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ON JP-SEMILATTICES OF BEGUM AND NOOR

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Abstract. In recent papers, S. N. Begum and A. S. A. Noor have studied join partial semilattices (JP-semilattices) defined as meet semilattices with an additional partial operation (join) satisfying certain axioms. We show why their axiom system is too weak to be a satisfactory basis for the authors' constructions and proofs, and suggest an additional axiom for these algebras. We also briefly compare axioms of JP-semilattices with those of nearlattices, another kind of meet semilattices with a partial join operation.

Keywords: JP-semilattice, meet semilattice, nearlattice, partial lattice

MSC 2010: 06A12, 06B75, 08A55

Introduction

S. N. Begum and A. S. A. Noor have introduced in [1], [2] a class of algebras, which they called join partial semilattices (JP-semilattices, for short), and obtained a number of interesting nontrivial results concerning modularity, distributivity, ideals, filters and congruences in such algebras as well as in pseudocomplemented JP-semilattices. Unfortunately, the very notion of a JP-semilattice is, in fact, vaguely defined; for this reason, many constructions and results in these papers may seem to be questionable. We aim in this note to clarify what JP-semilattices really could (or should) be and to improve the definition given by the authors.

A JP-semilattice was defined in [1] to be a partial algebra $(S, \wedge, \vee)$, where $(S, \wedge)$ is a meet semilattice and $\vee$ is a partial binary operation characterized by five axioms (we write here $s \downarrow t$, with arbitrary terms s, t, to mean '$s \vee t$ is defined')

(i) $x \downarrow x$ and $x \vee x = x$,

(ii) if $x \downarrow y$, then $y \downarrow x$ and $x \vee y = y \vee x$,

(iii) if $x \downarrow y$, $y \downarrow z$ and $x \vee y \downarrow z$, then $x \downarrow y \vee z$ and $(x \vee y) \vee z = x \vee (y \vee z)$,

(iv) if $x \mathrel{⫰} y$, then $x = x \wedge (x \vee y)$,
(v) if $y \mathrel{⫰} z$, then $x \wedge y \mathrel{⫰} x \wedge z$.

This definition is repeated in [2] without changes. Neither any informal description of these algebras nor a motivation for the choice of axioms is given. Still, it is noted on p. 57 in [1] that if $x \vee y$ exists, then $x \vee y$ is the supremum of $\{x, y\}$ with respect to the natural $\wedge$-semilattice ordering $\leqslant$ of S. As the following example shows, the operation $\vee$ in a JP-semilattice may actually be very weak.

Example. Let $(S, \wedge)$ be an arbitrary meet semilattice, and let the operation $\vee$ be defined on S by setting $x \vee y = z$ if and only if $x = y = z$. Then $(S, \wedge, \vee)$ is a JP-semilattice.

We first observe that the authors have used some natural assumptions on the operation $\vee$ which rule out algebras such as the one in this example and which are not, therefore, derivable from the axioms. We then propose an improved descriptive definition of JP-algebras, and show what additional axiom is necessary. Finally, as it is mentioned in [2] that every nearlattice (i.e., a meet semilattice having the upper bound property) is a JP-semilattice but not conversely, we also briefly compare axiom systems of these two classes of algebras.

Inadequacy of the axioms

In both papers [1], [2], various examples and counterexamples of JP-semilattices are pictured by Hasse diagrams of their underlying posets. However, the use of diagrams becomes problematic if the join $x \vee y$ may remain undefined when $\sup\{x, y\}$ is. This arouses suspicion that the list of axioms (i)–(v) is, in fact, not adequate for the class of algebras the authors have really had in mind.

The cause of the inadequacy is easily seen even independently of diagrams. A structure $(S, \vee)$ satisfying axioms (i)–(iii) is what could be called a weak partial semilattice (analogously to weak partial lattices discussed in [6]; see also [5, Definition 3.4]). As Lemma 16 in Sect. I.5 of [6] suggests, the relation $\leqslant'$ on S defined by

$$x \leqslant' y \quad \text{if and only if} \quad x \mathrel{⫰} y \ \text{ and } \ x \vee y = y$$

is a partial order, while $x \vee y$ is then the l.u.b. of x and y with respect to this order. Now, the orderings $\leqslant$ and $\leqslant'$ are not correlated properly in JP-semilattices (as the latter have been defined): apart from the absorption law (iv), also the dual law

(vi) $x \wedge y \mathrel{⫰} y$ and $y = (x \wedge y) \vee y$

should be added to JP-semilattice axioms. As usual, the conjunction of (iv) and (vi) is equivalent to the duality equivalence asserting that $x \leqslant y$ if and only if $x \leqslant' y$. An algebra $(S, \wedge, \vee)$ satisfying (i)–(iv) and (vi) is, in terms of [6], a weak partial lattice (with meet totally defined).

By the way, the condition (vi) is explicitly used in [1], for example, in the proofs of Proposition 3 and Theorem 6. Moreover, the proof of Proposition 12 in [1] rests on an implicit supposition that the l.u.b. of x and y w.r.t. $\leqslant$ is just the join $x \vee y$ needed there. It is the duality equivalence that justifies this supposition.

An improved definition

These observations suggest that the intention of authors in [1], [2] could actually have been to investigate a class of meet semilattices equipped with a partial operation $\vee$ defined as follows:

$$x \downarrow y \text{ if and only if } x \text{ and } y \text{ have a least upper bound, and } x \vee y = \sup\{x, y\}.$$

According to Lemma 21 in [6, Sect. I.5], an algebra obtained in this way is a particular partial lattice in the sense of that monograph. Let us call such an algebra a *join-enriched meet semilattice*. It is now easily seen that a meet semilattice with an additional partial operation $\vee$ is join-enriched if and only if it satisfies (i)–(iv) and (vi).

Now, the following improved definition of a JP-semilattice seems to be suitable to "legalize" the results presented in [1], [2].

Definition. A JP-semilattice is a join-enriched meet semilattice which satisfies (v).

The proper role of the axiom (v) is, however, not quite clear. An example in [1], [2] demonstrates that it is independent. The axiom was necessary, in particular, to state specific definitions of modular and distributive JP-semilattices (see [1] and, respectively, [2]; the formulation of the distributive law in [1] contains a sad misprint); these definitions are weaker than the standard ones for meet semilattices. Still, the existence condition provided by (v) could be incorporated directly in the defining conditions (cf. (iii), and (iii′) below).

It is worth to note that, in a modular JP-semilattice, (iv) (together with (v)) implies this weakened version of (vi): if $x \downarrow y$, then $x \downarrow x \wedge y$ and $x = (x \wedge y) \vee x$.

A comparison with nearlattices

A nearsemilattice was defined in [4] as a poset where every pair of elements bounded from above has a least upper bound. It may be considered as a partial algebra $(A, \vee)$, where $x \downarrow y$ if and only if x and y have a common upper bound and, if this is the case, $x \vee y = \sup\{x, y\}$. A nearlattice is then an algebra $(A, \wedge, \vee)$, where $(A, \wedge)$ is a meet semilattice and $(A, \vee)$ is a nearsemilattice with respect to the semilattice ordering (see references given in [2]).

An axiomatic description of nearsemilattices can be found in Section 2 of [3], where they have been discussed under the name 'partial semilattices'. Independently, a logically simpler axiom system for nearsemilattices, consisting of (i), (ii) and

(iii′) if $x \downarrow y$ and $x \vee y \downarrow z$, then $y \downarrow z$, $x \downarrow y \vee z$ and $(x \vee y) \vee z = x \vee (y \vee z)$,

was presented in [4, Proposition 2] (with some differences in notation). Together with (iv) and (vi), we thus have an axiom system for nearlattices. As (v) is satisfied in nearlattices, we may conclude that every nearlattice is a JP-semilattice. Of course, the converse does not hold; [2] contains an explicit counterexample. This is how the difference between the two versions of the associative law for partial join operations manifests itself.

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