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# OPTIMAL DESIGN OF AN ELASTIC BEAM WITH A UNILATERAL ELASTIC FOUNDATION: SEMICOERCIVE STATE PROBLEM

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*Abstract.* A design optimization problem for an elastic beam with a unilateral elastic foundation is analyzed. Euler-Bernoulli's model for the beam and Winkler's model for the foundation are considered. The state problem is represented by a nonlinear semicoercive problem of 4th order with mixed boundary conditions. The thickness of the beam and the stiffness of the foundation are optimized with respect to a cost functional. We establish solvability conditions for the state problem and study the existence of a solution to the optimization problem.

*Keywords*: shape optimization, semicoercive beam problem, unilateral foundation

*MSC 2010*: 49K15, 49J15, 65K10, 74B99, 74K10, 74P05

## 1. Introduction

Shape design optimization has been the subject of an extensive research and has many engineering applications. The present paper deals with an optimization of an elastic beam resting on a unilateral elastic foundation.

We do not consider the beam and its foundation to be individual deformable bodies as is usual in classical contact problems. The foundation is included into the model through a suitable response function $s$. Nowadays, the linear Winkler's model of foundation with the response function $s = qu$ is well known, see e.g. [7], [11], [2]. Unfortunately, in some cases the linear model is not suitable, especially, when the foundation is not firmly connected to the beam. Then the nonlinear (unilateral) model with the response function $s = qu^+$ is more realistic and the state prob-

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lem leads to a nonlinear differential equation. This kind of foundation is from the theoretical and practical point of view examined e.g. in [7], [13].

The beam thickness and the stiffness of the foundation play the role of design variables. The optimization problem is then formulated as a minimization of a cost functional on a set of admissible design variables. Many results have been achieved in this field. First of all let us mention [4] and [5]. Optimization of beams with a linear foundation is studied e.g. in [8] or [2]. Design optimization of a beam with unilateral supports is presented in [6]. A related problem, namely optimization of an axisymmetric plate on an elastic foundation, is treated in [12]. None of these papers concerns the beam optimization with semicoercive state problems.

The paper is organized as follows. First, we establish the solvability of the state problem. Due to prescribed boundary conditions, rigid displacements of the beam are allowed and the state problem is only semicoercive. Standard Friedrich's or Poincaré inequality cannot be used in the existence analysis as is usual for coercive problems. In this paper we decompose the space of kinematically admissible displacements and use the modified Poincaré inequality on appropriate subspaces. To obtain the coercivity and to prove the existence and uniqueness of a solution to the state problem, additional assumptions on the beam load must be introduced.

Further we analyze the existence of at least one solution of the beam optimization problem. The basic step in the analysis is the proof of the continuous dependence of the state on the design variable. Continuity of this mapping is sufficient for the existence of an optimal design. Nevertheless, our state problem is very close to problems governed by variational inequalities. Therefore, it is reasonable to assume that the optimization problem is nonsmooth, as is known from optimization with inequality constraints (see [4], [5]). The Lipschitz continuity of the control on the state mapping and the cost functional is established.

### 1.1. Notation

In this paper we will use the Lebesgue spaces $L^p(\Omega)$, $p \in [1,\infty]$, the Hilbert case of Sobolev spaces $H^k(\Omega)$, $k \in \mathbb{N}$, and the spaces $C^k(\overline{\Omega})$, $k \in \mathbb{N}$, of continuously differentiable functions up to order $k$ which can be continuously extended to $\overline{\Omega}$, where $\Omega$ is a nonempty open interval in $\mathbb{R}^1$. Their standard norms will be denoted by $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{k,2,\Omega}$ and $\|\cdot\|_{C^k(\overline{\Omega})}$, respectively. The $i$th seminorm in $H^k(\Omega)$ is denoted as $|\cdot|_{i,2,\Omega}$. For the standard scalar product in $L^2(\Omega)$ we will use notation $(\cdot,\cdot)_{2,\Omega}$. The standard scalar product in $H^k(\Omega)$, $k \in \mathbb{N}$ will be denoted by $(\cdot,\cdot)_{k,2,\Omega}$. Further, $C^{0,1}(\overline{\Omega})$ stands for the space of Lipschitz continuous functions in $\overline{\Omega}$. The space of polynomials of $k$th degree will be denoted by $P_k$.

## 2. Setting of the problem

Let us consider an elastic beam of length $l$ and of a rectangular cross section, represented by the interval $\Omega := (0, l)$. Along its entire length the beam is supported by a unilaterally elastic foundation. We suppose that the left end of the beam is allowed to move in the vertical direction but it cannot slope there while the right end is free. The beam is subject to a vertical load $f$. The classical theory (see e.g. [10]) yields the following boundary value problem for the deflection $u$:

Find $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ such that

$$\begin{cases} (\beta t^3(x)u''(x))'' + q(x)u^+(x) = f(x) \quad \forall\, x \in \Omega, \\ u'(0) = u'''(0) = u''(l) = u'''(l) = 0, \end{cases} \tag{2.1}$$

where $t$ and $q$ are functions representing the thickness of the beam and the stiffness coefficient of the foundation, respectively. The constant $\beta = \frac{2}{3}bE$ depends on the beam width $b$ and Young's modulus of elasticity $E$. By $u^+ := (u + |u|)/2$ we denote the positive part of $u$.

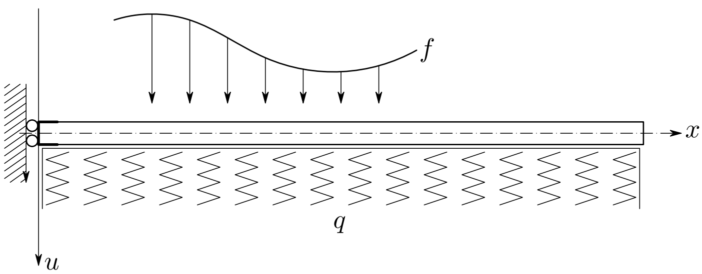


Figure 1. The beam with axes orientation

### 2.1. Variational formulation of the state problem

In practice, we often cannot guarantee that the parameters $\beta$, $f$, $t$, and $q$ are sufficiently smooth as is needed in the classical formulation (2.1). In what follows we will introduce the variational formulation of the problem. Let us suppose that $t \in L^\infty(\Omega)$, $q \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\beta \in L^\infty(\Omega)$ and let there exist a constant $\beta_0$ such that $0 < \beta_0 \leqslant \beta$ a.e. in $\Omega$. For purposes of the forthcoming analysis we will denote the pair $\{t, q\}$ by $e$. The space of kinematically admissible displacements is defined by

$$V = \{v \in H^2(\Omega) \colon\; v'(0) = 0\}. \tag{2.2}$$

$V$ is a closed subspace of $H^2(\Omega)$. The variational formulation of the state problem reads as follows:

$$\text{Find } u \in V \colon\; \mathcal{E}_e(u) \leqslant \mathcal{E}_e(v) \quad \forall\, v \in V, \tag{$\mathcal{P}(e)$}$$

where

$$\mathcal{E}_e(v) = \frac{1}{2}(a_t(v,v) + b_q(v^+,v^+)) - F(v) \tag{2.3}$$

is the functional of total potential energy of the system and

$$a_t(u,v) := \int_\Omega \beta t^3 u'' v'' \,\mathrm{d}x, \quad b_q(u,v) := \int_\Omega quv \,\mathrm{d}x, \quad u,v \in H^2(\Omega).$$

The bilinear forms $a_t, b_q$ correspond to the inner energy and the work of the foundation, respectively. The transversal beam load is represented by the continuous linear functional

$$F(v) := \int_\Omega fv \,\mathrm{d}x, \quad v \in H^2(\Omega).$$

### 2.2. Admissible design variables

The thickness $t$ and the stiffness coefficient $q$ will be the object of optimization. The set of admissible design variables is specified by the Cartesian product $U_{\mathrm{ad}} = U_{\mathrm{ad}}^t \times U_{\mathrm{ad}}^q$, where

$$U_{\mathrm{ad}}^t = \Big\{ t \in C^{0,1}(\overline{\Omega})\colon\ 0 < t_0 \leqslant t(x) \leqslant t_1,\ |t'(x)| \leqslant \gamma_2 \text{ in } \Omega,\ \int_\Omega t(x)\,\mathrm{d}x = \gamma_1 \Big\},$$
$$U_{\mathrm{ad}}^q = \{ q \in L^2(\Omega)\colon\ 0 < q_0 \leqslant q(x) \leqslant q_1 \text{ a.e. in } \Omega \}.$$

Positive constants $t_0$, $t_1$, $\gamma_1$, $\gamma_2$, $q_0$, and $q_1$ are chosen in such a way that $U_{\mathrm{ad}} \neq \emptyset$. It is also possible to keep one of the design variables fixed. This leads only to thickness or foundation stiffness optimization. The constraints appearing in the definition of $U_{\mathrm{ad}}$ are reasonable from the physical point of view and they play an important role in the mathematical analysis of the problem as well. The constraint $|t'(x)| \leqslant \gamma_2$ in $\Omega$ prevents thickness oscillations and $\int_\Omega t(x)\,\mathrm{d}x = \gamma_1$ keeps the beam volume fixed.

### 2.3. Cost functional and optimization problem

Finally, let us define a cost functional $I\colon U_{\mathrm{ad}} \times V \to \mathbb{R}^1$ and denote $J(e) := I(e, u(e))$, with $u(e)$ being a solution of the state problem $(\mathcal{P}(e))$, $e \in U_{\mathrm{ad}}$. The design optimization problem can be stated as follows:

$$\text{Find } e^* \in U_{\mathrm{ad}}\colon\ J(e^*) \leqslant J(e) \quad \forall e \in U_{\mathrm{ad}}. \tag{P}$$

In the next two sections, the existence and uniqueness of a solution to $(\mathcal{P}(e))$ and the existence of at least one solution of (P) will be analyzed.

## 3. State problem—existence and uniqueness results

In this section we shall analyze the existence and uniqueness of a solution to the state problem $(\mathcal{P}(e))$. Through the section we will consider $e \in U_{\rm ad}$ to be arbitrary but fixed, $\beta \in L^\infty(\Omega)$, $0 < \beta_0 \leqslant \beta$ a.e. in $\Omega$.

### 3.1. Some preliminaries

We start with auxiliary results that are important in what follows. First we introduce some properties of the positive part $u^+$ of a function $u \in H^2(\Omega)$. Recall that $\Omega = (0, l)$.

**Lemma 3.1.** *If $u \in H^2(\Omega)$, then the positive part*

$$u^+(x) = (u(x) + |u(x)|)/2, \quad x \in \Omega \tag{3.1}$$

*belongs to the space $H^1(\Omega)$ and $\|u^+\|_{1,2,\Omega} \leqslant \|u\|_{1,2,\Omega}$. Moreover, the following inequality holds:*

$$|u^+(x) - v^+(x)| \leqslant |u(x) - v(x)| \quad \forall\, u, v \in C(\overline{\Omega}),\ x \in \overline{\Omega}.$$

P r o o f. See [13]. $\square$

R e m a r k 3.1. If $u \in H^2(\Omega)$, then $\|u^+\|_{1,2,\Omega} \leqslant \|u\|_{2,2,\Omega}$.

**Lemma 3.2.** *Let $u_n, u \in H^2(\Omega)$ be such that $u_n \rightharpoonup u$ in $H^2(\Omega)$. Then $u_n \to u$ in $L^2(\Omega)$ and $u_n^+ \to u^+$ in $L^2(\Omega)$.*

P r o o f. The former part of the assertion is a consequence of the compactness of the embedding of $H^2(\Omega)$ into $L^2(\Omega)$, see e.g. [9]. The latter part follows from Lemma 3.1. $\square$

Next we introduce a lemma that will play a crucial role in the existence analysis for $(\mathcal{P}(e))$. In fact it is a modification of the well-known Poincaré inequality.

**Lemma 3.3** (Poincaré type inequality)**.** *Let $V$ be defined by (2.2). Then there exists a positive constant $c_P$ depending only on the interval $\Omega$ such that*

$$\|v\|^2_{2,2,\Omega} \leqslant c_P(|v|^2_{2,2,\Omega} + (v,1)^2_{2,\Omega}) \quad \forall\, v \in V. \tag{3.2}$$

P r o o f. Suppose that (3.2) does not hold. Then one can find a sequence $\{v_n\} \subset V$ such that

$$\frac{1}{n}\|v_n\|^2_{2,2,\Omega} > |v_n|^2_{2,2,\Omega} + (v_n,1)^2_{2,\Omega} \geqslant 0 \quad \forall\, n \geqslant 1. \tag{3.3}$$

First, we divide the inequality (3.3) by $\|v_n\|^2_{2,2,\Omega}$ and pass to the limit with $n \to \infty$. We have

$$\lim_{n\to\infty} |w_n|^2_{2,2,\Omega} = 0 \quad \text{and} \quad \lim_{n\to\infty} (w_n, 1)^2_{2,\Omega} = 0, \tag{3.4}$$

where $w_n := v_n/\|v_n\|_{2,2,\Omega}$. Clearly $\|w_n\|_{2,2,\Omega} = 1$ and $\{w_n\}$ is bounded in $H^2(\Omega)$. Hence one can find a subsequence of $\{w_n\}$ (denoted as the original sequence) and an element $w \in V$ such that $w_n \rightharpoonup w$ in $V$. Due to the Rellich theorem one has $w_n \to w$ in $H^1(\Omega)$. In view of (3.4) it holds that

$$\|w_n - w_m\|_{2,2,\Omega} \leqslant |w_n|_{2,2,\Omega} + |w_m|_{2,2,\Omega} + \|w_n - w_m\|_{1,2,\Omega} \to 0, \ m, n \to \infty.$$

Therefore $w_n \to w$ in $V$ and we have

$$0 = \lim_{n\to\infty} |w_n|^2_{2,2,\Omega} = |w|^2_{2,2,\Omega}$$

so that $w \equiv p \in P_0$. From

$$0 = \lim_{n\to\infty} (w_n, 1)^2_{2,\Omega} = (w, 1)^2_{2,\Omega}$$

it follows that $p = 0$. But this is a contradiction with $\|w_n\|_{2,2,\Omega} = 1$ and the fact that $w_n \to p$ in $V$. $\square$

### 3.2. Existence of a solution to $(\mathcal{P}(e))$

At the beginning of this section we recall the boundedness property of the bilinear forms $a_t, b_q$.

**Lemma 3.4.** *There exist positive constants $c_1$, $c_2$ such that*

$$|a_t(u, v)| \leqslant c_1\|u\|_{2,2,\Omega}\|v\|_{2,2,\Omega} \quad \forall\, u, v \in H^2(\Omega), \ \forall\, e \in U_{\mathrm{ad}},$$
$$|b_q(u^+, v)| \leqslant c_2\|u\|_{2,2,\Omega}\|v\|_{2,2,\Omega} \quad \forall\, u, v \in H^2(\Omega), \ \forall\, e \in U_{\mathrm{ad}}.$$

Further, we present properties of the functional $\mathcal{E}_e$ needed in what follows. Its Gâteaux differentiability, convexity and coercivity on the reflexive Banach space $V$ will be sufficient conditions for the existence of a solution to $(\mathcal{P}(e))$. It can be shown in a standard way that the functional $\mathcal{E}_e$ is Gâteaux differentiable on $H^2(\Omega)$. The Gâteaux derivative at any point $u \in H^2(\Omega)$ and in any direction $v \in H^2(\Omega)$ has the form

$$\mathcal{E}'_e(u; v) = a_t(u, v) + b_q(u^+, v) - F(v) \quad \forall\, u, v \in H^2(\Omega), \ \forall\, e \in U_{\mathrm{ad}}. \tag{3.5}$$

**Lemma 3.5.** *Let $e \in U_{\mathrm{ad}}$. Then the functional $\mathcal{E}_e$ is convex on $H^2(\Omega)$.*

P r o o f. Using the fact that $e \in U_{\text{ad}}$, $0 < \beta_0 \leqslant \beta$ a.e. in $\Omega$ and the inequality

$$(s^+ - t^+)(s - t) \geqslant (s^+ - t^+)^2 \quad \forall\, s, t \in \mathbb{R}^1$$

we have

$$\begin{aligned} \mathcal{E}'_e(u; u - v) - \mathcal{E}'_e(v; u - v) &= a_t(u - v, u - v) + b_q(u^+ - v^+, u - v) \\ &\geqslant a_t(u - v, u - v) + b_q(u^+ - v^+, u^+ - v^+) \\ &\geqslant \beta_0 t_0^3 |u - v|^2_{2,2,\Omega} + q_0 \|u^+ - v^+\|^2_{2,\Omega} \\ &\geqslant 0 \qquad\qquad \forall\, u, v \in H^2(\Omega),\ \forall\, e \in U_{\text{ad}}. \end{aligned}$$

This is a sufficient condition of convexity of the Gâteaux differentiable functional $\mathcal{E}_e$, see e.g. [3]. $\square$

According to the previous results we can introduce the equivalent weak formulation of the state problem $(\mathcal{P}(e))$:

$(\mathcal{P}'(e))$ Find $u \in V\colon\ a_t(u, v) + b_q(u^+, v) = F(v) \quad \forall\, v \in V.$

In our case the functional $\mathcal{E}_e$ is only semicoercive on $V$, i.e. there exists a constant $c > 0$ such that

$$a_t(v, v) + b_q(v^+, v) \geqslant c|v|^2_{2,2,\Omega} \quad \forall\, v \in V,\ \forall\, e \in U_{\text{ad}}.$$

To get the existence and uniqueness of a solution to $(\mathcal{P}(e))$, additional assumptions on the beam load $f$ eliminating the rigid motions have to be introduced. First of all let us define the convex closed cone of kinematically admissible rigid displacements

(3.6) $$\mathcal{R}_V = \{v \in V\colon\ a_t(v, v) + b_q(v^+, v) = 0\} = \{p \in P_0\colon\ p \leqslant 0\}.$$

Therefore, using the definition of the standard scalar product in $H^2(\Omega)$, the negative polar cone $\mathcal{R}_V^{\ominus}$ is characterized by

(3.7) $$\mathcal{R}_V^{\ominus} = \{v \in V\colon\ (v, p)_{2,2,\Omega} \leqslant 0\ \forall\, p \in \mathcal{R}_V\} = \{v \in V\colon\ (v, 1)_{2,\Omega} \geqslant 0\}.$$

**Theorem 3.1** (Necessary condition for the existence of a solution to $(\mathcal{P}(e))$). *Let there exist a solution of the state problem $(\mathcal{P}(e))$, $e \in U_{\text{ad}}$. Then the condition*

(S1) $$F(1) \geqslant 0$$

*has to be satisfied.*

P r o o f. Let $u \in V$ be a solution of $(\mathcal{P}(e))$. Inserting $v := p \in \mathcal{R}_V$ into $(\mathcal{P}'(e))$ we get

$$0 \geqslant b_q(u^+, p) = a_t(u, p) + b_q(u^+, p) = F(p) = pF(1) \quad \forall\, p \in \mathcal{R}_V.$$

$\square$

**Lemma 3.6.** *Let $\mathcal{R}_V$, $\mathcal{R}_V^{\ominus}$ be defined by (3.6) and (3.7), respectively. Then the space $V$ can be uniquely decomposed into the orthogonal sum $\mathcal{R}_V \oplus \mathcal{R}_V^{\ominus}$: $\forall\, v \in V\ \exists\, \{p, \overline{v}\} \in \mathcal{R}_V \times \mathcal{R}_V^{\ominus}$ such that*

$$v = p \oplus \overline{v}, \quad (p, \overline{v})_{2,2,\Omega} = p(1, \overline{v})_{2,\Omega} = 0. \tag{3.8}$$

P r o o f. For the proof we refer to [1]. $\square$

Taking into account the orthogonality conditions in (3.8) and the definitions of $\mathcal{R}_V$, $\mathcal{R}_V^{\ominus}$, we can deduce that only one of the following cases can occur:

$$p = 0 \quad \text{and} \quad (\overline{v}, 1)_{2,\Omega} \geqslant 0, \tag{A1}$$

$$p \leqslant 0 \quad \text{and} \quad (\overline{v}, 1)_{2,\Omega} = 0. \tag{A2}$$

**Lemma 3.7.** *Let $e \in U_{\text{ad}}$ and*

$$F(1) > 0. \tag{S2}$$

*Then the functional $\mathcal{E}_e$ is coercive on $V$.*

P r o o f. Let (S2) be satisfied. According to the decomposition (3.8), the functional $\mathcal{E}_e$ can be written as

$$\begin{aligned} 2\mathcal{E}_e(v) = 2\mathcal{E}_e(p + \overline{v}) &= a_t(\overline{v}, \overline{v}) + b_q(v^+, v^+) - 2F(p) - 2F(\overline{v}) \\ &\geqslant \beta_0 t_0^3 |\overline{v}|^2_{2,2,\Omega} + q_0 \|(p + \overline{v})^+\|^2_{2,\Omega} + 2|p|F(1) - 2F(\overline{v}). \end{aligned}$$

First, if (A1) holds then $p = 0$. Consequently $v \equiv \overline{v}$ and $(\overline{v}, 1)_{2,\Omega} \geqslant 0$. This implies that

$$0 \leqslant (\overline{v}, 1)^2_{2,\Omega} \leqslant (\overline{v}^+, 1)^2_{2,\Omega} \leqslant l \|\overline{v}^+\|^2_{2,\Omega}. \tag{3.9}$$

From (3.9) and (3.2) we obtain

$$\begin{aligned} 2\mathcal{E}_e(v) = 2\mathcal{E}_e(\overline{v}) &= a_t(\overline{v}, \overline{v}) + b_q(\overline{v}^+, \overline{v}^+) - 2F(\overline{v}) \\ &\geqslant \beta_0 t_0^3 |\overline{v}|^2_{2,2,\Omega} + q_0 \|\overline{v}^+\|^2_{2,\Omega} - 2F(\overline{v}) \\ &\geqslant \beta_0 t_0^3 |\overline{v}|^2_{2,2,\Omega} + \frac{q_0}{l} (\overline{v}, 1)^2_{2,\Omega} - 2F(\overline{v}) \\ &\geqslant \|\overline{v}\|_{2,2,\Omega} (c_1 \|\overline{v}\|_{2,2,\Omega} - 2\|f\|_{2,\Omega}), \end{aligned} \tag{3.10}$$

where $c_1 := (1/c_P) \min\{\beta_0 t_0^3, q_0/l\}$.

On the other hand, if (A2) holds, then $(\overline{v},1)_{2,\Omega}=0$ and $p\leqslant 0$. This and (3.2) yield

$$(3.11)\qquad \begin{aligned} 2\mathcal{E}_e(v) &= 2\mathcal{E}_e(p+\overline{v}) = a_t(\overline{v},\overline{v}) + b_q(v^+,v^+) - 2F(p) - 2F(\overline{v})\\ &\geqslant \beta_0 t_0^3|\overline{v}|^2_{2,2,\Omega} + q_0\|(p+\overline{v})^+\|^2_{2,\Omega} + 2|p|F(1) - 2F(\overline{v})\\ &\geqslant \beta_0 t_0^3|\overline{v}|^2_{2,2,\Omega} + 2|p|F(1) - 2F(\overline{v})\\ &= \beta_0 t_0^3|\overline{v}|^2_{2,2,\Omega} + (\overline{v},1)^2_{2,\Omega} + 2|p|F(1) - 2F(\overline{v})\\ &\geqslant c_2\|\overline{v}\|^2_{2,2,\Omega} + 2|p|F(1) - 2\|f\|_{2,\Omega}\|\overline{v}\|_{2,2,\Omega}, \end{aligned}$$

where $c_2 := (1/c_P)\min\{\beta_0 t_0^3, 1\}$. Due to the orthogonality of the decomposition (3.8), the relation $\|v\|^2_{2,2,\Omega} = \|\overline{v}\|^2_{2,2,\Omega} + \|p\|^2_{2,2,\Omega}$ is satisfied. Therefore, $\|v\|_{2,2,\Omega}\to\infty$ implies that either $\|\overline{v}\|^2_{2,2,\Omega}$ or $|p|$ converges to $\infty$. Using the assumption (S2) and (3.10), (3.11) we arrive at the assertion of the lemma. □

Now we can establish the main results of this section. The coercivity of $\mathcal{E}_e$ enables us to prove the following theorem.

**Theorem 3.2** (Necessary and sufficient condition for the existence and uniqueness of a solution to $(\mathcal{P}(e))$)**.** *The state problem $(\mathcal{P}(e))$ has a unique solution for any $e\in U_{\rm ad}$ if and only if the condition* (S2) *is satisfied. In addition,*

$$(\text{S3})\qquad \mu(M_u) > 0$$

*holds, where $M_u=\{x\in\Omega\colon u(x)>0\}$ and $\mu$ is the one-dimensional Lebesgue measure.*

P r o o f. *Necessity.* The first part of the proof will be done by contradiction. Let us suppose that $u\in V$ is a unique solution of $(\mathcal{P}(e))$ and (S2) does not hold. Then Theorem 3.1 implies $F(1)=0$. Taking $v\equiv p\in\mathcal{R}_V$ in $(\mathcal{P}'(e))$ we have

$$(3.12)\qquad a_t(u,p) + b_q(u^+,p) = F(p) = pF(1)\quad \forall\, p\in\mathcal{R}_V,\ p\neq 0,$$

so that $b_q(u^+,p)=0\ \forall\, p\in\mathcal{R}_V,\ p\neq 0$ implying $u^+=0$ in $\Omega$. Thus $u\leqslant 0$ in $\Omega$ and $u+p<0$ in $\Omega$, $\forall\, p\in\mathcal{R}_V,\ p\neq 0$. Therefore $b_q((u+p)^+,v)=0\ \forall\, p\in\mathcal{R}_V,\ p\neq 0$ and $\forall\, v\in V$. Then it is easy to see that $u+p$ is another solution of $(\mathcal{P}(e))$, which contradicts our assumption. Hence the condition (S2) must be satisfied.

*Sufficiency.* Let the condition (S2) be fulfilled. Due to Lemma 3.5 and Lemma 3.7 we already know that $\mathcal{E}_e$ is Gâteaux differentiable, convex and coercive on $V$, implying the existence of a solution $u\in V$ to $(\mathcal{P}(e))$, see [3].

Next, let $u \in V$ solve $(\mathcal{P}(e))$ and let $u \leqslant 0$ a.e. in $\Omega$. Then using $v := p \in \mathcal{R}_V$, $p \neq 0$ in $(\mathcal{P}'(e))$ we have

$$0 = b_q(u^+, p) = pF(1). \tag{3.13}$$

But (3.13) contradicts (S2). Thus the set $M_u = \{x \in \Omega\colon\ u(x) > 0\}$ has a positive Lebesgue measure.

Finally, let us assume that there exist solutions $u_1, u_2 \in V$ of $(\mathcal{P}'(e))$. Then

$$a_t(u_1, v) + b_q(u_1^+, v) = F(v) \quad \forall\, v \in V, \tag{3.14}$$

$$a_t(u_2, v) + b_q(u_2^+, v) = F(v) \quad \forall\, v \in V. \tag{3.15}$$

By subtracting (3.15) from (3.14) and choosing $v := u_1 - u_2$ we obtain

$$a_t(u_1 - u_2, u_1 - u_2) + b_q(u_1^+ - u_2^+, u_1 - u_2) = 0.$$

This definition of $a_t$, $b_q$ yields

$$u_1 - u_2 = p \in P_0 \quad \text{and} \quad u_1^+ - (u_1 - p)^+ = 0 \ \text{ a.e. in } \Omega.$$

Taking into account (S3) we obtain $p = 0$ and consequently $u_1 = u_2$ a.e. in $\Omega$. Therefore, the solution of $(\mathcal{P}(e))$ is unique. □

R e m a r k 3.2. The condition (S2) is the basic assumption in the existence and uniqueness analysis. In practice it means that the load resultant is oriented against the foundation. Therefore the rigid beam motions for which the foundation is not active are eliminated.

R e m a r k 3.3. We considered the particular boundary condition $u'(0) = 0$. However, this procedure can be applied, with small modifications, also to other types of boundary conditions. As an example let us mention the boundary condition $u(0) = 0$. In this case the state problem remains semicoercive.

## 4. Existence of solutions to (P)

We have proved the existence of a unique solution $u(e)$ to the state problem $(\mathcal{P}(e))$ provided that (S2) is satisfied. Next we shall prove the existence of at least one solution to the optimization problem (P). The continuous dependence of $u(e)$ on the design variable $e$ will be the key point of the analysis. First of all let us define convergence in the set $U^t_{\mathrm{ad}}$ as the uniform convergence of continuous functions in $\Omega$ and convergence in $U^q_{\mathrm{ad}}$ as the weak convergence in the Lebesgue space $L^2(\Omega)$:

$$e_n \to e \text{ in } U_{\mathrm{ad}} \Leftrightarrow t_n \rightrightarrows t \text{ in } \Omega \wedge q_n \rightharpoonup q \text{ in } L^2(\Omega), \tag{4.1}$$

where $e_n = \{t_n, q_n\}$, $e = \{t, q\}$. The following result is standard.

**Theorem 4.1.** *The set $U_{\mathrm{ad}}$ is compact in $C(\overline{\Omega}) \times L^2(\Omega)$ with respect to the convergence defined by* (4.1).

Assume that $\beta \in L^\infty(\Omega)$ and $F \in V^*$ satisfying $F(1) > 0$ are given. Then we know that for any $e \in U_{\mathrm{ad}}$ there exists a unique solution of $(\mathcal{P}(e))$ with the property (S3). The set of all such solutions will be denoted by $W$ in what follows:

$$W := \{\{u, t, q\} \in V \times U^t_{\mathrm{ad}} \times U^q_{\mathrm{ad}} \colon\ u = u(e) \text{ solves } (\mathcal{P}(e)),\ e = \{t, q\}\}.$$

The next lemma plays an important role in the existence analysis.

**Lemma 4.1.** *There exists a positive constant $c_1$ such that*

$$c_1 \|u\|^2_{2,2,\Omega} \leqslant a_t(u, u) + b_q(u^+, u) \quad \forall \{u, t, q\} \in W. \tag{4.2}$$

*In addition, $c_1$ does not depend on $\{u, t, q\} \in W$.*

Proof. Let us suppose that (4.2) does not hold. Then one can find a sequence $\{u_n, t_n, q_n\} \subset W$ such that

$$\frac{1}{n}\|u_n\|^2_{2,2,\Omega} > a_{t_n}(u_n, u_n) + b_{q_n}(u_n^+, u_n) \geqslant 0 \quad \forall n \geqslant 1. \tag{4.3}$$

Dividing (4.3) by $\|u_n\|^2_{2,2,\Omega}$ and letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} a_{t_n}(w_n, w_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} b_{q_n}(w_n^+, w_n) = 0,$$

where $w_n := u_n/\|u_n\|_{2,2,\Omega}$. Clearly $\|w_n\|_{2,2,\Omega} = 1$. Hence there exists a subsequence of $\{w_n\}$ (denoted as the original sequence) and an element $w \in V$ such that $w_n \rightharpoonup w$ in $V$. Therefore

$$0 = \lim_{n\to\infty} a_{t_n}(w_n, w_n) \geqslant t_0 \liminf_{n\to\infty} |w_n|^2_{2,2,\Omega} \geqslant t_0 |w|^2_{2,2,\Omega} \geqslant 0.$$

Thus $|w|^2_{2,2,\Omega} = 0$ so that $w \equiv p \in P_0$. Since $|w_n|^2_{2,2,\Omega} \to 0$ and $w_n \rightharpoonup p$ in $V$ we see that

$$w_n \to p \quad \text{in } V. \tag{4.4}$$

From

$$0 = \lim_{n\to\infty} b_{q_n}(w_n^+, w_n^+) \geqslant q_0 \liminf_{n\to\infty} \|w_n^+\|^2_{2,\Omega} = q_0\|w^+\|^2_{2,\Omega} \geqslant 0$$

it follows that $w \equiv p \leqslant 0$ in $\Omega$. From (4.4) and the compact embedding of $V$ into $C(\overline{\Omega})$ we have that $w_n \rightrightarrows p$ in $\overline{\Omega}$. From (S3) we know that $\forall\, n \geqslant 1,\ \exists\, x_n \in \Omega$ such that $w_n(x_n) > 0$. Without loss of generality we may assume $x_n \to x$ in $\overline{\Omega}$. Then $w_n(x_n) \to p(x) \geqslant 0$. Therefore $p = 0$ in $\Omega$. But this contradicts $\|w_n\|_{2,2,\Omega} = 1$ and (4.4). $\square$

**Lemma 4.2** (Continuous dependence). *Let $e_n, e \in U_{\rm ad}$, $e_n \to e$ and let $u_n := u(e_n) \in V$ be a solution of $(\mathcal{P}(e_n))$. Then there exists a function $u \in V$ such that*

$$u_n \to u \quad \text{in } V.$$

*Moreover, $u := u(e)$ is a solution of the state problem $(\mathcal{P}(e))$.*

Proof. Let $\{u(e_n), t_n, q_n\} \in W$. Using the definition of $(\mathcal{P}(e_n))$ and setting $v = u_n$ we have

$$c_1\|u_n\|^2_{2,2,\Omega} \leqslant a_{t_n}(u_n, u_n) + b_{q_n}(u_n^+, u_n) = F(u_n) \leqslant \|f\|_{2,\Omega}\|u_n\|_{2,2,\Omega},$$

making use of (4.2). Thus the sequence $\{u_n\}$ is bounded in $H^2(\Omega)$:

$$\exists\, c = \text{const.} > 0\colon\ \|u_n\|_{2,2,\Omega} \leqslant c\ \forall\, n \in \mathbb{N}. \tag{4.5}$$

Consequently, one can pass to a subsequence of $\{u_n\}$ (denoted as the original sequence) such that

$$u_n \rightharpoonup u \quad \text{in } V \tag{4.6}$$

for some $u \in V$. In order to show that $u$ solves $(\mathcal{P}(e))$ we use the definition of $(\mathcal{P}(e_n))$:

$$a_{t_n}(u_n, v) + b_{q_n}(u_n^+, v) = F(v) \quad \forall\, v \in V \tag{4.7}$$

and pass to the limit with $n \to \infty$. First of all we will focus on the first term in (4.7). We employ the boundedness of $\{u_n\}$, (4.1), and (4.6). It is readily seen that $\lim\limits_{n\to\infty}(a_{t_n}(u_n, v) - a_t(u_n, v)) = 0\ \forall\, v \in V$ so that

$$\lim_{n\to\infty} a_{t_n}(u_n, v) = \lim_{n\to\infty}(a_{t_n}(u_n, v) - a_t(u_n, v)) + \lim_{n\to\infty} a_t(u_n, v) = a_t(u, v).$$

To analyze the second term in (4.7) we make use of (4.1), (4.6), and Lemma 3.2. It is easy to see that $\lim\limits_{n\to\infty}(b_{q_n}(u_n^+,v)-b_{q_n}(u^+,v))=0\ \forall\, v\in V$ so that

$$\lim_{n\to\infty} b_{q_n}(u_n^+,v)=\lim_{n\to\infty}(b_{q_n}(u_n^+,v)-b_{q_n}(u^+,v))+\lim_{n\to\infty} b_{q_n}(u^+,v)=b_q(u^+,v).$$

Thus the limit element $u\in V$ satisfies

$$a_t(u,v)+b_q(u^+,v)=F(v)\quad \forall\, v\in V, \tag{4.8}$$

i.e. $u$ solves $(\mathcal{P}(e))$. Since $u(e)$ is unique, not only the subsequence but the whole sequence $\{u_n\}$ tends weakly to $u$ in $V$. Since $u_n\rightharpoonup u$ in $V$, due to the Rellich theorem one has that $u_n\to u$ in $H^1(\Omega)$. To prove the strong convergence it is sufficient to show the convergence of the seminorm $|u|_{a_t,\Omega}:=\sqrt{a_t(u,u)}$, i.e. $a_t(u_n,u_n)\to a_t(u,u)$ as $n\to\infty$. From (4.8) and the definition of $(\mathcal{P}(e))$, $(\mathcal{P}(e_n))$ it follows that

$$a_{t_n}(u_n,u_n)+b_{q_n}(u_n^+,u_n)=F(u_n)\to F(u)=a_t(u,u)\ +\ b_q(u^+,u) \tag{4.9}$$

as $n\to\infty$. It is not difficult to see that $\lim\limits_{n\to\infty}(b_{q_n}(u_n^+,u_n)-b_q(u^+,u))=0$. Therefore $\lim\limits_{n\to\infty}(a_{t_n}(u_n,u_n)-a_t(u,u))=0$ and consequently

$$a_t(u_n,u_n)=a_t(u_n,u_n)\pm a_{t_n}(u_n,u_n)\to a_t(u,u),\quad n\to\infty, \tag{4.10}$$

by virtue of $a_{t_n}(u_n,u_n)-a_t(u,u)\to 0$, $n\to\infty$. The assertion of the theorem is now proved. ∎

**Lemma 4.3.** *There exists a constant $c_2>0$ such that $\forall\,\{u_i,t_i,q_i\}\in W$, $i=1,2$*

$$c_2\|u_1-u_2\|^2_{2,2,\Omega}\leqslant a_{t_1}(u_1-u_2,u_1-u_2)+b_{q_1}(u_1^+-u_2^+,u_1-u_2). \tag{4.11}$$

*The constant $c_2$ does not depend on $\{u_i,t_i,q_i\}\in W$, $i=1,2$.*

Proof. Assume that (4.11) does not hold. Then there exist sequences $\{u_{1,n},t_{1,n},q_{1,n}\},\{u_{2,n},t_{2,n},q_{2,n}\}\subset W$ such that

$$\begin{aligned}\frac{1}{n}\|u_{1,n}-u_{2,n}\|^2_{2,2,\Omega}&>a_{t_{1,n}}(u_{1,n}-u_{2,n},u_{1,n}-u_{2,n})\\&\quad+b_{q_{1,n}}(u_{1,n}^+-u_{2,n}^+,u_{1,n}-u_{2,n})\geqslant 0\quad\forall\, n\geqslant 1.\end{aligned} \tag{4.12}$$

According to (4.5) the sequences $\{u_{1,n}\}$, $\{u_{2,n}\}$ are bounded in $H^2(\Omega)$. Thus one can find their subsequences (denoted as the original sequences) and functions $\hat u_1$, $\hat u_2$

such that $u_{i,n} \rightharpoonup \hat{u}_i$ in $H^2(\Omega)$, $i=1,2$ implying $u_{i,n} \rightrightarrows \hat{u}_i$ in $\overline{\Omega}$, $i=1,2$. Inserting $v=1$ into $(\mathcal{P}'(e_i))$ we obtain for $i=1,2$

$$q_1 \int_\Omega u_i^+ \,\mathrm{d}x = q_1 \lim_{n\to\infty} \int_\Omega u_{i,n}^+ \,\mathrm{d}x \geqslant \lim_{n\to\infty} \int_\Omega q_{i,n} u_{i,n}^+ \,\mathrm{d}x = F(1) > 0.$$

Hence, we can find sets $M_1, M_2 \subset \overline{\Omega}$ with positive one-dimensional Lebesgue measures such that $u_{i,n} > 0$, $\hat{u}_i > 0$ in $M_i$, $i=1,2$ for $n$ large enough.

Dividing (4.12) by $\|u_{1,n} - u_{2,n}\|_{2,2,\Omega}^2$ $(\neq 0)$ we have

$$a_{t_{1,n}}(w_{1,n}-w_{2,n}, w_{1,n}-w_{2,n}) \to 0 \quad \text{and} \quad b_{q_{1,n}}(w_{1,n}^+-w_{2,n}^+, w_{1,n}-w_{2,n}) \to 0, \tag{4.13}$$

where $w_{i,n} := u_{i,n}/\|u_{1,n}-u_{2,n}\|_{2,2,\Omega}$, $i=1,2$. Clearly $\|w_{1,n}-w_{2,n}\|_{2,2,\Omega} = 1$. Hence there exist subsequences of $\{w_{i,n}\}$, $i=1,2$ (denoted as the original sequences) and an element $w \in V$ such that $w_{1,n} - w_{2,n} \rightharpoonup w$ in $V$. Thus

$$0 = \lim_{n\to\infty} a_{t_{1,n}}(w_{1,n}-w_{2,n}, w_{1,n}-w_{2,n}) \geqslant t_0 \liminf_{n\to\infty} |w_{1,n}-w_{2,n}|_{2,2,\Omega}^2 \geqslant t_0 |w|_{2,2,\Omega}^2 \geqslant 0.$$

Therefore $|w|_{2,2,\Omega}^2 = 0$, i.e. $w \equiv p \in P_0$ and in addition $w_{1,n} - w_{2,n} \to p$ in $H^2(\Omega)$. Consequently, (4.13) yields

$$w_{1,n} - w_{2,n} \to p \text{ in } H^2(\Omega) \quad \text{and} \quad w_{1,n}^+ - w_{2,n}^+ \to 0 \text{ in } L^2(\Omega). \tag{4.14}$$

First suppose that

$$\exists\, c > 0\colon\ \|u_{1,n} - u_{2,n}\|_{2,2,\Omega} \geqslant c \quad \forall\, n \in \mathbb{N}. \tag{4.15}$$

Then $\{w_{1,n}\}$, $\{w_{2,n}\}$ are bounded in $H^2(\Omega)$ and there exist subsequences (denoted as the original sequences) converging weakly to $\hat{w}_1$, $\hat{w}_2$ in $H^2(\Omega)$. Hence (4.14) leads to

$$\hat{w}_1 - \hat{w}_2 = p \quad \text{and} \quad \hat{w}_1^+ - (\hat{w}_1^+ - p) = 0 \text{ a.e. in } \Omega. \tag{4.16}$$

As $\hat{u}_1 > 0$ in $M_1$, also $\hat{w}_1 > 0$ in $M_1$. From this and (4.16) we have $p = 0$ in $\Omega$ on the one hand and $\|p\|_{2,2,\Omega} = 1$ on the other hand as follows from (4.14) and the fact that $\|w_{1,n} - w_{2,n}\|_{2,2,\Omega} = 1$.

If (4.15) is not satisfied then $\|u_{1,n} - u_{2,n}\|_{2,2,\Omega} \to 0$. Thus $\hat{u}_1 = \hat{u}_2$ in $\Omega$. Denote by $M_{1,2} \subseteq \Omega$ a subinterval where $\hat{u}_1$, $u_{i,n}$, $i=1,2$ are positive for $n$ large enough. This implies that $w_{i,n} > 0$, $i=1,2$ in $M_{1,2}$. Then

$$w_{1,n} - w_{2,n} = w_{1,n}^+ - w_{2,n}^+ \to 0 \text{ a.e. in } M_{1,2} \quad \text{as} \quad n \to \infty. \tag{4.17}$$

From (4.17) and (4.14) it follows that $p = 0$ in $\Omega$, contradicting

$$\|w_{1,n} - w_{2,n}\|_{2,2,\Omega} = 1 \quad \forall\, n \in \mathbb{N}.$$

$\square$

Let us now recall that the optimization problems with the state relations given by a variational inequality are in general nonsmooth, see [4] or [5]. Our state problem is represented by a nonlinear variational equation which contains the non-differentiable term $b_q$. Accordingly, one can assume that the problem (P) is non-differentiable as well. In Lemma 4.2 the continuity of the mapping $u\colon e \mapsto u(e)$ was established. In what follows we shall prove that this mapping is even Lipschitz continuous in $U_{\rm ad}$.

**Lemma 4.4.** *The mapping $u\colon e \mapsto u(e)$, where $u(e)$ is a solution of $(\mathcal{P}(e))$, is Lipschitz continuous in $U_{\rm ad}$, i.e., there exists a constant $K_1 > 0$ such that for any $e_1 = \{t_1, q_1\}, e_2 = \{t_2, q_2\} \in U_{\rm ad}$:*

$$\|u(e_1) - u(e_2)\|_{2,2,\Omega} \leqslant K_1(\|t_1 - t_2\|_{C(\overline{\Omega})} + \|q_1 - q_2\|_{2,\Omega}).$$

Proof. Let $e_1, e_2 \in U_{\rm ad}$ and let $u_1 := u(e_1)$, $u_2 := u(e_2)$ be solutions of $(\mathrm{P}(e_1))$, $(\mathrm{P}(e_2))$, respectively. Subtracting $(\mathcal{P}(e_2))$ from $(\mathrm{P}(e_1))$ we have

$$a_{t_1}(u_1, v) - a_{t_2}(u_2, v) + b_{q_1}(u_1^+, v) - b_{q_2}(u_2^+, v) = 0 \quad \forall\, v \in V. \tag{4.18}$$

Adding the terms $a_{t_2}(u_1, v)$, $-b_{q_2}(u_1^+, v)$ to both sides of (4.18) we obtain

$$\begin{aligned} &a_{t_2}(u_1 - u_2, v) + b_{q_2}(u_1^+ - u_2^+, v) \\ &\qquad = (a_{t_2} - a_{t_1})(u_1, v) + (b_{q_2} - b_{q_1})(u_1^+, v) \quad \forall\, v \in V. \end{aligned} \tag{4.19}$$

Inserting $v = u_1 - u_2$ into (4.19) and using (4.11) we obtain

$$c\|u_1 - u_2\|_{2,2,\Omega}^2 \leqslant a_{t_2}(u_1 - u_2, u_1 - u_2) + b_{q_2}(u_1^+ - u_2^+, u_1 - u_2), \tag{4.20}$$

where $c$ is a positive constant independent of $\{u_1, t_1, q_1\}, \{u_2, t_2, q_2\} \in W$. The right-hand side of (4.19) can be estimated as follows:

$$(a_{t_2} - a_{t_1})(u_1, u_1 - u_2) \leqslant c\|t_1 - t_2\|_{C(\overline{\Omega})}\|u_1\|_{2,2,\Omega}\|u_1 - u_2\|_{2,2,\Omega}, \tag{4.21}$$

$$(b_{q_2} - b_{q_1})(u_1^+, u_1 - u_2) \leqslant \|q_1 - q_2\|_{2,\Omega}\|u_1^+\|_{1,2,\Omega}\|u_1 - u_2\|_{2,2,\Omega}. \tag{4.22}$$

Therefore the assertion of the lemma is a consequence of (4.19)–(4.22) and of the uniform boundedness of $u(e)$, $e \in U_{\rm ad}$. □

To ensure the existence of a solution to (P), it remains to assume the lower semi-continuity of the cost functional $I$:

(I1) If $e$, $e_n \in U_{\rm ad}$, $e_n \to e$ in $U_{\rm ad}$ and $v$, $v_n \in V$, $v_n \to v$ in $V$, then

$$\liminf_{n\to\infty} I(e_n, v_n) \geqslant I(e, v).$$

**Theorem 4.2.** *Let the cost functional $I$ satisfy* (I1). *Then there exists at least one solution of* (P).

P r o o f. Let $\{e_n\} \subset U_{\rm ad}$ be a minimization sequence of (P):

$$\lambda := \inf_{e \in U_{\rm ad}} I(e, u(e)) = \lim_{n \to \infty} I(e_n, u(e_n)).$$

The compactness of $U_{\rm ad}$ (see Theorem 4.1) implies the existence of a subsequence (denoted as the original sequence) $\{e_n\} \subset U_{\rm ad}$ and an element $e^* \in U_{\rm ad}$ such that $e_n \to e^*$ in $U_{\rm ad}$. Making use of Lemma 4.2 we obtain $u(e_n) \to u(e^*)$ in $V$, where $u(e_n)$, $u(e^*)$ solve $\mathcal{P}(e_n)$ and $\mathcal{P}(e^*)$, respectively. Due to (I1) we have

$$\lambda = \liminf_{n \to \infty} I(e_n, u(e_n)) \geqslant I(e^*, u(e^*)) \geqslant \lambda;$$

i.e., $e^*$ is a solution of (P). $\square$

In addition, let us suppose that $I$ is Lipschitz continuous in $U_{\rm ad} \times V$:
(I2) There exists a constant $c > 0$ such that $\forall\, e_1, e_2 \in U_{\rm ad}$ and $\forall\, v_1, v_2 \in V$ we have

$$|I(e_1, v_1) - I(e_2, v_2)| \leqslant c(\|v_1 - v_2\|_{2,2,\Omega} + \|t_1 - t_2\|_{C(\overline{\Omega})} + \|q_1 - q_2\|_{2,\Omega}).$$

**Lemma 4.5.** *Let $I$ satisfy* (I2). *Then the functional $J(e) := I(e, u(e))$, with $u(e)$ being a solution of $(\mathcal{P}(e))$, is Lipschitz continuous in $U_{\rm ad}$, i.e. there exists a constant $K_2 > 0$ such that $\forall\, e_1, e_2 \in U_{\rm ad}$:*

$$|J(e_1) - J(e_2)| \leqslant K_2(\|t_1 - t_2\|_{C(\overline{\Omega})} + \|q_1 - q_2\|_{2,\Omega}).$$

P r o o f. The assertion directly follows from (I2) and Lemma 4.4. $\square$

R e m a r k 4.1. As an example of the cost functional satisfying (I1) and (I2) we mention the compliance of the beam:

$$I(e, v) = \int_\Omega f v \,\mathrm{d}x.$$

In fact, the minimization of the compliance is equivalent to the maximization of the total potential energy evaluated at the equilibrium state $u(e)$.

## 5. Conclusion

Design optimization problems for an elastic beam on a unilateral elastic foundation of Winkler's type is studied in this paper. In order to eliminate the rigid displacements from the state problem, the decomposition of the space of kinematically admissible displacements has been used. We have formulated additional conditions on the resultant of the beam load which ensures the coercivity $\mathcal{E}_e$ and the existence and uniqueness of a solution to $(\mathcal{P}(e))$. The approach has been used for a boundary condition $u'(0) = 0$. Nonethless, it can be also used, with some modifications, for other types of boundary conditions such as $u(0) = 0$ for which the state problem remains semicoercive. The existence of optimal thickness of the beam and optimal stiffness of its foundation has been proved using standard approach based on the continuous dependence of the arguments (Lemma 4.2), compactness of $U_{\rm ad}$ and lower semicontinuity of the cost functional. We proved that the control of the state mapping is Lipschitz continuous.

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*References*

[1] *J. P. Aubin*: Applied Functional Analysis, 2nd edition. Wiley-Interscience, New York, 2000.
[2] *J. Chleboun*: Optimal design of an elastic beam on an elastic basis. Apl. Mat. *31* (1986), 118–140.
[3] *S. Fučík, A. Kufner*: Nonlinear Differential Equations. Studies in Applied Mechanics, Vol. 2. Elsevier, New York, 1980.
[4] *J. Haslinger, R. A. E. Mäkinen*: Introduction to Shape Optimization: Theory, Approximation and Computation. SIAM, Philadelphia, 2003.
[5] *J. Haslinger, P. Neittaanmäki*: Finite Element Approximation for Optimal Shape, Material and Topology Design. J. Wiley & Sons, Chichester, 1996.
[6] *I. Hlaváček, I. Bock, J. Lovíšek*: Optimal control of a variational inequality with applications to structural analysis. I: Optimal design of a beam with unilateral supports. Appl. Math. Optim. *11* (1984), 111–143.
[7] *J. V. Horák, I. Netuka*: Mathematical model of pseudointeractive set: 1D body on nonlinear subsoil. I. Theoretical aspects. Engineering Mechanics *14* (2007), 3311–3325.
[8] *J. V. Horák, R. Šimeček*: ANSYS implementation of shape design optimization problems. ANSYS conference 2008, 16. ANSYS FEM Users' Meeting, Luhačovice, 5.–7. November 2008. SVS-FEM, Brno, 2008, released on CD.
[9] *A. Kufner, O. John, S. Fučík*: Function Spaces. Noordhoof International Publishing/ Academia Praha, Nordhoof/Prague, 1977.
[10] *J. Nečas, I. Hlaváček*: Mathematical Theory of Elastic and Elasto-Plastic Bodies. An Introduction. Elsevier, Amsterdam, 1980.
[11] *H. Netuka, J. V. Horák*: System beam-spring-foundation after two years. Proceedings, Conference "Olomouc Days of Applied Mathematics ODAM 2007". 2007, pp. 18–42. (In Czech.)

[12] *P. Salač*: Shape optimization of elastic axisymmetric plate on an elastic foundation. Appl. Math. *40* (1995), 319–338.
[13] *S. Sysala*: Unilateral subsoil of Winkler's type: Semi-coercive beam problem. Appl. Math. *53* (2008), 347–379.

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