M})^*$, it is clear that all AFC' sets mentioned above, including a set X from Theorem 14, are everywhere meager. Nevertheless the following question of Kraszewski (see Problem 1 in [5]) remains open.

Problem 18. *Is there an AFC' set which is not a member of the class $\mathcal{EM}$?*

2. To conclude this paper, we present relations between various ideals which appeared above (see Definition 1 and 2). Suppose that $\to$ denotes the inclusion and $\not\leftarrow$ means that the reverse inclusion cannot be proved in ZFC. Recall (see the first part of this paper) that Zindulka's result states that $\mathcal{E}^* = \mathcal{M}^*$.

Proposition 19. *The following diagram of inclusions holds.*

$$\begin{tikzcd}
 & & & & (\mathcal{E},\mathcal{M})^* \arrow[dl, shift left] \arrow[d, shift left, "/" marking] \arrow[r, shift right, "/" marking] & SFC \arrow[l, shift right] \\
\mathcal{N}^* \arrow[r, shift left] & (\mathcal{M}\cap\mathcal{N})^* \arrow[l, shift left, "?"] \arrow[r, shift left] & \mathcal{E}^* = \mathcal{M}^* \arrow[l, shift left, "/" marking] \arrow[r, shift left] & (\mathcal{E},\mathcal{M}\cap\mathcal{N})^* \arrow[l, shift left, "/" marking] \arrow[ur, shift left, "/" marking] \arrow[dr, shift left, "/" marking] & \\
 & & & & (\mathcal{E},\mathcal{N})^* \arrow[ul, shift left] \arrow[u, shift left, "/" marking] & = SMZ
\end{tikzcd}$$

Proof: Most inclusions are immediate consequences of the definitions. By Shelah's characterization of sets in $\mathcal{N}^*$ (see Theorem 2.7.18 in [1]) which implies $\mathcal{N}^* \to \mathcal{M}^*$, we have that $\mathcal{N}^* \to (\mathcal{M}\cap\mathcal{N})^*$. For the reverse inclusion see Problem 20 below. In Theorem 22 we prove that there may be a particularly small meager additive set which does not belong to $(\mathcal{M}\cap\mathcal{N})^*$. This implies that $(\mathcal{M}\cap\mathcal{N})^* \not\leftarrow \mathcal{E}^* = \mathcal{M}^*$. Theorem 23 below provides the proof of the inclusion $(\mathcal{M}\cap\mathcal{N})^* \to \mathcal{E}^* = \mathcal{M}^*$. Moreover, $\mathcal{E}^* = \mathcal{M}^* \not\leftarrow (\mathcal{E},\mathcal{M}\cap\mathcal{N})^*$ follows from Theorem 10. By Theorem on page 172 in [10] or Theorem 8.1.18 in [1], we have $(\mathcal{E},\mathcal{N})^* = SMZ$. Since SMZ sets do not have to be meager, we obtain $(\mathcal{E},\mathcal{M}\cap\mathcal{N})^* \not\leftarrow (\mathcal{E},\mathcal{N})^* = SMZ$. As mentioned in the previous remarks, we have $SFC \to (\mathcal{E},\mathcal{M})^*$. Crossed arrow in $SFC \not\leftarrow (\mathcal{E},\mathcal{M})^*$ was explained earlier in (5), page 182 in [10], and can be found implicit in Remark 11. The fact that SFC sets (hence sets in $(\mathcal{E},\mathcal{M})^*$) do not have to be measure zero yields $(\mathcal{E},\mathcal{M}\cap\mathcal{N})^* \not\leftarrow (\mathcal{E},\mathcal{M})^*$. By the same argument as above, both $(\mathcal{E},\mathcal{M})^* \not\leftarrow (\mathcal{E},\mathcal{N})^*$ and $(\mathcal{E},\mathcal{M})^* \not\rightarrow (\mathcal{E},\mathcal{N})^*$ hold. The author of this paper does not currently know whether $(\mathcal{E},\mathcal{M})^*$ can be equal to the collection of all countable sets of reals (see Problem 21 below). However, all the other nodes in the diagram are equal to the countable sets of reals under Borel Conjecture, or dual Borel Conjecture in case of SFC. $\square$

Problem 20. *Is it consistent with ZFC that the class $(\mathcal{M}\cap\mathcal{N})^*$ contains sets that are not in $\mathcal{N}^*$?*

Problem 21. *Is there a model of ZFC in which every element of the class $(\mathcal{E},\mathcal{M})^*$ is at most countable?*

Theorem 22. *Assume CH (the continuum hypothesis), or $\mathfrak{p} = \mathfrak{c}$. Then there exist a γ-set $X \subseteq 2^\omega$ and a set $A \subseteq 2^\omega$, $A \in \mathcal{M}\cap\mathcal{N}$, such that*

$$X + A \notin \mathcal{M}\cap\mathcal{N}.$$

In particular, the set X is meager additive (but not in $(\mathcal{M}\cap\mathcal{N})^*$).

Proof: This is exactly the same as the proof of Theorem 2.1 from [2] with one modification. Suppose that $\{k_n\}_{n\in\omega}$ is an increasing sequence of natural numbers such that the set

$$A' = \bigcap_{m \in \omega} \bigcup_{n \geq m} \{x \in 2^\omega : x \restriction [k_n, k_{n+1}) \equiv \mathbb{0}\}$$

has measure zero. Let

$$A = A' \cap \{x \in 2^\omega : \forall_n^\infty \ x \restriction [k_n, k_{n+1}) \not\equiv \mathbb{1}\}.$$

Clearly, $A \in \mathcal{M} \cap \mathcal{N}$. Let

$$B = \{x \in 2^\omega : \forall_n^\infty \ (x \restriction [k_n, k_{n+1}) \not\equiv \mathbb{0} \text{ and } x \restriction [k_n, k_{n+1}) \not\equiv \mathbb{1})\}.$$

Similar to the A', the set $2^\omega \setminus B$ has measure zero. Therefore, $\mu(B) = 1$. Suppose that $\{y_\alpha\}_{\alpha < \mathfrak{c}}$ is a bijective enumeration of B. Applying Lemma 2.2 from [2], we construct by induction on α a γ-set $X = Q \cup X' \subseteq 2^\omega$, where Q is the set of sequences eventually equal to zero, and $X' = \{x_\alpha\}_{\alpha < \mathfrak{c}}$. Assume that for $\alpha < \lambda < \mathfrak{c}$, x_α is already given, so that:

$$\begin{aligned}
&\forall_{\alpha < \lambda} \ \exists_n^\infty \ x_\alpha \restriction [k_n, k_{n+1}) \equiv \mathbb{1}, \\
&\forall_{\alpha < \lambda} \ \forall_n^\infty \ x_{\alpha+1}(n) \leq x_\alpha(n), \ \text{and} \\
&\forall_{\kappa < \lambda,\, \kappa \in \mathrm{Lim}}, \ \forall_{\alpha < \kappa} \ \forall_n^\infty \ x_\kappa(n) \leq x_\alpha(n).
\end{aligned}$$

Using CH, or $\mathfrak{p} = \mathfrak{c}$ (see Lemma 2.3 in [2] in case $\lambda \in \mathrm{Lim}$) we find x_λ such that:

$$\exists_n^\infty \ x_\lambda \restriction [k_n, k_{n+1}) \equiv \mathbb{1}, \quad \text{and} \quad \forall_{\alpha < \lambda} \forall_n^\infty \ x_\lambda(n) \leq x_\alpha(n).$$

We may assume without loss of generality that $\forall_{n \in \omega} \ (x_\lambda \restriction [k_n, k_{n+1}) \equiv \mathbb{0}$ or $x_\lambda \restriction [k_n, k_{n+1}) \equiv \mathbb{1})$. Thus we can choose x_λ, so that it satisfies the following two additional conditions: there exists an infinite $a \subseteq \omega$ with $\omega \setminus a$ infinite such that $\forall_{n \in a} \ x_\lambda \restriction [k_n, k_{n+1}) = y_\lambda$, and $\forall_{n \in \omega \setminus a} \ (x_\lambda \restriction [k_n, k_{n+1}) \equiv \mathbb{0}$ or $x_\lambda \restriction [k_n, k_{n+1}) \equiv \mathbb{1})$. It follows by the definition of the set B that

$$x_\lambda + y_\lambda \in A.$$

Consequently, $B \subseteq X + A$, and since every γ-set is meager additive (see Theorem 6 in [4]), we have the theorem. □

Next theorem contrasts with Theorem 10 above.

Theorem 23. $(\mathcal{M} \cap \mathcal{N}, \mathcal{M})^* \subseteq \mathcal{E}^* = \mathcal{M}^*$.

Proof: We follow closely the notation and the proof of Lemma 2.7.5 from [1], and we finesse it with one minor observation. Assume $X \in (\mathcal{M} \cap \mathcal{N}, \mathcal{M})^*$. We will show that X is meager additive using the characterization (see below) in Theorem 2.7.17 from [1].

For $f \in \omega^{\omega\uparrow}$ and $x \in 2^\omega$, let

$$B_{f,x} = \{y \in 2^\omega : \forall_n^\infty \ y \restriction [f(n), f(n+1)) \neq x \restriction [f(n), f(n+1))\}.$$

It was shown in Theorem 2.2.4, [1] that every meager set in 2^ω is contained in a set of the above form. Given $f \in \omega^{\omega\uparrow}$ with $f(n+1) \geq f(n)+n$ for every $n \in \omega$, let $H_f = \{x \in 2^\omega : \exists_n^\infty \ x \upharpoonright [f(n), f(n+1)) \equiv \mathbb{1}\}$. Put $\overline{B}_{f,\mathbb{O}} = B_{f,\mathbb{O}} \cap H_f$. Obviously, $\overline{B}_{f,\mathbb{O}} \in \mathcal{M} \cap \mathcal{N}$. By the assumption that $X \in (\mathcal{M} \cap \mathcal{N}, \mathcal{M})^*$, $X + \overline{B}_{f,\mathbb{O}}$ is meager. This implies that $X + \overline{B}_{f,\mathbb{O}} \subseteq B_{g,y}$, for some $g \in \omega^{\omega\uparrow}$ and $y \in 2^\omega$. We will show that for every $x \in X$,

$$\begin{aligned} &\forall_n^\infty \ \exists k \ (g(n) \leq f(k) < f(k+1) \leq g(n+1) \quad \text{and} \\ &x \upharpoonright [f(k), f(k+1)) = y \upharpoonright [f(k), f(k+1))) \end{aligned}$$

which finishes the proof by Theorem 2.7.17 from [1].

So, assume that the above condition is not true for some $x \in X$. Then the set

$$\begin{aligned} S = \{n \in \omega : \forall k (g(n) \leq f(k) < f(k+1) \leq g(n+1) \rightarrow \\ x \upharpoonright [f(k), f(k+1)) \neq y \upharpoonright [f(k), f(k+1)))\} \end{aligned}$$

is infinite. Let $a \subseteq S$ be infinite with $\omega \setminus a$ infinite. Define $t \in 2^\omega$ as follows: for each $n \in a$, let $t \upharpoonright [g(n), g(n+1)) = y \upharpoonright [g(n), g(n+1))$ (this ensures $t \notin B_{g,y}$); for $n \notin a$, let $t \upharpoonright [g(n), g(n+1)) = x \upharpoonright [g(n), g(n+1)) + \mathbb{1}$. We can choose a sufficiently fast growing function g such that each interval $[g(n), g(n+1))$ contains an interval $[f(k), f(k+1))$, so $t + x \in H_f$; it is also easy to see that $t + x \in B_{f,\mathbb{O}}$ (using the definition of S). Thus $t \in x + \overline{B}_{f,\mathbb{O}}$ which brings us to a contradiction as $t \notin B_{g,y}$. □

With the hope that it may help to find a solution of Problem 21 above we end this article with stating a combinatorial characterization of sets belonging to the class $(\mathcal{E}, \mathcal{M})^*$.

Let $\widehat{F} \subseteq \omega^{\omega\uparrow}$ be the set of functions f such that we have $\forall_{n\in\omega} \ f(n+1) \geq f(n) + n$. If $f \in \widehat{F}$, then we define

$$\begin{aligned} \Omega_f = \Bigg\{ h : h \text{ is a function with } \operatorname{dom}(h) = \omega, \\ \forall_{n\in\omega} \ h(n) \subseteq 2^{[f(n), f(n+1))}, \ \text{ and } \ \forall_{n\in\omega} \ \frac{|h(n)|}{2^{f(n+1)-f(n)}} \leq \frac{1}{2^n} \Bigg\}. \end{aligned}$$

Theorem 24. *c*$X \in (\mathcal{E}, \mathcal{M})^*$ *if and only if* $\forall_{f\in\widehat{F}} \ \forall_{h\in\Omega_f} \ \exists_{g\in\omega^{\omega\uparrow}} \ \exists_{y\in 2^\omega} \ \forall_{x\in X} \ \forall_n^\infty$ $\exists k \ (g(n) \leq f(k) < f(k+1) \leq g(n+1)$ *and* $x \upharpoonright [f(k), f(k+1)) \notin h(k) + y \upharpoonright [f(k), f(k+1)))$.

Proof: We prove the non-trivial direction. Suppose that $X \in (\mathcal{E}, \mathcal{M})^*$, and that the above characterization fails for $f \in \widehat{F}$ and $h \in \Omega_f$. Applying the characterization of sets in $\mathcal{E}$ from Chapter 2.6 in [1], we define $F \in \mathcal{E}$ to be equal to the set $\{x \in 2^\omega : \forall_n^\infty x \upharpoonright [f(n), f(n+1)) \in h(n)\}$. By assumption, $X + F \subseteq B_{g,y}$ for some $g \in \omega^{\omega\uparrow}$ and $y \in 2^\omega$ (see the proof of Theorem 23). Clearly, we may suppose that the range of g is included in the range of f. Fix $x \in X$ for which the assertion of the theorem fails. Put

$$S = \{n : \forall k\ (g(n) \leq f(k) < f(k+1) \leq g(n+1) \to x \restriction [f(k), f(k+1)) \in h(k) + y \restriction [f(k), f(k+1)))\}.$$

Obviously, S is infinite, thus we can proceed as in the proof of Theorem 23 above to get a contradiction. $\square$

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References

[1] Bartoszyński T., Judah H., *Set Theory*, AK Peters, Wellesley, Massachusetts, 1995.
[2] Bartoszyński T., Recław I., *Not every γ-set is strongly meager*, Contemp. Math., 192, Amer. Math. Soc. Providence, RI, 1996, pp. 25–29.
[3] Bartoszyński T., Shelah S., *Strongly meager sets of size continuum*, Arch. Math. Logic **42** (2003), 769–779.
[4] Galvin F., Miller A., *γ-sets and other singular sets of real numbers*, Topology Appl. **17** (1984), 145–155.
[5] Kraszewski J., *Everywhere meagre and everywhere null sets*, Houston J. Math. **35** (2009), no. 1, 103–111.
[6] Miller A., *Special subsets of the real line*, in Handbook of Set-Theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, 1984, pp. 201–233.
[7] Nowik A., *Remarks about transitive version of perfectly meager sets*, Real Anal. Exchange **22** (1996/97), no. 1, 406–412.
[8] Nowik A., Scheepers M., Weiss T., *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), 301–324.
[9] Nowik A., Weiss T., *Some remarks on totally imperfect sets*, Proc. Amer. Math. Soc. **132** (2004), no. 1, 231–237.
[10] Pawlikowski J., *A characterization of strong measure zero sets*, Israel J. Math. **93** (1996), 171–183.
[11] Pawlikowski J., Sabok M., *Two stars*, Arch. Math. Logic **47** (2008), no. 7–8, 673–676.
[12] Zindulka O., *Small sets of reals through the prism of fractal dimensions*, preprint, 2010.
[13] *Cohen reals and strong measure zero sets* – MathOverflow.15. http://mathoverflow.net/questions/63497/cohen-reals-and-strong-measure-zero-sets

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