# IMPULSIVE STABILIZATION OF HIGH-ORDER NONLINEAR RETARDED DIFFERENTIAL EQUATIONS

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*Abstract.* In this paper, impulsive stabilization of high-order nonlinear retarded differential equations is investigated by using Lyapunov functions and some analysis methods. Our results show that several non-impulsive unstable systems can be stabilized by imposition of impulsive controls. Some recent results are extended and improved. An example is given to demonstrate the effectiveness of the proposed control and stabilization methods.

*Keywords*: high-order nonlinear retarded differential equation, Lyapunov function, impulsive stabilization, exponential stability

*MSC 2010*: 34A37, 34K20

## 1. Introduction and Preliminaries

Over the last decade, impulsive control and impulsive stabilization for delay differential equations have attracted a great deal of attention due to its potential applications in many fields such as biological systems, chemical reactions, dosage supply in pharmacokinetics, ecosystems management, dynamic portfolio management and stabilization and synchronization in chaotic secure communication systems and other chaos systems [5], [6], [10], [13], [15], [16]. For example, in pest management, the impulse models the process of periodic release of infective pests at fixed moments to control pests population size, which can be described by an impulsive control system [16]. In such system, the equilibrium solution that can be effectively controlled by impulses denotes that the pests population size can be kept at acceptably low levels

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in the long term. Moreover, in many cases, impulsive control can produce better performance than continuous control; even in some cases, only impulsive approaches can be used. A representative example is that a central bank cannot change its interest rate everyday in order to regulate the money supply in a financial market [15]. In recent years, various results for impulsive control systems have been reported, see [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [14]. Since impulses can make unstable systems stable, and stable systems can become unstable after impulse effects, it is reasonable to ask whether the solutions of high-order nonlinear retarded differential equations have similar properties. However, to the best of author's knowledge, there is almost no result on impulsive stabilization of high-order nonlinear retarded differential equations.

In this paper, we consider the high-order nonlinear retarded differential equations

$$x^{(n)}(t)+a(t)x^{(\gamma)}(t)+b(t)x^{\varrho}(t)+\sum_{i=1}^{N} f_i(t,x^{\delta}(g_i(t)))=0, \quad t\geqslant t_0, \tag{1.1}$$

and the corresponding equations with impulses

$$\begin{cases} x^{(n)}(t)+a(t)x^{(\gamma)}(t)+b(t)x^{\varrho}(t)+\sum\limits_{i=1}^{N} f_i(t,x^{\delta}(g_i(t)))=0,\ t\geqslant t_0,\ t\neq t_k,\\ x(t_k)=I_k(x(t_k^-)), x^{(j)}(t_k)=J_{jk}(x^{(j)}(t_k^-)),\ j=1,2,\dots,n-1,\ t=t_k. \end{cases} \tag{1.2}$$

We also consider other high-order nonlinear retarded differential equations

$$x^{(n)}(t)+a(t)x^{(\gamma)}(t)+b(t)x^{\varrho}(t)+\sum_{i=1}^{N}\int_{g_i(t)}^{t} f_i(t-u,x^{\delta}(u))\,\mathrm{d}u=0,\ t\geqslant t_0, \tag{1.3}$$

and the corresponding equations with impulses

$$\begin{cases} x^{(n)}(t)+a(t)x^{(\gamma)}(t)+b(t)x^{\varrho}(t)+\sum\limits_{i=1}^{N}\int_{g_i(t)}^{t} f_i(t-u,x^{\delta}(u))\,\mathrm{d}u=0,\ t\geqslant t_0,\\ x(t_k)=I_k(x(t_k^-)), x^{(j)}(t_k)=J_{jk}(x^{(j)}(t_k^-)),\ j=1,2,\dots,n-1,\ t=t_k. \end{cases} \tag{1.4}$$

The following assumptions will be needed throughout the paper:

($A_1$) The impulsive sequence $t_k$ satisfies $0\leqslant t_0<t_1<\ldots<t_k<\ldots$, $\lim\limits_{k\to+\infty} t_k=+\infty$;

($A_2$) $I_k, J_{jk}\colon \mathbb{R}\to\mathbb{R}$ are continuous and $I_k(0)=J_{jk}(0)=0$, $k\in\mathbb{Z}_+$;

($A_3$) $a(t), b(t)\colon [t_0,\infty)\to\mathbb{R}$ are continuous functions;

($A_4$) $f_i\colon [t_0,\infty)\times\mathbb{R}\to\mathbb{R}$, $f_i(t,0)=0$, $t\geqslant t_0$. There exists a sequence of functions $p_i(t)$, where $p_i\colon [t_0,\infty)\to\mathbb{R}$ are continuous such that for all $t\geqslant t_0$, $|f_i(t,x)|\leqslant |p_i(t)||x|$, $i=1,2,\dots,N$, where $N\geqslant 1$;

($\mathrm{A}_5$) $g_i\colon [t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous satisfying $0\leqslant t-g_i(t)<\infty$ for all $t\geqslant t_0\geqslant 0$, $i=1,2,\ldots,N$;

($\mathrm{A}_6$) $\gamma$, $\varrho$, $\delta$ are constants and $1\leqslant\gamma<n$, $\varrho\geqslant 1$, $\delta\geqslant 1$, $\gamma\in\mathbb{Z}_+$;

($\mathrm{A}_7$) $x'(t)$ denotes the right hand derivative of $x(t)$, i.e., $x'(t_k)=x'(t_k^+)=\lim\limits_{h\to 0^+}(x(t_k+h)-x(t_k^+))/h$, and $x''(t)=[x'(t)]'$, …, $x^{(n)}(t)=[x^{(n-1)}(t)]'$.

For any $\sigma\geqslant 0$, let $\tau_i^\sigma=\sup\limits_{t\geqslant\sigma}\{t-g_i(t)\}$, $\tau^\sigma=\max\limits_{1\leqslant i\leqslant N}\tau_i^\sigma$ and let $\Phi(\sigma)$ denote the set of functions $\varphi\colon[\sigma-\tau^\sigma,\sigma]\to\mathbb{R}$ which have at most finitely many discontinuity points of the first kind and are right continuous at these points.

So for any $\sigma\geqslant 0$ and $\varphi\in\Phi$, we can define the initial value condition of (1.1), (1.2), (1.3), (1.4)

$$x(t)=\varphi(t),\quad t\in[\sigma-\tau^\sigma,\sigma];\quad x^{(j)}(\sigma)=x_{j0},\quad j=1,2,\ldots,n-1. \tag{1.5}$$

If we let $n=2$, $\gamma=1$, $\delta=\varrho=1$, $N=1$, $g_1(t)=t-\tau_1$, $f_1(t,x)=p(t)x$, then (1.1) and (1.3) reduce to the differential equations

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)+p(t)x(t-\tau_1)=0,\ t\geqslant t_0,\\ x(t)=\varphi(t),\ t_0-\tau_1\leqslant t\leqslant t_0,\ x'(t_0)=x_0\end{cases}$$

and

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)+\displaystyle\int_{t-\tau_1}^{t}p(t-u)x(u)\,\mathrm{d}u=0,\ t\geqslant t_0,\\ x(t)=\varphi(t),\ t_0-\tau_1\leqslant t\leqslant t_0,\ x'(t_0)=x_0.\end{cases}$$

The existence of solutions and impulsive stabilization of these equations was extensively investigated in [14].

If we let $a(t)=0$, $f_i=0$, then (1.1) or (1.3) reduces to the differential equation

$$\begin{cases} x^{(n)}(t)+b(t)x^{\varrho}(t)=0,\ t\geqslant t_0,\\ x(t_0)=x_0,\ x^{(j)}(t_0)=x_{j0},\ j=1,2,\ldots,n-1.\end{cases} \tag{1.6}$$

The stabilization of the solutions of (1.6) with impulse has been investigated in [9].

In the paper [4], the authors study the impulsive stabilization for the following second order delay differential equations:

$$\begin{cases} x''(t)+f(t,x(t),x'(t))+g(t,x(t),x(t-\tau))=0,\ t\geqslant t_0,\\ x(t)=\varphi(t),\ t_0-\tau\leqslant t\leqslant t_0,\ x'(t_0)=y_0\end{cases}$$

with some necessary assumptions which include: $f,g\colon[t_0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous such that $f(t,0,0)=g(t,0,0)=0$ and there exist constants $F>0$, $G>0$ such that for all $t\geqslant t_0$ and $u,v\in\mathbb{R}$, $|f(t,u,v)|\leqslant F|u|$, $|g(t,u,v)|\leqslant G|v|$.

In the present paper, we deal with the equations (1.1) and (1.3), which are more general than those in [14] and [9]. To some extent the systems studied in [4] are more general than the systems we study (when $n=2$, $\gamma=1$, $\delta=\sigma=1$, $N=1$, $g_1(t)=t-\tau_1$ in (1.1)), but there still exist some cases to which the methods cannot be applied, for example $f(t,x,x')=x^3+x'$. The results we study can solve it.

In this paper, since (1.1) may reduce to first order impulsive differential equation, we can obtain a global existence result of the solution of system (1.1), see [11], [2], [1]. So we always assume the solution of (1.1) to exist globally in this paper.

**Definition 1.1.** For any $\sigma \geqslant 0$ and $\varphi \in \Phi$, a function $x\colon [\sigma-\tau^\sigma, \sigma+a) \to \mathbb{R}$, $a>0$, is said to be a solution of (1.2) and (1.5) through $(\sigma, \varphi, \{x_{j0}\}_{j=1}^{n-1})$ if

(i) $x(t)$ and $x^{(j)}(t)$, $j=1,2,\ldots,n-1$, are continuous on $[\sigma-\tau^\sigma, \sigma+a)\setminus\{t_k; k\in\mathbb{Z}_+\}$ and are right continuous at $t_k$;

(ii) $x(t)$ satisfies (1.1) and (1.5);

(iii) $x(t)$ and $x^{(j)}(t)$, $j=1,2,\ldots,n-1$ fulfil (1.2) for each $k\in\mathbb{Z}_+$.

Remark 1.1. The definition of a solution of (1.4) and (1.5) is similar to Definition 1.1, we omit it.

Remark 1.2. In the present paper, for convenience we use $\tau$ instead of $\tau^\sigma$.

**Definition 1.2.** The zero solution of (1.1) is said to be exponentially stabilized by impulses, if there exist $\alpha>0$, a sequence $\{t_k\}_{k=1}^\infty$, $I_k$, $J_{jk}$ satisfying ($A_1$) and ($A_2$) such that for all $\varepsilon>0$ there exists a $\delta^*>0$ such that, when the solution $x(t)$ of (1.1) through $(\sigma, \varphi, \{x_{j0}\}_{j=1}^{n-1})$ fulfils

$$\left(\|\varphi\|_{t_0}^2 + \sum_{j=1}^{n-1}(x_{j0})^2\right)^{\frac{1}{2}} \leqslant \delta^*, \tag{1.7}$$

then

$$\left(x^2(t) + \sum_{j=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} \leqslant \varepsilon \exp[-\alpha(t-t_0)], \quad t \geqslant t_0. \tag{1.8}$$

where $\|\varphi\|_t = \sup\limits_{t-\tau\leqslant s\leqslant t} |\varphi(s)|$.

**Definition 1.3.** The zero solution of (1.1) is said to be exponentially stabilized by periodic impulses if there exist $\alpha>0$, a sequence $\{t_k\}_{k=1}^\infty$ satisfying ($A_1$) and $t_k-t_{k-1}=d$ ($>0$ constant), $I_k$, $J_{jk}$ satisfying ($A_2$) and

$$I_k(x)=I(x), J_{jk}(x)=J(x), \quad x\in\mathbb{R},$$

such that for all $\varepsilon>0$ there exists a $\delta>0$ such that the solution $x(t)$ of (1.1) through $(\sigma, \varphi, \{x_{j0}\}_{j=1}^{n-1})$ fulfils (1.7) and (1.8).

## 2. Main results

**Theorem 2.1.** *Assume that conditions* ($A_3$)–($A_7$) *hold. Moreover, suppose that*

($A_8$) *there exists a constant* $\beta > 0$ *such that* $g_i'(t) \geqslant \beta$, $t \geqslant 0$, $i = 1, 2, \ldots, N$;

($A_9$) *there exist constants* $a, b, p_i \geqslant 0$, $i = 1, 2, \ldots, N$ *such that* $|a(t)| \leqslant a$, $|b(t)| \leqslant b$, $|p_i(t)| \leqslant p_i$;

($A_{10}$)

$$\frac{\tau}{\beta}\sum_{i=1}^{N} p_i < \exp\Bigl[-\Bigl(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\Bigr)\tau\Bigr].$$

*Then the zero solution of* (1.1) *can be exponentially stabilized by impulses.*

P r o o f. Since ($A_{10}$) holds, there exist $\alpha > 0$ and $\lambda \geqslant \tau$ such that

$$\frac{1}{\beta}\sum_{i=1}^{N} p_i\tau_i \leqslant \frac{\tau}{\beta}\sum_{i=1}^{N} p_i \leqslant \exp[-2\alpha(\lambda+\tau)]\exp\Bigl[-\Bigl(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\Bigr)\lambda\Bigr].$$

Then one may choose a sequence $\{t_k\}_{k=1}^{\infty}$ satisfying ($A_1$) and $\tau \leqslant t_{k+1} - t_k \leqslant \lambda$, $t_0 = \sigma$. Note that $g_i'(t) \geqslant \beta > 0$, hence $g$ is nondecreasing in $t$ for $t \geqslant 0$. Thus one may choose a sequence $\{\eta_k\}_{k=1}^{\infty}$, $\eta_k \in (0,1)$ such that, when the solution $x(t) = x(t, t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ through $(t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ fulfils

$$\Bigl(x^2(t_k) + \sum_{j=1}^{n-1}(x^{(j)}(t_k))^2\Bigr)^{\frac{1}{2}} \leqslant \eta_k,$$

then

$$\Bigl(x^2(t) + \sum_{j=1}^{n-1}(x^{(j)}(t))^2\Bigr)^{\frac{1}{2}} < 1, \quad t \in [t_k, t_{k+1}). \tag{2.1}$$

Let

$$|I_k(u)| = \mathrm{d}_k|u|, \quad |J_{jk}(v)| = \mathrm{d}_k|v|,$$

$$\mathrm{d}_k = \min\Bigl\{\eta_k\exp[\alpha(t_1-\sigma)],\ \Bigl(\Gamma_k - \sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\Bigr)^{\frac{1}{2}}\Bigr\},$$

$$\Gamma_k = \exp[-2\alpha(t_{k+1}-t_k+\tau)]\exp\Bigl[-\Bigl(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\Bigr)(t_{k+1}-t_k)\Bigr].$$

It is obvious that $\mathrm{d}_k \geqslant 0$.

For any $\varepsilon \in (0,1)$, let

$$\delta^* = \min\left\{\eta_0, \varepsilon, \varepsilon\left(1+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\right)^{-1/4} \times \exp[-\alpha(t_1-\sigma)]\exp\left[-\frac{1}{2}\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_1-\sigma)\right]\right\}.$$

Next, we prove that each solution $x(t)=x(t,\sigma,\varphi,\{x_{j0}\}_{j=1}^{n-1})$ of (1.2) and (1.5) with

$$\left(\|\varphi\|_\sigma^2+\sum_{j=1}^{n-1}(x_{j0})^2\right)^{\frac{1}{2}} \leqslant \delta^*$$

satisfies

$$\left(x^2(t)+\sum_{j=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} \leqslant \varepsilon\exp[-\alpha(t-t_0)], \quad t \geqslant t_0,$$

where $\|\varphi\|_t = \sup\limits_{t-\tau\leqslant s\leqslant t}|\varphi(s)|$.

First, for $t\in[\sigma,t_1)$, we choose a Lyapunov function

$$V(t)=x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\int_{g_i(t)}^{t}x^{2\delta}(s)\,\mathrm{d}s.$$

It follows from conditions (A$_8$)–(A$_{10}$) and (2.1) that

(1) $V(t)\geqslant x^2(t)+\sum\limits_{i=1}^{n-1}(x^{(j)}(t))^2$;

(2)

$$\begin{aligned}
V(t) &\leqslant x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\int_{t-\tau_i}^{t}x^{2\delta}(s)\,\mathrm{d}s\\
&\leqslant x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\sup_{t-\tau_i\leqslant s\leqslant t}|x(s)|^{2\delta}\\
&\leqslant x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\|x\|_t^{2\delta}\\
&\leqslant x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\|x\|_t^{2}\\
&\leqslant \left(1+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\right)\left(\|x\|_t^2+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right),
\end{aligned}$$

where $\|x\|_t=\sup\limits_{t-\tau\leqslant s\leqslant t}|x(s)|$.

(3) We denote by $V'(t)$ the right upper derivative of $V(t)$ along the solution of (1.1)–(1.3). Then

$$
\begin{aligned}
V'(t) &= 2x(t)x'(t) + 2x'(t)x''(t) + \ldots + 2x^{(n-1)}(t)x^{(n)}(t) \\
&\quad + \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(g_i(t))g_i'(t) \\
&= 2x(t)x'(t) + 2x'(t)x''(t) + \ldots \\
&\quad + 2x^{(n-1)}(t)\Big\{ - a(t)x^{(\gamma)}(t) - b(t)x^{\varrho}(t) - \sum_{i=1}^{N} f_i\big(t, x^{\delta}(g_i(t))\big)\Big\} \\
&\quad + \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(g_i(t))g_i'(t) \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 \\
&\quad + a\big([x^{(n-1)}(t)]^2 + [x^{(\gamma)}(t)]^2\big) + b\big([x^{(n-1)}(t)]^2 + x^2(t)\big) \\
&\quad + 2|x^{(n-1)}(t)| \sum_{i=1}^{N} |p_i(t)||x^{2\delta}(g_i(t))| + \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(g_i(t))g_i'(t) \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 \\
&\quad + a\big([x^{(n-1)}(t)]^2 + [x^{(\gamma)}(t)]^2\big) + b\big([x^{(n-1)}(t)]^2 + x^2(t)\big) \\
&\quad + \sum_{i=1}^{N} p_i\big([x^{(n-1)}(t)]^2 + x^{2\delta}(g_i(t))\big) + \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(g_i(t))g_i'(t) \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 \\
&\quad + a\big([x^{(n-1)}(t)]^2 + [x^{(\gamma)}(t)]^2\big) + b\big([x^{(n-1)}(t)]^2 + x^2(t)\big) \\
&\quad + \sum_{i=1}^{N} p_i[x^{(n-1)}(t)]^2 + \sum_{i=1}^{N} \frac{p_i}{\beta} x^2(t) + \sum_{i=1}^{N} p_i x^{2\delta}(g_i(t))\Big(1 - \frac{g_i'(t)}{\beta}\Big) \\
&\leqslant \left(1 + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right) x^2(t) + 2[x'(t)]^2 + \ldots + 2[x^{(\gamma-1)}(t)]^2 + (a+2)[x^{(\gamma)}(t)]^2 \\
&\quad + 2[x^{(\gamma+1)}(t)]^2 + \ldots + \left(2 + a + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)[x^{(n-1)}(t)]^2 \\
&\leqslant \left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right) V(t),
\end{aligned}
$$

which implies that

$$
V(t) \leqslant V(t_0) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t - \sigma), \quad t \in [\sigma, t_1).
$$

Thus for $t \in [\sigma, t_1)$ we get

$$\begin{aligned}
x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2 &\leqslant V(t) \leqslant V(t_0) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t - \sigma) \\
&< V(t_0) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t_1 - \sigma) \\
&\leqslant \left(1 + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)\left(\|x\|_\sigma^2 + \sum_{i=1}^{n-1} (x^{(j)}(\sigma))^2\right) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t_1 - \sigma) \\
&= \left(1 + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)\left(\|\varphi\|_\sigma^2 + \sum_{j=1}^{n-1} x_{j0}^2\right) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t_1 - \sigma) \\
&\leqslant \left(1 + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)\delta^{*2} \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)(t_1 - \sigma) \\
&\leqslant \varepsilon^2 \exp[-2\alpha(t_1 - \sigma)] < \varepsilon^2 \exp[-2\alpha(t - \sigma)],
\end{aligned}$$

which implies that

$$\left(x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < \varepsilon \exp[-\alpha(t - \sigma)], \quad t \in [\sigma, t_1).$$

Especially,

$$\left(x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right)^{\frac{1}{2}} \leqslant \varepsilon \exp[-\alpha(t_1 - \sigma)].$$

It then follows that

$$\begin{aligned}
\left(x^2(t_1) + \sum_{i=1}^{n-1} (x^{(j)}(t_1))^2\right)^{\frac{1}{2}} &= d_1\left(x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right)^{\frac{1}{2}} \\
&\leqslant d_1\varepsilon \exp[-\alpha(t_1 - \sigma)] < d_1 \exp[-\alpha(t_1 - \sigma)] \leqslant \eta_1,
\end{aligned}$$

which implies that, for $t \in [t_1, t_2)$,

$$\left(x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < 1. \tag{2.2}$$

For $t \in [t_1, t_2)$, we still choose a Lyapunov function

$$V(t) = x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2 + \sum_{i=1}^{N} \frac{p_i}{\beta} \int_{g_i(t)}^{t} x^{2\delta}(s)\, \mathrm{d}s.$$

Then we have

$$
\begin{aligned}
&x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\\
&\leqslant V(t)\\
&\leqslant V(t_1)\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t-t_1)\\
&< V(t_1)\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&=\left[x^2(t_1)+\sum_{i=1}^{n-1}(x^{(j)}(t_1))^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\int_{g_i(t_1)}^{t_1}x^{2\delta}(s)\,\mathrm{d}s\right]\\
&\quad\times\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\left\{d_1^2\left[x^2(t_1^-)+\sum_{i=1}^{n-1}(x^{(j)}(t_1^-))^2\right]+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\sup_{t_1-\tau_i\leqslant t\leqslant t_1}x^{2\delta}(t)\right\}\\
&\quad\times\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\left\{d_1^2\left[x^2(t_1^-)+\sum_{i=1}^{n-1}(x^{(j)}(t_1^-))^2\right]+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\sup_{t_1-\tau\leqslant t\leqslant t_1}x^{2}(t)\right\}\\
&\quad\times\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\left\{d_1^2\sup_{t_1-\tau\leqslant t\leqslant t_1}\left[x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right]+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\sup_{t_1-\tau\leqslant t\leqslant t_1}x^{2}(t)\right\}\\
&\quad\times\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\left(d_1^2+\sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\right)\sup_{t_1-\tau\leqslant t\leqslant t_1}\left[x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right]\\
&\quad\times\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\Gamma_1\varepsilon^2\exp(-2\alpha(t_1-\sigma-\tau))\exp\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_2-t_1)\\
&\leqslant\varepsilon^2\exp[-2\alpha(t_2-\sigma)]\\
&<\varepsilon^2\exp[-2\alpha(t-\sigma)],\quad t\in[t_1,t_2).
\end{aligned}
$$

Hence,

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}}<\varepsilon\exp[-\alpha(t-\sigma)],\quad t\in[t_1,t_2).$$

Arguing as before, by induction hypothesis we may prove, in general, that for $k\geqslant 1$,

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}}<\varepsilon\exp[-\alpha(t-\sigma)],\quad t\in[t_k,t_{k+1}).$$

Therefore, we finally obtain

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}}<\varepsilon\exp[-\alpha(t-\sigma)],\quad t\geqslant\sigma,$$

and the proof is therefore complete. □

R e m a r k 2.1. In Theorem 2.1, we choose linear functions $I_k(u)=d_ku$, $J_{jk}(v)=d_kv$. In fact, from the procedure in the proof of Theorem 2.1, it is not difficult to realize that we only need $I_k(u)$, $J_{jk}(v)$ to satisfy: $|I_k(u)|\leqslant d_k|u|$, $|J_{jk}(v)|\leqslant d_k|v|$.

R e m a r k 2.2. If $\gamma=n-1$ in Theorem 2.1, then the condition $|a(t)|\leqslant a$ can be replaced by $a(t)\geqslant 0$.

R e m a r k 2.3. Suppose that all the conditions in Theorem 2.1 hold. Then the procedure in Theorem 2.1 can be used to prove the exponential stabilization for the high-order nonlinear retarded differential equations

$$x^{(n)}(t)+h(t,x(t),x^{(\gamma)}(t))+\sum_{i=1}^{N}f_i(t,x^{\delta}(g_i(t)))=0,\ t\geqslant t_0,$$

and the corresponding equations with impulses

$$\begin{cases}x^{(n)}(t)+h(t,x(t),x^{(\gamma)}(t))+\sum\limits_{i=1}^{N}f_i(t,x^{\delta}(g_i(t)))=0, & t\geqslant t_0,\ t\neq t_k,\\ x(t_k)=I_k(t_k^-),\ x^{(j)}(t_k)=J_{jk}(t_k^-), & j=1,2,\ldots,n-1,\ t=t_k,\end{cases}$$

where there exist two continuous functions $h_1$, $h_2$ such that $|h(t,x,y)|\leqslant|h_1(t)||x|+|h_2(t)||y|$.

R e m a r k 2.4. In Remark 2.3, we can further consider the high-order nonlinear retarded differential equations

$$x^{(n)}(t)+h(t,x(t),x^{(\gamma)}(t))+f(t,x(g_1(t)),x(g_2(t)),\ldots,x(g_N(t)))=0,\quad t\geqslant t_0,$$

and the corresponding equations with impulses

$$\begin{cases} x^{(n)}(t) + h(t, x(t), x^{(\gamma)}(t)) + f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_N(t))) = 0, \\ \hfill t \geqslant t_0,\ t \neq t_k, \\ x(t_k) = I_k(t_k^-), x^{(j)}(t_k) = J_{jk}(t_k^-), \quad j = 1, 2, \dots, n-1,\ t = t_k, \end{cases}$$

where there exists a sequence of continuous functions $\omega_i$, $i = 1, 2, \dots, N$ such that $|f(t, \mu_1, \mu_2, \dots, \mu_N)| \leqslant \sum\limits_{i=1}^{N} |\omega_i(t)||\mu_i|$.

**Theorem 2.2.** *(Case: $\delta = \varrho = 1$.) Assume that the conditions in Theorem* 2.1 *hold. Then the zero solution of* (1.1) *can be exponentially stabilized by periodic impulses.*

P r o o f. Here one may choose a sequence $\{t_k\}_{k=1}^{\infty}$ satisfying ($\mathrm{A}_1$) and $t_{k+1} - t_k = \lambda \geqslant \tau$, $t_0 = \sigma$. Note that since $\delta = \varrho = 1$, we only need to choose

$$|I_k(u)| = d|u|, \quad |J_{jk}(v)| = d|v|, \quad d = \left(\Gamma - \sum_{i=1}^{N} \frac{p_i \tau_i}{\beta}\right)^{\frac{1}{2}},$$
$$\Gamma = \exp[-2\alpha(\lambda + \tau)] \exp\left[-\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)\lambda\right].$$

Since this proof is similar to the proof of Theorem 2.1, we omit it partly. Finally, it can be deduced that each solution $x(t) = x(t, \sigma, \varphi, \{x_{j0}\}_{j=1}^{n-1})$ of (1.2) and (1.5) through $(\sigma, \varphi, \{x_{j0}\}_{j=1}^{n-1})$ fulfils

$$\left(\|\varphi\|_{\sigma}^2 + \sum_{j=1}^{n-1} (x_{j0})^2\right)^{\frac{1}{2}} \leqslant \delta^*,$$

hence

$$\left(x^2(t) + \sum_{j=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} \leqslant \varepsilon \exp[-\alpha(t - t_0)], \quad t \geqslant t_0$$

where $\|\varphi\|_t = \sup\limits_{t-\tau \leqslant s \leqslant t} |\varphi(s)|$. □

**Theorem 2.3.** *(Case: either $\delta$ or $\varrho > 1$.) Assume that the conditions in Theorem 2.1 still hold. Moreover, suppose that*

($A_{11}$) *there exists a nonnegative constant $\theta$ such that $g_i(t+\theta) = g_i(t) + \theta$;*

($A_{12}$) *there exists a nonnegative constant $\mu$ such that $a(t+\mu) = a(t)$, $b(t+\mu) = b(t)$, $f(t+\mu, x) = f(t,x)$.*

*Then the zero solution of* (1.1) *can be exponentially stabilized by periodic impulses.*

Proof. Suppose that $\tau < T$, $T = \theta\mu$. Or else, we can choose $kT$ to replace $T$ such that $kT > \tau$, $k \in \mathbb{Z}_+$. Then let $t_{k+1} - t_k = T$, $t_0 = \sigma$. Similarly to the proof of Theorem 2.1, for $t \in [t_1, t_2)$ we choose

$$d_1 = \min\left\{\eta_1 \exp[\alpha T],\ \left(\Gamma_1 - \sum_{i=1}^{N} \frac{p_i \tau_i}{\beta}\right)^{\frac{1}{2}}\right\},$$

$$\Gamma_1 = \exp[-\alpha(T+\tau)] \exp\left[-\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\right)T\right].$$

Since $\eta_1$ depends only on system (1.1), $d_1$ also depends only on system (1.1). It then follows from Theorem 2.1 that the solution $x(t) = x(t, t_1, x(t_1), \{x_{j1}\}_{j=1}^{n-1})$ through $(t_1, x(t_1), \{x_{j1}\}_{j=1}^{n-1})$ fulfils

$$\left(x^2(t_1) + \sum_{j=1}^{n-1} (x^{(j)}(t_1))^2\right)^{\frac{1}{2}} \leqslant \eta_1,$$

hence

$$\left(x^2(t) + \sum_{j=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < 1, \quad t \in [t_1, t_2).$$

Next we prove that, for $t \in [t_k, t_{k+1})$, the solution $x(t) = x(t, t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ through $(t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ fulfils

$$\left(x^2(t_k) + \sum_{j=1}^{n-1} (x^{(j)}(t_k))^2\right)^{\frac{1}{2}} \leqslant \eta_1,$$

hence

$$\left(x^2(t) + \sum_{j=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < 1, \quad t \in [t_k, t_{k+1}).$$

Since $t_{k+1} - t_k = T$, $[t_k, t_{k+1}) = [t_1 + (k-1)T, t_1 + kT)$. We define $\Omega(t) = x(t + (k-1)T)$ where $x(t + (k-1)T) = x(t + (k-1)T, t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$. Hence, we note that the solution $x(t+(k-1)T)$ of (1.1) begins at $t=t_k$ and ends at $t=t_{k+1}$, which means that the function $\Omega(t)$ begins at $t=t_1$ and ends at $t=t_2$. On the other hand, in view of ($A_{11}$) and ($A_{12}$) we get

$$\begin{aligned}
&\Omega^{(n)}(t) = x^{(n)}(t+(k-1)T)\\
&\quad = -a(t+(k-1)T)x^{(\gamma)}(t+(k-1)T) - b(t+(k-1)T)x^{\varrho}(t+(k-1)T)\\
&\qquad - \sum_{i=1}^{N} f_i(t+(k-1)T, x^{\delta}(g_i(t+(k-1)T)))\\
&\quad = -a(t)x^{(\gamma)}(t+(k-1)T) - b(t)x^{\varrho}(t+(k-1)T) - \sum_{i=1}^{N} f_i(t, x^{\delta}(g_i(t)+(k-1)T))\\
&\quad = -a(t)\Omega^{(\gamma)}(t) - b(t)\Omega^{\varrho}(t) - \sum_{i=1}^{N} f_i(t, \Omega^{\delta}(g_i(t))).
\end{aligned}$$

Hence, $\Omega(t)$ is a solution of (1.1) which begins at $t=t_1$ and ends at $t=t_2$. As mentioned above, we obtain

$$\begin{aligned}
\left(\Omega^2(t_1) + \sum_{j=1}^{n-1}(\Omega^{(j)}(t_1))^2\right)^{\frac{1}{2}} &= \left(x^2(t_1+(k-1)T) + \sum_{j=1}^{n-1} x^{(j)}(t_1+(k-1)T)^2\right)^{\frac{1}{2}}\\
&= \left(x^2(t_k) + \sum_{j=1}^{n-1}(x^{(j)}(t_k))^2\right)^{\frac{1}{2}} \leqslant \eta_1.
\end{aligned}$$

Consequently, for $t\in[t_1,t_2)$,

$$\left(x^2(t+(k-1)T) + \sum_{j=1}^{n-1}(x^{(j)}(t+(k-1)T))^2\right)^{\frac{1}{2}} = \left(\Omega^2(t) + \sum_{j=1}^{n-1}(\Omega^{(j)}(t))^2\right)^{\frac{1}{2}} < 1, \quad t\in[t_1,t_2),$$

which implies that

$$\left(x^2(t) + \sum_{j=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} < 1, \quad t\in[t_k,t_{k+1}).$$

Then we only need to choose

$$d_k = d = \min\left\{\eta_1\exp[\alpha T], \left(\Gamma - \sum_{i=1}^{N}\frac{p_i\tau_i}{\beta}\right)^{\frac{1}{2}}\right\},$$

$$\Gamma = \exp[-\alpha(T+\tau)]\exp\left[-\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)T\right].$$

We see that $d_k$ depends only on system (1.1). The rest of the proof is similar to Theorem 2.1 and we omit it here. The proof is complete. □

R e m a r k 2.5. Let $f_i = a = 0$ in Theorem 2.3. The corresponding results have been investigated in [9]. If $n = 2$, $\gamma = 1$, $\varrho = 1$, $f_i(t,x) = p(t)x$, $g_i(t) = t - \tau$, the corresponding results have been given in [14].

Next we consider high-order nonlinear retarded differential equations (1.3) and the corresponding equations with impulses (1.4).

**Theorem 2.4.** *Assume that conditions* ($\mathrm{A}_3$)–($\mathrm{A}_9$) *hold. Moreover, suppose that* ($\mathrm{A}_{11}$)

$$\frac{\tau^2}{2\beta}\sum_{i=1}^{N} p_i < \exp\Big[-\Big(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\Big)\tau\Big].$$

*Then the zero solution of* (1.3) *can be exponentially stabilized by impulses.*

P r o o f. Since ($\mathrm{A}_{11}$) holds, there exist $\alpha > 0$ and $\lambda \geqslant \tau$ such that

$$\frac{1}{2\beta}\sum_{i=1}^{N} p_i\tau_i^2 \leqslant \tau\sum_{i=1}^{N} p_i \leqslant \exp[-\alpha(\lambda+\tau)]\exp\Big[-\Big(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\Big)\lambda\Big].$$

Similarly to Theorem 2.1, we choose a sequence $\{t_k\}_{k=1}^{\infty}$ satisfying ($\mathrm{A}_1$) and $\tau \leqslant t_{k+1} - t_k \leqslant \lambda$, $t_0 = \sigma$. Considering condition ($\mathrm{A}_8$), we choose a sequence $\{\eta_k\}_{k=1}^{\infty}$, $\eta_k \in (0,1)$ such that, when the solution $x(t) = x(t, t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ through $(t_k, x(t_k), \{x_{jk}\}_{j=1}^{n-1})$ fulfils

$$\Big(x^2(t_k) + \sum_{j=1}^{n-1}(x^{(j)}(t_k))^2\Big)^{\frac{1}{2}} \leqslant \eta_k,$$

then

$$\Big(x^2(t) + \sum_{j=1}^{n-1}(x^{(j)}(t))^2\Big)^{\frac{1}{2}} < 1, \quad t \in [t_k, t_{k+1}).$$

Let

$$|I_k(u)| = d_k|u|, \quad |J_{jk}(v)| = d_k|v|,$$

$$d_k = \min\Big\{\eta_k\exp[\alpha(t_1-\sigma)], \Big(\Gamma_k - \sum_{i=1}^{N}\frac{p_i}{2\beta}\tau_i^2\Big)^{\frac{1}{2}}\Big\},$$

$$\Gamma_k = \exp[-2\alpha(t_{k+1} - t_k + \tau)]\exp\Big[-\Big(2 + a + b + \sum_{i=1}^{N}\frac{p_i}{\beta}\tau_i\Big)(t_{k+1} - t_k)\Big],$$

which implies that $d_k \geqslant 0$.

For any $\varepsilon \in (0,1)$, let

$$\delta^* = \min\left\{\eta_0, \varepsilon, \varepsilon\left(1+\sum_{i=1}^{N}\frac{p_i}{2\beta}\tau_i^2\right)^{-1/4} \times \exp[-\alpha(t_1-\sigma)]\exp\left[-\frac{1}{2}\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)(t_1-\sigma)\right]\right\}.$$

Next, we prove that each solution $x(t) = x(t,\sigma,\varphi,\{x_{j0}\}_{j=1}^{n-1})$ of (1.4) and (1.5) with

$$\left(\|\varphi\|_\sigma^2 + \sum_{j=1}^{n-1}(x_{j0})^2\right)^{\frac{1}{2}} \leqslant \delta^*$$

satisfies

$$\left(x^2(t) + \sum_{j=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} \leqslant \varepsilon\exp[-\alpha(t-t_0)], \quad t \geqslant t_0,$$

where $\|\varphi\|_t = \sup\limits_{t-\tau\leqslant s\leqslant t}|\varphi(s)|$.

First, for $t \in [\sigma, t_1)$ we choose a Lyapunov function

$$V(t) = x^2(t) + \sum_{i=1}^{n-1}(x^{(j)}(t))^2 + \sum_{i=1}^{N}\frac{p_i}{\beta}\int_{g_i(t)}^{t}\int_{u}^{t}x^{2\delta}(s)\,\mathrm{d}s\,\mathrm{d}u.$$

Then $V(t)$ satisfies:

(1) $V(t) \geqslant x^2(t) + \sum\limits_{i=1}^{n-1}(x^{(j)}(t))^2$;

(2)

$$\begin{aligned}
V(t) &\leqslant x^2(t) + \sum_{i=1}^{n-1}(x^{(j)}(t))^2 + \sum_{i=1}^{N}\frac{p_i}{\beta}\sup_{g_i(t)\leqslant s\leqslant t}|x(s)|^{2\delta}\int_{g_i(t)}^{t}\int_{u}^{t}\mathrm{d}s\,\mathrm{d}u\\
&\leqslant x^2(t) + \sum_{i=1}^{n-1}(x^{(j)}(t))^2 + \sum_{i=1}^{N}\frac{p_i}{\beta}\sup_{g_i(t)\leqslant s\leqslant t}|x(s)|^{2\delta}\frac{(t-g_i(t))^2}{2}\\
&\leqslant x^2(t) + \sum_{i=1}^{n-1}(x^{(j)}(t))^2 + \sum_{i=1}^{N}\frac{p_i}{\beta}\sup_{t-\tau_i\leqslant s\leqslant t}|x(s)|^{2\delta}\frac{\tau_i^2}{2}\\
&\leqslant x^2(t) + \sum_{i=1}^{n-1}(x^{(j)}(t))^2 + \sum_{i=1}^{N}\frac{p_i}{2\beta}\tau_i^2\|x\|_t^2\\
&\leqslant \left(1+\sum_{i=1}^{N}\frac{p_i}{2\beta}\tau_i^2\right)\left(\|x\|_t^2 + \sum_{i=1}^{n-1}(x^{(j)}(t))^2\right),
\end{aligned}$$

where $\|x\|_t = \sup\limits_{t-\tau\leqslant s\leqslant t}|x(s)|$.

(3) We denote by $V'(t)$ the right upper derivative of $V(t)$ along the solution of (1.4) and (1.5). Then

$$
\begin{aligned}
V'(t) &= 2x(t)x'(t) + 2x'(t)x''(t) + \ldots + 2x^{(n-1)}(t)x^{(n)}(t) \\
&\qquad + \sum_{i=1}^{N} \frac{p_i}{\beta} x^{2\delta}(t)(t - g_i(t)) - \sum_{i=1}^{N} \frac{p_i}{\beta} g_i'(t) \int_{g_i(t)}^{t} x^{2\delta}(s)\,\mathrm{d}s \\
&\leqslant 2x(t)x'(t) + 2x'(t)x''(t) + \ldots + 2x^{(n-1)}(t)\Big\{ - a(t)x^{(\gamma)}(t) - b(t)x^{\varrho}(t) \\
&\quad - \sum_{i=1}^{N} \int_{g_i(t)}^{t} f_i(t-u, x^{\delta}(u))\,\mathrm{d}u \Big\} + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} g_i'(t) \int_{g_i(t)}^{t} x^{2\delta}(s)\,\mathrm{d}s \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 + a([x^{(n-1)}(t)]^2 \\
&\quad + [x^{(\gamma)}(t)]^2) + b([x^{(n-1)}(t)]^2 + x^{2\varrho}(t)) \\
&\quad + 2|x^{(n-1)}(t)| \sum_{i=1}^{N} \int_{g_i(t)}^{t} |p_i(t-u)||x^{2\delta}(u)|\,\mathrm{d}u \\
&\quad + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} g_i'(t) \int_{g_i(t)}^{t} x^{2\delta}(s)\,\mathrm{d}s \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 + a([x^{(n-1)}(t)]^2 \\
&\quad + [x^{(\gamma)}(t)]^2) + b([x^{(n-1)}(t)]^2 + x^2(t)) + \sum_{i=1}^{N} p_i \int_{g_i(t)}^{t} ([x^{(n-1)}(t)]^2 + x^{2\delta}(u))\,\mathrm{d}u \\
&\quad + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i x^{2\delta}(t) - \sum_{i=1}^{N} \frac{p_i}{\beta} g_i'(t) \int_{g_i(t)}^{t} x^{2\delta}(s)\,\mathrm{d}s \\
&\leqslant x^2(t) + [x'(t)]^2 + \ldots + [x^{(n-2)}(t)]^2 + [x^{(n-1)}(t)]^2 + a([x^{(n-1)}(t)]^2 \\
&\quad + [x^{(\gamma)}(t)]^2) + b([x^{(n-1)}(t)]^2 + x^2(t)) + \sum_{i=1}^{N} p_i \tau_i [x^{(n-1)}(t)]^2 \\
&\quad + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i x^2(t) + \sum_{i=1}^{N} p_i \int_{g_i(t)}^{t} x^{2\delta}(u)\,\mathrm{d}u \Big(1 - \frac{g_i'(t)}{\beta}\Big) \\
&\leqslant \Big(1 + b + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i\Big) x^2(t) + 2[x'(t)]^2 + \ldots + 2[x^{(\gamma-1)}(t)]^2 + (a+2)[x^{(\gamma)}(t)]^2 \\
&\quad + 2[x^{(\gamma+1)}(t)]^2 + \ldots + \Big(2 + a + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i\Big) [x^{(n-1)}(t)]^2 \\
&\leqslant \Big(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta} \tau_i\Big) V(t),
\end{aligned}
$$

which implies that

$$V(t) \leqslant V(t_0) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t-\sigma),\ \ t \in [\sigma, t_1).$$

Thus for $t \in [\sigma, t_1)$ we get

$$\begin{aligned}
x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2 &\leqslant V(t) \leqslant V(\sigma) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t-\sigma) \\
&< V(\sigma) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_1-\sigma) \\
&\leqslant \left(1 + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2\right)\left(\|x\|_\sigma^2 + \sum_{i=1}^{n-1}(x^{(j)}(\sigma))^2\right) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_1-\sigma) \\
&= \left(1 + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2\right)\left(\|\varphi\|_\sigma^2 + \sum_{j=1}^{n-1} x_{j0}^2\right) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_1-\sigma) \\
&\leqslant \left(1 + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2\right)\delta^{*2} \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_1-\sigma) \\
&= \varepsilon^2 \exp[-2\alpha(t_1-\sigma)] < \varepsilon^2 \exp[-2\alpha(t-\sigma)],
\end{aligned}$$

which implies that

$$\left(x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < \varepsilon \exp[-\alpha(t-\sigma)],\ \ t \in [\sigma, t_1).$$

Especially,

$$\left(x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right)^{\frac{1}{2}} < \varepsilon \exp[-\alpha(t_1-\sigma)].$$

It then follows that

$$\begin{aligned}
\left(x^2(t_1) + \sum_{i=1}^{n-1} (x^{(j)}(t_1))^2\right)^{\frac{1}{2}} &= d_1 \left(x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right)^{\frac{1}{2}} \\
&< d_1 \varepsilon \exp[-\alpha(t_1-\sigma)] < d_1 \exp[-\alpha(t_1-\sigma)] = \eta_1.
\end{aligned}$$

Thus we obtain that for $t \in [t_1, t_2)$,

$$\left(x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right)^{\frac{1}{2}} < 1.$$

For $t \in [t_1, t_2)$, we choose a Lyapunov function

$$V(t) = x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2 + \sum_{i=1}^{N} \frac{p_i}{\beta} \int_{g_i(t)}^{t} \int_{u}^{t} x^{2\delta}(s)\,\mathrm{d}s\,\mathrm{d}u.$$

Then we have

$$\begin{aligned}
x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2 &\leqslant V(t) \leqslant V(t_1) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t - t_1) \\
&< V(t_1) \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&= \left[x^2(t_1) + \sum_{i=1}^{n-1} (x^{(j)}(t_1))^2 + \sum_{i=1}^{N} \frac{p_i}{\beta} \int_{g_i(t_1)}^{t_1} \int_{u}^{t_1} x^{2\delta}(s)\,\mathrm{d}s\,\mathrm{d}u\right] \\
&\quad \times \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \left\{d_1^2\left[x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right] + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2 \sup_{t_1 - \tau_i \leqslant t \leqslant t_1} x^{2\delta}(t)\right\} \\
&\quad \times \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \left\{d_1^2\left[x^2(t_1^-) + \sum_{i=1}^{n-1} (x^{(j)}(t_1^-))^2\right] + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2 \sup_{t_1 - \tau \leqslant t \leqslant t_1} x^{2}(t)\right\} \\
&\quad \times \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \left\{d_1^2 \sup_{t_1 - \tau \leqslant t \leqslant t_1} \left[x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right] + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2 \sup_{t_1 - \tau \leqslant t \leqslant t_1} x^{2}(t)\right\} \\
&\quad \times \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \left(d_1^2 + \sum_{i=1}^{N} \frac{p_i}{2\beta}\tau_i^2\right) \sup_{t_1 - \tau \leqslant t \leqslant t_1} \left[x^2(t) + \sum_{i=1}^{n-1} (x^{(j)}(t))^2\right] \\
&\quad \times \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \Gamma_1 \varepsilon^2 \exp[-2\alpha(t_1 - \sigma - \tau)] \exp\left(2 + a + b + \sum_{i=1}^{N} \frac{p_i}{\beta}\tau_i\right)(t_2 - t_1) \\
&\leqslant \varepsilon^2 \exp[-2\alpha(t_2 - \sigma)] < \varepsilon^2 \exp[-2\alpha(t - \sigma)], \quad t \in [t_1, t_2).
\end{aligned}$$

Hence,

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} < \varepsilon\exp[-\alpha(t-\sigma)], \quad t\in[t_1,t_2).$$

By the induction hypothesis, we may prove that, for $k\geqslant 1$,

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} < \varepsilon\exp[-\alpha(t-\sigma)],\ t\in[t_k,t_{k+1}).$$

Therefore, we arrive at

$$\left(x^2(t)+\sum_{i=1}^{n-1}(x^{(j)}(t))^2\right)^{\frac{1}{2}} < \varepsilon\exp[-\alpha(t-\sigma)],\ t\geqslant\sigma,$$

and the proof is complete. □

R e m a r k 2.6. Remarks 2.1–2.3 are still valid for Theorem 2.4.

## 3. An example

E x a m p l e. Consider the equation

$$\text{(3.1)}\qquad \begin{cases} x^{(6)}(t)+x^{(3)}(t)+0.9x(t)-0.5x(t-0.01)-0.5x(t-0.02) \\ \qquad +a(t)x^3(t-0.015)=0,\ t\geqslant 0, \\ x(t)=\varphi(t),\ -0.02\leqslant t\leqslant 0,\ x^{(j)}(0)=x_{j0},\ j=1,\dots,5, \end{cases}$$

where $a(t)\in\Gamma$, $\Gamma=\{s(t)\in[0,\infty)\colon\ |s(t)|\leqslant 0.5\}$.

When $a(t)=0\in\Gamma$, then its characteristic equation is

$$\lambda^6+\lambda^3-0.5\mathrm{e}^{-0.01\lambda}-0.5\mathrm{e}^{-0.02\lambda}+0.9=0.$$

Using the software Mathematica, we obtain a characteristic root with positive real part. Hence the non-impulsive equation (4.1) is unstable for some $a(t)\in\Gamma$. But if we take $\beta=1$, $\tau=\lambda=0.02$, $\alpha=\frac{1}{2}$, $p_i=0.5$, $i=1,2,3$, $N=3$, then it is easy to check that

$$\begin{aligned}\frac{\tau}{\beta}\sum_{i=1}^{N}p_i=0.03<\mathrm{e}^{-0.02}\mathrm{e}^{-0.108} &= \exp[-\alpha(\lambda+\tau)]\exp\left[-\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)\lambda\right] \\ &< \exp\left[-\left(2+a+b+\sum_{i=1}^{N}\frac{p_i}{\beta}\right)\tau\right].\end{aligned}$$

We choose a sequence $\{t_k\}_{k=1}^{\infty}$ satisfying $(\mathrm{A}_1)$ and $t_{k+1} - t_k = 0.02$, $t_0 = 0$, $k \in \mathbb{Z}_+$, such that

$$x(t_k) = dx(t_k^-),$$
$$x^{(j)}(t_k) = dx^{(j)}(t_k^-), \quad j = 1, 2, \ldots, 5,$$

where $d = \min\{\eta_1 \exp[0.01], \sqrt{\mathrm{e}^{-0.128} - 0.0225}\}$ and $\eta_1$ depends only on the first equation of (4.1) on $[0.02, 0.04]$. Then the hypotheses in Theorem 2.3 are satisfied and hence the unstable equation (4.2) can be exponentially stabilized by periodic impulses for all $a(t) \in \Gamma$.

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