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ON A VARIATIONAL APPROACH TO TRUNCATED PROBLEMS OF MOMENTS

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Abstract. We characterize the existence of the L^1 solutions of the truncated moments problem in several real variables on unbounded supports by the existence of the maximum of certain concave Lagrangian functions. A natural regularity assumption on the support is required.

Keywords: problem of moments, representing measure

MSC 2010: 44A60, 49J99

1. Introduction

The present paper is concerned with the truncated problem of moments in several real variables, in the following context. Let $n \in \mathbb{N}$ and fix a closed subset $T \neq \emptyset$ of $\mathbb{R}^n$, a finite subset $I \subset (\mathbb{Z}_+)^n$ with $0 \in I$ and a set $g = (g_i)_{i \in I}$ of real numbers with $g_0 = 1$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Typically a problem of moments [1] requires to establish whether there exist Borel measures $\nu \geqslant 0$ on $\mathbb{R}^n$, supported on T, such that $\int_T |t^i| \, \mathrm{d}\nu(t) < \infty$ and $\int_T t^i \, \mathrm{d}\nu(t) = g_i$ for all $i \in I$. As usual $t^i = t_1^{i_1} \dots t_n^{i_n}$ where $t = (t_1, \dots, t_n)$ is the variable in $\mathbb{R}^n$ and $i = (i_1, \dots, i_n)$ is a multiindex. In this case we call ν a representing measure of g, and g_i the moments of ν. We are interested in those measures $\nu = f \, \mathrm{d}t$ that are absolutely continuous with respect to the n-dimensional Lebesgue measure $\mathrm{d}t = \mathrm{d}t_1 \dots \mathrm{d}t_n$, in which case we call f a representing density of g. Namely, the (class of equivalence of the) Lebesgue integrable function f is $\geqslant 0$ almost everywhere (a.e.) on T, has finite moments of orders $i \in I$ and

$$\int_T t^i f(t) \, \mathrm{d}t = g_i \quad (i \in I). \tag{1}$$

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Our main result is Theorem 3, the statement of which relies on the following rather known idea. Given partial information in the integral form $\int_T t^i \,\mathrm{f}\, \varrho \,\mathrm{d}t = g_i$ about representing densities f on a space $(T, \varrho\,\mathrm{d}t)$ endowed with a reference density ϱ does not determine them uniquely. An approach favorite to physicists and statisticians is, when ϱ is a probability density, to choose that particular density f_* minimizing the entropy functional $h(\mathrm{f}) = \int_T (\mathrm{f} \ln \mathrm{f}) \varrho \,\mathrm{d}t$ amongst all solutions of the moments constraints. This uniquely selects the unbiased probability distribution f_* (that proves to have the form $\mathrm{f}_*(t) = \mathrm{e}^{\sum_{i\in I} \lambda_i^* t^i}$) on the knowledge of the prescribed average values g_i of t^i, where t is considered as a T-valued random variable with repartition ϱ [6], [9], [18], [20]. Under suitable hypotheses, f_* turns to exist whenever problem (1) is feasible, even for more general reference measures. A main tool to this aim is Fenchel duality [8], [24], [26], [27], that deals with minimizing such convex functionals $h\colon X \to \mathbb{R} \cup \{\infty\}$ on convex subsets of locally convex spaces X, in connection with the dual problem of maximizing $-h^*$, where $h^*\colon X^* \to \mathbb{R} \cup \{\infty\}$ is the convex conjugate of h, called also its Legendre-Fenchel transform [26], [27], defined on the dual X^* of X by $h^*(y) = \sup\{\langle x, y\rangle - h(x)\colon h(x) < \infty\}$. Typically $\inf h = \max(-h^*)$ and, briefly speaking, minimizing $h(f) = \int_T \mathrm{f} \ln \mathrm{f} \varrho \,\mathrm{d}t$ as above (that is, maximizing the corresponding $-h^*$) is to find $\lambda^* = (\lambda_i^*)_{i\in I}$ maximizing $L(\lambda) = \sum\limits_{i\in I} g_i \lambda_i - \int_T \mathrm{e}^{\sum_{i\in I} \lambda_i t^i} \varrho \,\mathrm{d}t$. Many results exist in this direction [3], [5]–[9], [16], [17], [21], [22], [23]. Additional hypotheses are always necessary when the conclusion $\inf h = \min h$ is sought for, since there are data g for which the primal attainment (that is, the existence of f_* such that $\inf h = h(f_*)$) fails [16], [17] although problem (1) has solutions.

By Theorem 3 we prove that the feasibility of problem (1) is equivalent to the boundedness from above $\sup L < \infty$ with attainment $\sup L = \max L$ for the concave function L (the Lagrangian). This holds no matter whether $\inf h$ is attained or not (the general theory still provides us with $\inf h = \max L$).

Initiated by Stieltjes, Hausdorff, Hamburger and Riesz, the area of the truncated problems of moments knows various other approaches, based for instance on operator methods, or sums-of-squares representations of positive polynomials [10]–[14], [19], [25]. Although important, these topics remain beyond the aim of this work.

The author got the idea to consider L instead of h from the works [5] where a similar characterization exists, and [16], [17], drawn to his attention by professor Mihai Putinar. Our statement and proof are rather general, independent of these cited works.

2. Main results

We recall that a linear Riesz functional φ_γ [12] associated with a set $\gamma = (\gamma_i)_{i\in J}$ of real numbers γ_i for $J \subset \mathbb{Z}_+^n$ is defined on the polynomials p from the linear span of $X_1^{i_1} \dots X_n^{i_n}$ where $i = (i_1, \dots, i_n) \in J$ by $\varphi_\gamma X^i = \gamma_i$. One calls φ_γ T-positive [12] if $\varphi_\gamma p \geqslant 0$ whenever $p(t) \geqslant 0$ for all $t \in T$. If γ has representing measures $\nu \geqslant 0$ on T, φ_γ is T-positive since $\varphi_\gamma p = \int_T p \,\mathrm{d}\nu$ for any such polynomial p. In the full case $J = \mathbb{Z}_+^n$ the T-positivity condition is sufficient for the existence of the representing measures, by the Riesz-Haviland theorem [15]. An analogue of this theorem [12] for the truncated case $I = \{i\colon\ |i| \leqslant 2k\}$ characterizes the existence of the representing measures by the existence of T-positive extensions of φ_γ to the space of polynomials of degree $\leqslant 2k+2$. For later use, we state below a version of these results (Theorem 1) and a Fenchel theoretic result of dual attainment (Theorem 2).

Definitions. We call T *regular* [4] if for any $t \in T$ and $\varepsilon > 0$ the Lebesgue measure of the set $\{x \in T\colon\ \|x - t\| < \varepsilon\}$ is positive. As usual $\|t\| = \Big(\sum\limits_{\iota=1}^{n} t_\iota^2\Big)^{1/2}$. For any $i \in I$ set $\sigma_i = \{j \in \mathbb{Z}_+^n\colon\ j_k = \text{either } 0 \text{ or } i_k,\ 1 \leqslant k \leqslant n\}$. We call I *regular* [4] if $\sigma_i \subset I$ for all $i \in I$. Define Γ, $G \subset \mathbb{R}^N$ ($N = \operatorname{card} I$) by $\Gamma = \{\gamma = (\gamma_i)_{i\in I}\colon \exists$ measures $\nu \geqslant 0$ on T with $\int_T t^i \,\mathrm{d}\nu(t) = \gamma_i,\ i \in I\}$ and $G = \{\gamma = (\gamma_i)_{i\in I} \neq 0\colon \exists\ f \in L_+^1(T, \mathrm{d}t)$ such that $\int_T t^i f(t) \,\mathrm{d}t = \gamma_i,\ i \in I\}$. The notation $L^p(T, \mu)$, $L^p(\mu)$ for a measure μ on T, $1 \leqslant p \leqslant \infty$ has the usual meaning. In particular, $L_+^1(T, \mu)$ is the set of all $f \in L^1(T, \mu)$, $f \geqslant 0$ μ-a.e. For $\gamma = (\gamma_i)_{i\in I}$, φ_γ is the linear functional defined on the span $P_I \subset \mathbb{R}[X_1, \dots, X_n]$ of all X^i with $i \in I$ by $\varphi_\gamma X^i = \gamma_i$. Set $\mathrm{e}_\iota = (0, \dots, \overset{\iota}{1}, \dots, 0)$ for $1 \leqslant \iota \leqslant n$.

By [4, Theorem 6] the convex cone G is the dense interior of the cone Γ.

Theorem 1 [4, Theorem 7]**.** *Let $T \subset \mathbb{R}^n$ be a closed regular set, $I \subset \mathbb{Z}_+^n$ a finite regular set and $g = (g_i)_{i\in I}$ a set of numbers with $g_0 = 1$. Then $g \in G \Leftrightarrow \varphi_g\, p > 0$ for every $p \in P_I \setminus \{0\}$ such that $p(t) \geqslant 0$ for all $t \in T$.*

Theorem 2 [8, Corollary 2.6]**.** *Let $\mathcal{T}$ be a space with finite measure $\mu \geqslant 0$, $1 \leqslant p \leqslant \infty$ and $a_i \in L^q(\mu)$, $g_i \in \mathbb{R}$ for $i \in I$ = finite where $1/p + 1/q = 1$. Let $\varphi\colon\ \mathbb{R} \to (-\infty, \infty]$ be proper, convex, lower semicontinuous with $\varphi|_{(0,\infty)} < \infty$. If there are $x \in L^p(\mu)$, $x > 0$ a.e. such that $\varphi \circ x \in L^1(\mu)$ and $\int_{\mathcal{T}} a_i x \,\mathrm{d}\mu = g_i$, then the quantities*

$$P = \inf\bigg\{ \int_{\mathcal{T}} \varphi(x(t)) \,\mathrm{d}\mu(t)\colon\ x \in L^p(\mu), x \geqslant 0 \text{ a.e.},\ \varphi \circ x \in L^1(\mu), \int_{\mathcal{T}} a_i x \,\mathrm{d}\mu = g_i\ \forall i \bigg\},$$

$$D = \max\left\{\sum_{i\in I} g_i\lambda_i - \int_T \varphi^*\Big(\sum_{i\in I}\lambda_i a_i(t)\Big)\,\mathrm{d}\mu(t)\colon\ \lambda_i\in\mathbb{R},\ \varphi^*\circ\sum_{i\in I}\lambda_i a_i\in L^1(\mu)\right\}$$

are equal, $-\infty\leqslant P = D < \infty$ and the maximum D is attained.

Theorem 3 is a reminiscent to [3, Theorem 4], where $\int_T \mathrm{f}\ln\mathrm{f}\varrho\,\mathrm{d}t$ is minimized subject to $\int_T t^i\mathrm{f}\varrho\,\mathrm{d}t = g_i$ under stronger hypotheses on ϱ, like $\varrho(t)\sim \mathrm{e}^{-\varepsilon\|t\|^p}$ with $p > 2k$ (to fit the notation in [3], let $a = 1$ and our $f := \varrho\mathrm{f}$, whence $L_{\varrho,a,g}(\lambda) = L(\lambda-\lambda_0)+1$, where $\lambda_0 = (\lambda_{0i})_{i\in I}$ with $\lambda_{0i} = \delta_{i,0}$ and $\delta_{i,j}$ is Kronecker's symbol, $\delta_{i,j} = 1$ if $i = j$ and 0 if $i\neq j$). Although we do not obtain here the existence of a maximum entropy solution f_*, our present hypotheses on ϱ are weaker and condition $g\in G$ is characterized in Lagrangian terms.

Theorem 3. *Let $T\subset\mathbb{R}^n$ be a closed regular set. Let $I\subset\mathbb{Z}_+^n$ be a finite regular set such that $\max\limits_{i\in I}|i| = 2k$ where $k\in\mathbb{N}$. Assume $2k\mathrm{e}_\iota\in I$ $(1\leqslant\iota\leqslant n)$. Let $g = (g_i)_{i\in I}$ be a set of numbers with $g_0 = 1$. Fix $\varrho\in L^1(T,\mathrm{d}t)$, $\varrho > 0$ a.e. The following statements* (a) *and* (b) *are equivalent:*

(a) *There exist functions $f\in L^1_+(T,\mathrm{d}t)$ such that $\int_T |t^i|f(t)\,\mathrm{d}t<\infty$ and*

$$\int_T t^i f(t)\,\mathrm{d}t = g_i\quad (i\in I).$$

(b) *The functional $L\colon\ \mathbb{R}^N\to\mathbb{R}\cup\{-\infty\}$ defined by*

$$L(\lambda) = \sum_{i\in I} g_i\lambda_i - \int_T \mathrm{e}^{\sum_{i\in I}\lambda_i t^i}\varrho(t)\,\mathrm{d}t,\qquad \lambda = (\lambda_i)_{i\in I}$$

is bounded from above and $\sup L$ is attained at a (unique) point λ^*.

P r o o f. Since $L(0) > -\infty$, $L\not\equiv-\infty$. Since $g_0 = 1$, each of the conditions (a) and (b) implies that T has positive Lebesgue measure, finite or not. Hence by means of Jensen's inequality one can show that L is strictly concave. Then whenever $\sup L$ is finite and attained at some point λ^*, this λ^* is unique.

(a) $\Rightarrow$ (b) The regularity condition on T is not necessary for this implication. Let $\mu = \tilde\varrho\,\mathrm{d}t$ be the measure on T with density $\tilde\varrho := \varrho\mathrm{e}^{-\sum_{\iota=1}^n t_\iota^{2k}}$. Then $0<\mu(T)<\infty$. Since (1) has a solution f, hence $\tilde f := f/\tilde\varrho$ satisfies

$$\int_T t^i\tilde f(t)\,\mathrm{d}\mu(t) = g_i\quad (i\in I). \tag{2}$$

By [8, Theorem 2.9], see also [4, Lemma 4] for $\beta = 0$, problem (2) has also a solution $f_0\in L^\infty(T)$ with $f_0 > 0$ a.e. The conclusion $\sup L<\infty$ may hold either directly by Theorem 2, or by an elementary argument as shown below. Let $x = f_0(t)$ a.e. and $y = \|f_0\|_\infty + 1$ in the inequalities $-\mathrm{e}^{-1} \leqslant x \ln x \leqslant y \ln y$ for $0 \leqslant x \leqslant y$, $y \geqslant 1$, then integrate with respect to μ. Hence $f_0 \ln f_0 \in L^1(T, \mu)$. Fix $\lambda = (\lambda_i)_{i \in I}$. Let $x = f_0(t)$ and $y = \sum\limits_{i \in I} \lambda_i t^i$ in the simple version $x \ln x - x \geqslant xy - \mathrm{e}^y$ of Fenchel's inequality [27], then integrate. It follows, using (2) for f_0, that

$$\int_T f_0 \ln f_0 \,\mathrm{d}\mu - \int_T f_0 \,\mathrm{d}\mu \geqslant \sum_{i \in I} g_i \lambda_i - \int_T \mathrm{e}^{\sum_{i \in I} \lambda_i t^i} \,\mathrm{d}\mu(t) = L(\lambda - \lambda_0) + \sum_{i \in I} g_i \lambda_{0i}$$

where $\lambda_0 = (\lambda_{0i})_{i \in I}$ with $\lambda_{0i} = \sum\limits_{\iota=1}^{n} \delta_{i,\, 2k\mathrm{e}_\iota}$ and $\delta_{i,j}$ is Kronecker's symbol. Since λ was arbitrary, we get $\sup\limits_{\lambda} L(\lambda) < \infty$. Now for the attainment $\sup L = \max L$, we need Theorem 2 as follows. Use $|t_j| \leqslant \left(\sum\limits_{\iota=1}^{n} t_\iota^{2k} \right)^{1/2k}$,

$$|t^i| = |t_1|^{i_1} \ldots |t_n|^{i_n} \leqslant \left(\sum_{\iota=1}^{n} t_\iota^{2k} + 1 \right)^{|i|/2k} \leqslant \sum_{\iota=1}^{n} t_\iota^{2k} + 1 \qquad (|i| \leqslant 2k)$$

and $\nu + 1 \leqslant \mathrm{e}^\nu$ for $\nu = \sum\limits_{\iota=1}^{n} t_\iota^{2k}$ to get $\int_T |t^i| \,\mathrm{d}\mu(t) \leqslant \int_T \varrho \,\mathrm{d}t < \infty$ for $i \in I$. Then let $\mathcal{T} = T$, the measure $\mu = \tilde{\varrho} \,\mathrm{d}t$, $p = \infty$, the moment functions $a_i(t) = t^i$ and the integrand φ be defined by $\varphi(x) = x \ln x$ for $x > 0$, $\varphi(0) = 0$ and $\varphi(x) = +\infty$ for $x < 0$. The feasibility hypotheses is fulfilled by $x = f_0$. The convex conjugate $\varphi^*(y) = \sup\limits_{x \geqslant 0}(xy - x \ln x)$ of φ is given by $\varphi^*(y) = \mathrm{e}^{y-1}$ for $y \in \mathbb{R}$. We get the attainment $D = \sup \mathcal{L}$ for $\mathcal{L}(\lambda) = L(\lambda - \lambda'_0) + \sum\limits_{i \in I} g_i \lambda'_{0i}$ where $\lambda'_0 = (\lambda'_{0i})_{i \in I}$ with $\lambda'_{0i} = \lambda_{0i} + \delta_{i,0}$. Thus we obtain a λ^* such that $\sup L = L(\lambda^*)$.

(b) $\Rightarrow$ (a) Let $\lambda^* \in \mathbb{R}^N$ be such that $\sup L = L(\lambda^*)$. We prove that φ_g satisfies the positivity condition in Theorem 1. Let $p = \sum\limits_{i \in I} \lambda_i X^i$, $p \not\equiv 0$ be arbitrary such that $p(t) \leqslant 0$ for $t \in T$. We show that $\varphi_g p < 0$. The vector $\lambda := (\lambda_i)_{i \in I}$ is $\neq 0$. For any $r > 0$, set $e_r(t) = \mathrm{e}^{r \sum_{i \in I} \lambda_i t^i}$. Thus $e_r(t) \leqslant 1$ for $t \in T$. Then the integral term $\int_T e_r \varrho \,\mathrm{d}t$ of $L(r\lambda) = r \sum\limits_{i \in I} g_i \lambda_i - \int_T e_r \varrho \,\mathrm{d}t$ remains bounded as $r \to \infty$. Hence $\varphi_g p = \sum\limits_{i \in I} g_i \lambda_i \leqslant 0$, for otherwise the linear term $r \varphi_g p$ of $L(r\lambda)$ would give $\sup L = \infty$ which is false. Assume that $\varphi_g p = 0$. Then the restriction of the function L to the half-line $l := \{r\lambda \colon r > 0\}$ is given by the function $r \mapsto -\int_T e_r \varrho \,\mathrm{d}t$. This function is finite, bounded and strictly monotonically increasing on $(0, \infty)$. Use to this aim that $0 < e_r \leqslant 1$, $\int_T \varrho \,\mathrm{d}t < \infty$, $e_r = \mathrm{e}^{rp}$ with $p \leqslant 0$ and $L|_l$ is strictly concave. Then a finite limit $\lim_{r\to\infty} L(r\lambda) = \sup_l L$ exists, in particular $\sup_{r\geqslant 1} |L(r\lambda)| < \infty$. For $a > 0$,

$$\infty > L(\lambda^* + a\lambda) = \sum_{i\in I} g_i\lambda_i^* + a\sum_{i\in I} g_i\lambda_i - \int_T \mathrm{e}^{\sum_{i\in I}\lambda_i^* t^i}\mathrm{e}^{a\sum_{i\in I}\lambda_i t^i}\varrho(t)\,\mathrm{d}t$$
$$\geqslant \sum_{i\in I} g_i\lambda_i^* + r\cdot 0 - \int_T \mathrm{e}^{\sum_{i\in I}\lambda_i^* t^i}\varrho(t)\,\mathrm{d}t = L(\lambda^*) = \max L \geqslant L(0) > -\infty$$

because $\sum_{i\in I} g_i\lambda_i = 0$ and $\sum_{i\in I}\lambda_i t^i \leqslant 0$ for all $t \in T$. Hence L is finite at every point of the half-line $\{\lambda^* + a\lambda\}_{a>0}$. Note that λ^* cannot be colinear with λ due to the behaviour of L on l: namely, $\lambda^* \notin l$ because L reaches its global maximum only in λ^* while $L|_l$ increases strictly along l as $r \to \infty$. Also $\lambda^* \notin \{0\} \cup (-l)$, for otherwise the concavity of the restriction $L|_{\mathbb{R}\lambda}\colon \mathbb{R}\lambda \to \{-\infty\}\cup\mathbb{R}$ of L to the line $\mathbb{R}\lambda$ would imply, for some $r \geqslant 0$ with $\lambda^* = -r\lambda$, that $L(r\lambda) \geqslant L(0) = L(\frac{1}{2}(\lambda^* + r\lambda)) \geqslant \frac{1}{2}(L(\lambda^*) + L(r\lambda))$, whence $L(\lambda^*) \leqslant L(r\lambda) < \sup L|_l \leqslant \sup L = L(\lambda^*)$, which is impossible. Thus $\lambda^* \notin \mathbb{R}\lambda$. Then a 2-dimensional drawing shows that for every $r > 1$ there is a unique point x_r of intersection of the segments $(\lambda^*, r\lambda)$ and $(\lambda, \lambda^* + \lambda)$. Write to this aim $x_r = s\lambda^* + (1-s)r\lambda = s'\lambda + (1-s')(\lambda^* + \lambda)$ with coefficients $s = s_r$, $s' = s'_r$, use the linear independence of λ^*, λ and get $s = (r-1)/r$, $s' = 1 - s$ whence s, $s' \in (0,1)$ and $\lim_{r\to\infty} s'_r = 0$. Then $\lim_{r\to\infty} x_r = \lambda^* + \lambda$. The concavity (and hence, continuity [27]) of L on the segment $(\lambda, \lambda^* + \lambda]$ gives $\lim_{r\to\infty} L(x_r) = L(\lambda^* + \lambda) < L(\lambda^*)$ with strict inequality, because the point λ^* of maximum of L is unique. But $L(x_r) = L(s\lambda^* + (1-s)r\lambda) \geqslant sL(\lambda^*) + (1-s)L(r\lambda)$ and letting $r \to \infty$ we derive, using $\lim_{r\to\infty} s_r = 1$ and $\sup_{r\geqslant 1}|L(r\lambda)| < \infty$, that $\lim_{r\to\infty} L(x_r) \geqslant L(\lambda^*)$. We got a contradiction. Hence $\varphi_g p < 0$. The feasibility of problem (1) follows then by Theorem 1. □

R e m a r k s. Since λ^* may be on the boundary of $\operatorname{dom} L := \{\lambda\colon L(\lambda) > -\infty\}$, one cannot prove (b) ⇒ (a) by differentiating under the integral in λ^*, and the h-minimization may fail [17]. Additional hypotheses may compel λ^* to be interior to $\operatorname{dom} L$ [16] in which case the entropy minimization can be obtained [24], providing the particular solution $f_*(t) = \mathrm{e}^{\sum_{i\in I}\lambda_i^* t^i}$, see for instance [3]. For example let $T = \mathbb{R}^n$, $I = \{i\colon |i| \leqslant 2k\}$ and $\varrho(t) = \mathrm{e}^{-\|t\|^{2k}}$. By Theorem 3, problem (1) is feasible if and only if L is bounded from above and attains its maximum at a point λ^*, even when a minimum entropy solution does not exist. By Fatou's lemma and Lebesgue's dominated convergence theorem, $f_0 := \mathrm{e}^{\sum_{|i|\leqslant 2k}\lambda_i^* t^i}$ has finite moments of order $\leqslant 2k$, we can get $\int t^i f_0\,\mathrm{d}t = g_i$ for $|i| < 2k$ and $\int t_\iota^{2k} f_0\,\mathrm{d}t \leqslant g_{2k\mathrm{e}_\iota}$ $(1 \leqslant \iota \leqslant n)$, but the equalities (1) may fail for $|i| = 2k$ [17]. By integration in polar coordinates, the homogeneous polynomial $p := \sum_{|i|=2k}\lambda_i^* X^i$ is shown to always satisfy $p(t) \leqslant 0$ on $\mathbb{R}^n$;

if moreover $p(t) < 0$ for all $t \neq 0$, then λ^* is interior to $\operatorname{dom} L$ and f_0 is indeed a solution of problem (1), $f_0 = f_*$. We omit the details and refer the reader to [16], [17].

Note also that whenever ϱ is at our disposal, various choices may be tried [3] to facilitate the numerical maximization of $L = L_\varrho$.

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