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AN INTERSECTION THEOREM FOR SET-VALUED MAPPINGS

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Abstract. Given a nonempty convex set X in a locally convex Hausdorff topological vector space, a nonempty set Y and two set-valued mappings $T\colon X \rightrightarrows X$, $S\colon Y \rightrightarrows X$ we prove that under suitable conditions one can find an $x \in X$ which is simultaneously a fixed point for T and a common point for the family of values of S. Applying our intersection theorem we establish a common fixed point theorem, a saddle point theorem, as well as existence results for the solutions of some equilibrium and complementarity problems.

Keywords: intersection theorem, fixed point, saddle point, equilibrium problem, complementarity problem

MSC 2010: 47H10, 49J53

1. Introduction and preliminaries

Many existence problems in mathematics can be reduced to the following, so called intersection problem: Let X and Y be nonempty sets and $S\colon Y \rightrightarrows X$ a set-valued mapping with nonempty values. Now the question is: when does the family $\{S(y)\colon y \in Y\}$ have nonempty intersection, that is $\bigcap_{y \in Y} S(y) \neq \emptyset$? An intersection point can be viewed as a fixed point, a coincidence point, an equilibrium point, a saddle point etc. In 1961 [8], Ky Fan extended the famous Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem to arbitrary topological vector spaces obtaining a remarkable intersection theorem for the family of values of a set-valued mapping. Since then many intersection theorems have appeared (see [1], [5], [7], [13], [15], [17]). The main theorem of our paper fits into this interesting group of results. More precisely, given a nonempty convex set X in a locally convex Hausdorff topological vector space, a nonempty set Y and two set-valued mappings $T\colon X \rightrightarrows X$, $S\colon Y \rightrightarrows X$ we prove that under suitable conditions one can find an $x \in X$ which is simultaneously a fixed point for T and a common point for the family of values of S. Applying our intersection theorem we establish a common fixed point theorem, a saddle point theorem, as well as existence results for the solutions of some equilibrium and complementarity problems.

For the remainder of this section we present some definitions and known results. If X and Y are topological spaces, a set-valued mapping $T\colon X \rightrightarrows Y$ is said to be: (i) *upper semicontinuous* if for every closed subset B of Y the set $\{x \in X\colon T(x) \cap B \neq \emptyset\}$ is closed; (ii) *closed* if its graph (that is, the set $\operatorname{Gr} T = \{(x, y) \in X \times Y\colon y \in T(x)\}$) is a closed subset of $X \times Y$; (iii) *compact* if $T(X)$ is contained in a compact subset of Y.

The following lemma collects known facts about upper semicontinuous mappings (see [2]):

Lemma 1.1. *Let $T\colon X \rightrightarrows Y$ be a set-valued mapping with nonempty values.*

(i) *If T has compact values, then T is upper semicontinuous if and only if for every net $\{x_t\}$ in X converging to $x \in X$ and for any net $\{y_t\}$ with $y_t \in T(x_t)$ there exist $y \in T(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y.*
(ii) *If T is closed and compact then it is upper semiconinuous.*

The proof of the main result of the paper will be relied on the following

Lemma 1.2. *If X is a convex set in a locally convex topological vector space and $T\colon X \rightrightarrows X$ is a compact upper semicontinuous mapping with nonempty compact convex values then T has a fixed point.*

The previous lemma is the well-known Himmelberg fixed point theorem [10]. When X is compact, it reduces to the Kakutani-Fan-Glicksberg fixed point theorem.

Let X be a convex set in a vector space, Z a vector space and C a convex cone in Z. A set-valued mapping $F\colon X \rightrightarrows Z$ is said to be:

(i) *C-convex* if $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C$ for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$;
(ii) *C-concave* if $F(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda F(x_1) + (1-\lambda)F(x_2) + C$ for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$;
(iii) *C-quasiconvex* if for each $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, either $F(x_1) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C$ or $F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C$;
(iv) *convex* (*concave*, *quasiconvex*) if it is C-convex (C-concave, C-quaiconvex, respectively) with $C = \{0_Z\}$ (0_Z being the zero element of the vector space Z).

In the case of the single-valued mappings (functions) the corresponding concepts are obtained by replacing the inclusions $\subseteq$ by $\in$.

2. Main result

Given a set-valued mapping $S\colon\ Y \rightrightarrows X$, the fiber of S on $x \in X$ is the set defined by $S^{-1}(x) = \{y \in Y\colon\ x \in S(y)\}$.

Theorem 2.1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space, Y a nonempty convex subset of a topological vector space and $T\colon\ X \rightrightarrows X$, $S\colon\ Y \rightrightarrows X$ two set-valued mappings. Assume that:*

(i) *T is a closed compact mapping with convex values;*
(ii) *the set $D = \{x \in X\colon\ x \in T(x)\}$ is convex;*
(iii) *for all $x \in X$ and $y \in Y$, $T(x) \cap S(y) \neq \emptyset$;*
(iv) *S is a closed mapping and for all $x \in X$ and all $y \in Y$ the sets $S(y)$ and $Y \setminus S^{-1}(x)$ are convex sets.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x}) \cap \bigcap\limits_{y\in Y} S(y)$.

Proof. Notice first that, since T is a closed compact mapping, its values are compact. Then, D is nonempty compact and convex. Indeed, D is nonempty, by Himmelberg fixed point theorem. Since T is closed, it follows that D is closed. Thus, D is compact, as a closed subset of the compact set $\overline{T(X)}$.

Suppose that the conclusion of the theorem is false. Hence, for each $x \in D$ there is $y \in Y$ such that $x \notin S(y)$. Set $S^{\complement}(y) = D \setminus S(y)$. Then $D = \bigcup\limits_{y\in Y} S^{\complement}(y)$. Since the set-valued mapping S is closed, for any $y \in Y$ the set $S^{\complement}(y)$ is open in D. Since D is compact, there exists a finite covering $\{S^{\complement}(y_1), \dots, S^{\complement}(y_n)\}$ of D and a partition of unity $\{\beta_1, \dots, \beta_n\}$ corresponding to this finite covering (i.e., each β_i is a continuous function from D into $[0,1]$ which vanishes outside of $S^{\complement}(y_i)$ and $\sum\limits_{i=1}^{n} \beta_i(x) = 1$ for all $x \in D$). Set $p(x) = \sum\limits_{i=1}^{n} \beta_i(x) y_i$ for all $x \in D$. Then p is a continuous mapping from D into Y.

Define a set-valued mapping $P\colon\ D \multimap D$ by

$$P(x) = S(p(x)) \cap D,\ x \in D.$$

We show that P has nonempty values. Let $x \in D$ be arbitrarily chosen. Define a new set-valued mapping $Q_x\colon\ X \multimap X$ by $Q_x(u) = T(u) \cap S(p(x)) \in X$. Then Q_x has nonempty (by (iii)) convex values. Since T is a closed mapping and $S(p(x))$ is a closed set, it follows readily that the mapping Q_x is closed. By Lemma 1.1, Q_x is upper semicontinuous. By the Himmelberg fixed point theorem, Q_x has a fixed point u_x. Then $u_x \in T(u_x)$, that is $u_x \in D$ and $u_x \in S(p(x))$. Thus $u_x \in P(x)$, hence P has nonempty values.

Applying now the Kakutani-Fan-Glicksberg fixed point theorem, we get a point $x_0 \in D$ for which $x_0 \in P(x_0)$. Then $x_0 \in S(p(x_0))$, i.e., $p(x_0) \in S^{-1}(x_0)$. On the other hand, if $\beta_i(x_0) > 0$, then $x_0 \in S^{\complement}(y_i)$, that is $y_i \in Y \setminus S^{-1}(x_0)$. Since $Y \setminus S^{-1}(x_0)$ is a convex set, $p(x_0) \in \operatorname{co}\{y_i\colon \beta_i(x_0) > 0\} \subseteq Y \setminus S^{-1}(x_0)$. The obtained contradiction proves the theorem. $\square$

Remark 1. Condition (ii) in Theorem 2.1 is fulfilled if the set-valued mapping T is convex. Indeed, in this case, for $x_1, x_2 \in D$ and $\lambda \in [0,1]$ we have

$$\lambda x_1 + (1-\lambda)x_2 \in \lambda T(x_1) + (1-\lambda)T(x_2) \subseteq T(\lambda x_1 + (1-\lambda)x_2),$$

hence $\lambda x_1 + (1-\lambda)x_2 \in D$.

Remark 2. Let $Q\colon X \rightrightarrows Y$ be a set-valued mapping such that $S\colon Y \rightrightarrows X$ defined by $S(y) = \{x \in X\colon y \notin Q(x)\}$ satisfies conditions (iii) and (iv) in Theorem 2.1. Then Theorem 2.1 provides a point $\bar{x} \in X$ which is simultaneously a fixed point for T (that is, $\bar{x} \in T(\bar{x})$) and a maximal element for Q (that is, $Q(\bar{x}) = \emptyset$).

When the convex set X is compact and $T(x) = X$ for all $x \in X$, Theorem 2.1 reduces to

Corollary 2.2. *Let X, Y be as in Theorem 2.1 and let S be a closed set-valued mapping from Y into X. If $S(y)$ and $Y \setminus S^{-1}(x)$ are convex sets for all $x \in X$ and all $y \in Y$ then $\bigcap\limits_{y \in Y} S(y) \neq \emptyset$.*

3. Applications

The first application is a common fixed point theorem.

Theorem 3.1. *Let X be a nonempty convex set in a normed vector space, $T\colon X \rightrightarrows X$ a closed compact mapping with convex values and $f\colon X \to X$ a continuous function. Assume that:*

(i) *the set $\{x \in X\colon x \in T(x)\}$ is convex;*
(ii) *for each $x, y \in X$ there exists $u \in T(x)$ such that $\|u - f(u)\| \leqslant \|y - f(u)\|$;*
(iii) *for each $y \in X$ the set $\{x \in X\colon \|x - f(x)\| \leqslant \|y - f(x)\|\}$ is convex.*

Then T and f have a common fixed point, that is, there exists $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x} \in T(\bar{x})$.

Proof. Let $S\colon X \rightrightarrows X$ be defined by

$$S(y) = \{u \in X\colon \|u - f(u)\| \leqslant \|y - f(u)\|\} \quad \text{for all } y \in X.$$

As the function f is continuous, S is closed. By (iii), S has convex values. For each $u \in X$, the set $X \setminus S^{-1}(u) = \{y \in X \colon \|y - f(u)\| < \|u - f(u)\|\}$ is convex. Indeed, if $\|y_i - f(u)\| < \|u - f(u)\|$, $i = 1, 2$, then for any $\lambda \in [0, 1]$ we have $\|\lambda y_1 + (1 - \lambda) y_2 - f(u)\| \leqslant \lambda \|y_1 - f(u)\| + (1 - \lambda)\|y_2 - f(u)\| < \|u - f(u)\|$. By Theorem 2.1, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $\|\bar{x} - f(\bar{x})\| \leqslant \|y - f(\bar{x})\|$ for every $y \in X$. Taking $y = f(\bar{x})$ we get $\|\bar{x} - f(\bar{x})\| \leqslant 0$, that is $\bar{x} = f(\bar{x})$. $\square$

Recall that $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is a *saddle point for a real function* f defined on $X_1 \times X_2$ if $f(x_1, \bar{x}_2) \leqslant f(\bar{x}_1, x_2)$ for all $x_1 \in X_1$, $x_2 \in X_2$. We extend this concept as follows:

Given two functions $f, g \colon X_1 \times X_2 \to \mathbb{R}$, we say that a point $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is a *saddle point for the pair* (f, g) if $f(x_1, \bar{x}_2) \leqslant g(\bar{x}_1, x_2)$ for all $x_1 \in X_1$, $x_2 \in X_2$.

Example. Let $X_1 = [0, 1]$, $X_2 = \mathbb{R}$ and for each $(x_1, x_2) \in X_1 \times X_2$, let $f(x_1, x_2) = x_1 - \mathrm{e}^{x_1 x_2}$ and $g(x_1, x_2) = \mathrm{e}^{x_1 x_2} - x_1 - x_2$. For each $x_1 \in X_1$, $x_2 \in X_2$ we have

$$f(x_1, 0) = x_1 - 1 \leqslant f(1, 0) = 0 = g(1, 0) \leqslant \mathrm{e}^{x_2} - 1 - x_2 = g(1, x_2).$$

Hence $(1, 0)$ is a saddle point for the pair (f, g).

It is clear that in the case $f = g$ the concept above reduces to the classical concept of a saddle point. It is also worth mentioning that the existence of a saddle point for the pair (f, g) implies the inequality $\inf\limits_{x_2 \in X_2} \sup\limits_{x_1 \in X_1} f(x_1, x_2) \leqslant \sup\limits_{x_1 \in X_1} \inf\limits_{x_2 \in X_2} g(x_1, x_2)$.

Applying Theorem 2.1 we establish sufficient conditions for a point $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ to be simultaneously a fixed point for a set-valued mapping $T \colon X_1 \times X_2 \rightrightarrows X_1 \times X_2$ and a saddle point for a pair of functions (f, g).

Theorem 3.2. *Let X_1, X_2 be nonempty convex sets in two locally convex Hausdorff topological vector spaces, let $T \colon X_1 \times X_2 \rightrightarrows X_1 \times X_2$ be a set-valued mapping, $f, g \colon X_1 \times X_2 \to \mathbb{R}$ continuous functions. Assume that:*

(i) *T is a closed compact mapping with convex values;*
(ii) *the set $\{(x_1, x_2) \in X_1 \times X_2 \colon (x_1, x_2) \in T(x_1, x_2)\}$ is convex;*
(iii) *for every $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ there is $(u_1, u_2) \in T(x_1, x_2)$ such that $f(y_1, u_2) \leqslant g(u_1, y_2)$;*
(iv) *f is concave in the first variable and convex in the second variable;*
(v) *g is convex in the first variable and concave in the second variable.*

Then there exists $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ which is a fixed point for T and a saddle point for the pair (f, g).

P r o o f. We will apply Theorem 2.1 when $X = Y = X_1 \times X_2$ and the set-valued mapping S is defined by

$$S(y_1, y_2) = \{(x_1, x_2) \in X_1 \times X_2 \colon f(y_1, x_2) \leqslant g(x_1, y_2)\}.$$

One can see that condition (ii) here is as (ii) in Theorem 2.1. The set-valued mapping S is closed, as f and g are continuous functions. We now show that for $(y_1, y_2) \in X_1 \times X_2$, the set $S(y_1, y_2)$ is convex. If $(x_1, x_2), (x_1', x_2') \in S(y_1, y_2)$, then $f(y_1, x_2) \leqslant g(x_1, y_2)$, $f(y_1, x_2') \leqslant g(x_1', y_2)$. For any $\lambda \in [0, 1]$ we have:

$$\begin{aligned} &f(y_1, \lambda x_2 + (1-\lambda)x_2') \leqslant \lambda f(y_1, x_2) + (1-\lambda) f(y_1, x_2') \\ &\leqslant \lambda g(y_1, x_2) + (1-\lambda) g(y_1, x_2') \leqslant g(y_1, \lambda x_2 + (1-\lambda)x_2'). \end{aligned}$$

Similarly one can prove that $X_1 \times X_2 \setminus S^{-1}(x_1, x_2)$ is a convex set for each $(x_1, x_2) \in X_1 \times X_2$. The conclusion follows now from Theorem 2.1. $\square$

Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space, let Y be a nonempty convex subset of a topological vector space, Z a topological vector space and $F \colon X \times Y \rightrightarrows Z$, $C \colon X \rightrightarrows Z$ two set-valued mappings such that $C(x)$ is a nonempty closed convex cone with nonempty interior. We investigate the existence of a solution for the following vector equilibrium problem:

(EP) Find $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $F(\bar{x}, y) \not\subseteq \operatorname{int} C(\bar{x})$, for all $y \in Y$.

The above problem has been introduced in [6] as an intermediary problem between an equilibrium and a quasi equilibrium problem.

Theorem 3.3. *Assume that conditions* (i) (ii) *in Theorem* 2.1 *as well as conditions* (iii) *and* (iv) *below are fulfilled:*

(iii) *for each $(x, y) \in X \times Y$ there exists $u \in T(x)$ such that $F(u, y) \not\subseteq \operatorname{int} C(u)$;*

(iv) *the set $\{(x, y) \in X \times Y \colon F(x, y) \not\subseteq \operatorname{int} C(x)\}$ is closed in $X \times Y$ and for all $x \in X$ and $y \in Y$ the sets $\{v \in Y \colon F(x, v) \subseteq \operatorname{int} C(x)\}$ and $\{u \in X \colon F(u, y) \not\subseteq \operatorname{int} C(u)\}$ are convex;*

Then (EP) has a solution.

P r o o f. Apply Theorem 2.1 when the mapping S is defined by:

$$S(y) = \{x \in X \colon F(x, y) \not\subseteq \operatorname{int} C(x)\}. \tag{1}$$

$\square$

The requirements encompassed in condition (iv) of the previous theorem can be replaced by suitable conditions on the mappings F and C.

Theorem 3.4. *Assume that the set-valued mapping T satisfies conditions* (i) *and* (ii) *in Theorem* 2.1 *and F and C satisfy the following conditions:*

(a) *F is upper semicontinuous with nonempty compact values on $X \times Y$, convex in the first variable and $C(x)$-concave in the second variable, for each $x \in X$;*
(b) *the set-valued mapping $W\colon X \rightrightarrows Z$ defined by $W(x) = Z \setminus \operatorname{int} C(x)$ is closed and convex;*
(c) *for each $(x, y) \in X \times Y$ there exists $u \in T(x)$ such that $F(u, y) \not\subseteq \operatorname{int} C(u)$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $F(\bar{x}, y) \not\subseteq \operatorname{int} C(\bar{x})$ for all $y \in Y$.

P r o o f. It suffices to show that the set-valued mapping S defined by (1) satisfies (iv) in Theorem 2.1. We claim that the mapping S is closed. Indeed, let $\{(y_t, x_t)\}_{t\in\Delta}$ be a net in $\operatorname{Gr} S$ converging to $(y, x) \in Y \times X$. For each $t \in \Delta$ there exists $z_t \in F(x_t, y_t) \cap W(x_t)$. Since F is upper semicontinuous with nonempty compact values, there exist $z \in F(x, y)$ and a subnet $\{z_{t_\alpha}\}$ of $\{z_t\}$ converging to z. As the mapping W is closed, $z \in W(x)$. Thus, $F(x, y) \cap W(x) \neq \emptyset$, hence $(y, x) \in \operatorname{Gr} S$.

Let $y \in X$ and $x_1, x_2 \in S(y)$. If $z_i \in F(x_1, y) \cap W(x_i)$ for $i = 1, 2$, since $F(\cdot, y)$ and W are convex mappings, for any $\lambda \in [0, 1]$ we have $\lambda z_1 + (1-\lambda) z_2 \in F(\lambda x_1 + (1-\lambda)x_2, y) \cap W(\lambda x_1 + (1-\lambda)x_2)$, hence $S(y)$ is convex.

For any $x \in X$, the set $X \setminus S^-(x) = \{y \in Y\colon F(x, y) \subseteq \operatorname{int} C(x)\}$ is convex. Indeed, if $y_1, y_2 \in X \setminus S^-(x)$ then for any $\lambda \in [0, 1]$ we have:

$$\begin{aligned} F(x, \lambda y_1 + (1-\lambda)y_2) &\subseteq \lambda F(x, y_1) + (1-\lambda)F(x, y_2) + C(x) \\ &\subseteq \lambda \operatorname{int} C(x) + (1-\lambda) \operatorname{int} C(x) + C(x) \subseteq \operatorname{int} C(x). \end{aligned}$$

Now we apply Theorem 2.1. □

In our opinion it is worth comparing the previous theorem with Theorem 6 in [6], which has the same conclusion. There, $X = Y$ and condition (c) is replaced by another one involving the inward set $\mathbb{O}(T(x); x)$. Moreover, the techniques of the proofs are different.

Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E, let Z be a topological vector space, C a closed, convex, pointed cone in Z (recall that a cone is pointed if $C \cap (-C) = \{0_Z\}$) and $f\colon X \times X \to Z$. We consider the following vector complementarity problem as a general model for a large class of problems of this type:

(VCP) Find $\bar{x} \in X$ such that $f(\bar{x}, \bar{x}) = 0_Z$ and $f(\bar{x}, y) \in C$, for all $y \in Y$.

It is well known that the cone C induces a partial order on Z defined by $z_1 \leqslant_C z_2$ if and only if $z_2 - z_1 \in C$. Thus, problem (VCP) can be formulated as follows: find $\bar{x} \in X$ such that $f(\bar{x}, \bar{x}) = 0_Z$ and $f(\bar{x}, y) \geqslant_C 0$, for all $y \in Y$.

Usually, in a complementarity problem, X is a closed convex cone but there is also a large number of papers (see [3], [4], [9], [12], [16]) in which X is just a convex set. The next theorem establishes sufficient conditions under which problem (VCP) has a solution.

Theorem 3.5. *Assume that the convex set X is compact and the following conditions are fulfilled:*

(i) *f is continuous on $X \times X$, affine in the first variable and C-quasiconvex in the second variable;*
(ii) *the set $\{x \in X\colon\ f(x,x) \in -C\}$ is convex;*
(iii) *for each $x, y \in X$ there exists $u \in X$ such that $f(u,x) \in -C$ and $f(u,y) \in C$.*

Then problem (VCP) *has a solution.*

P r o o f. Consider the set-valued mappings $T, S\colon\ X \rightrightarrows X$ defined by

$$T(x) = \{u \in X\colon\ f(u,x) \in -C\},$$
$$S(y) = \{u \in X\colon\ f(u,y) \in C\}.$$

As f is continuous, the set-valued mappings T and S are closed. Since f is affine in the first variable, T and S has convex values.

We now show that for $u \in X$, the set $X \setminus S^{-1}(u) = \{y \in X\colon\ f(u,y) \notin C\}$ is convex. Assume that there exists $y_1, y_2 \in X$ such that $f(u,y_1) \notin C$, $f(u,y_2) \notin C$ and for some $\lambda \in (0,1)$, $f(u, \lambda y_1 + (1-\lambda)y_2) \in C$. Since f is C-quasiconvex in the second variable, for some $i \in \{1,2\}$ we have $f(u,y_i) \in f(u, \lambda y_1 + (1-\lambda)y_2) + C \subseteq C$; a contradiction.

Observe that conditions (ii) and (iii) are nothing other than the conditions imposed in Theorem 2.1. By Theorem 2.1, there exists $\bar{x} \in X$ such that $f(\bar{x},\bar{x}) \in -C$ and $f(\bar{x},y) \in C$ for all $y \in X$. The cone C being pointed implies $f(\bar{x},\bar{x}) \in (-C) \cap C = \{0\}$. $\square$

R e m a r k 3. One can easily prove that condition (ii) in Theorem 3.5 is fulfilled when the function $X \ni x \to f(x,x)$ is C-convex.

A nonempty subset B of a cone X is a *base* of X if $X = \bigcup_{\lambda \geqslant 0} \lambda B$. Recall that in a Hausdorff locally convex space any proper pointed convex cone X which is locally compact has a compact convex base which is the intersection of X with a closed hyperplane (see [14]). It is also worth recalling that in a reflexive Hausdorff locally convex space a closed convex cone X admits a compact convex base whenever its dual has nonempty interior (see [11]). Further, we discuss the existence of a solution of problem (VCP) when X is a closed convex cone in a locally convex Hausdorff topological vector space.

Theorem 3.6. *Assume that the closed convex cone X has a compact convex base. If the function f satisfies conditions* (ii), (iii) *of Theorem* 3.5 *as well as the following conditions:*

(i′) *f is continuous on $X \times X$, linear in the first variable and C-quasiconvex in the second variable;*

(iv) *the set $\{x \in X\colon\ f(x,x) = 0_Z\}$ is compact,*

then problem (VCP) *has a solution.*

Proof. Let B be a compact convex base of X. Denote by $\mathcal{K}$ the family of all compact convex subsets K of X satisfying $\{0_Z\} \cup B \subseteq K$.

Let $K \in \mathcal{K}$ be arbitrarily fixed. Obviously, K is a base of X. Since f is linear in the first variable, by (ii), for each $x, y \in K$ there exists $u \in K$ such that $f(u,x) \in -C$ and $f(u,y) \in C$. By Theorem 3.5 there exists $x_K \in K$ satisfying $f(x_K, x_K) = 0_Z$ and $f(x_K, y) \in C$ for all $y \in K$.

It is clear that for any K, $K' \in \mathcal{K}$, $\operatorname{co}(K \cup K') \in \mathcal{K}$. Consequently, the ordered set $(\mathcal{K}, \subseteq)$ is directed to the right. Since $\{x_K\}_{K\in\mathcal{K}}$ is a net in the compact $D := \{x \in X\colon\ f(x,x) = 0_Z\}$ we may assume without loss of generality that it converges to an element $\bar{x}$ of D. We claim that $f(\bar{x}, y) \in C$ for every $y \in X$. Indeed, we have $f(x_K, y) \in C$ for every $K \in \mathcal{K}$ that contains y and since f is continuous and the cone C is closed, passing to the limit we get $f(\bar{x}, y) \in C$. □

References

[1] *R. P. Agarwal, M. Balaj, D. O'Regan*: Common fixed point theorems and minimax inequalities in locally convex Hausdorff topological vector spaces. Appl. Anal. *88* (2009), 1691–1699.

[2] *C. D. Aliprantis, K. C. Border*: Infinite Dimensional Analysis. A Hitchhiker's Guide. 3rd ed. Springer, Berlin, 2006.

[3] *Q. H. Ansari, A. P. Farajzadeh, S. Schaible*: Existence of solutions of vector variational inequalities and vector complementarity problems. J. Glob. Optim. *45* (2009), 297–307.

[4] *Q. H. Ansari, J. C. Yao*: An existence result for the generalized vector equilibrium problem. Appl. Math. Lett. *12* (1999), 53–56.

[5] *M. Balaj*: An intersection theorem with applications in minimax theory and equilibrium problem. J. Math. Anal. Appl. *336* (2007), 363–371.

[6] *M. Balaj*: A fixed point-equilibrium theorem with applications. Bull. Belg. Math. Soc. - Simon Stevin *17* (2010), 919–928.

[7] *M. Balaj, D. O'Regan*: Inclusion and intersection theorems with applications in equilibrium theory in G-convex spaces. J. Korean Math. Soc. *47* (2010), 1017–1029.

[8] *K. Fan*: A generalization of Tychonoff's fixed point theorem. Math. Ann. *142* (1961), 305–310.

[9] *A. P. Farajzadeh, M. A. Noor, S. Zainab*: Mixed quasi complementarity problems in topological vector spaces. J. Glob. Optim. *45* (2009), 229–235.

[10] *C. J. Himmelberg*: Fixed points of compact multifunctions. J. Math. Anal. Appl. *38* (1972), 205–207.

[11] *V. Jeyakumar, W. Oettli, M. Natividad*: A solvability theorem for a class of quasiconvex mappings with applications to optimization. J. Math. Anal. Appl. *179* (1993), 537–546.

[12] *S. A. Khan*: Generalized vector implicit quasi complementarity problems. J. Glob. Optim. *49* (2011), 695–705.

[13] *P. Q. Khanh, N. H. Quan*: Intersection theorems, coincidence theorems and maximal-element theorems in GFC-spaces. Optimization *59* (2010), 115–124.

[14] *G. Köthe*: Topological Vector Spaces I. Springer, Berlin, 1969.

[15] *K. Q. Lan*: An intersection theorem for multivalued maps and applications. Comput. Math. Appl. *48* (2004), 725–729.

[16] *B. S. Lee, A. P. Farajzadeh*: Generalized vector implicit complementarity problems with corresponding variational inequality problems. Appl. Math. Lett. *21* (2008), 1095–1100.

[17] *H. Lu, D. Tang*: An intersection theorem in L-convex spaces with applications. J. Math. Anal. Appl. *312* (2005), 343–356.

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