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ON NON-PERIODIC GROUPS WHOSE FINITELY GENERATED SUBGROUPS ARE EITHER PERMUTABLE OR PRONORMAL

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Abstract. The current article considers some infinite groups whose finitely generated subgroups are either permutable or pronormal. A group G is called a generalized radical, if G has an ascending series whose factors are locally nilpotent or locally finite. The class of locally generalized radical groups is quite wide. For instance, it includes all locally finite, locally soluble, and almost locally soluble groups. The main result of this paper is the following

Theorem. Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable. If G is non-periodic then every subgroup of G is permutable.

Keywords: pronormal subgroup, permutable subgroup, finitely generated subgroup, abnormal subgroup

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0. Introduction

One of the oldest and most efficient approaches to the investigation of groups consists in the studying of groups whose all proper subgroups possess some general property. At the early age of group theory, this approach was introduced in the classical works of R. Dedekind [2], G. A. Miller, H. Moreno [17], and O. Yu. Schmidt [22]. The cited papers were crucial for the farther development of group theory. They had numerous continuations in which the families of the subgroups satisfying the imposed general properties became proper subfamilies of all subgroups. This remark is valid for finite groups (see, for example, the book of L. A. Shemetkov [24, Chapter VI]), for infinite groups (see, for example, the survey [3]). However, we should admit that the case when all subgroups of a group possess the same is quite rare. More often we deal with the cases when subgroups of a group possess different, sometime even opposite properties. For example, a group simultaneously can have subnormal and selfnormalizing subgroups, normal and abnormal subgroups, and so on. It is logical to consider the case when subgroups of a group split into two classes (families) each of which possess one of two given properties. One of the first articles adopting this approach was the article of A. Fattahi [7] in which the finite groups whose subgroups are either normal or abnormal have been described. Recall that a subgroup H of a group G is *abnormal* if for every element g of G the subgroup $\langle H, H^g \rangle$ contains this element g (see for example, [24, Definition 17.1]). G. Ebert and S. Bauman [4], generalizing the results of A. Fattahi, considered finite groups all subgroups of which are either subnormal or abnormal. Infinite groups with this property have been later studied by M. De Falco, L. A. Kurdachenko and I. Ya. Subbotin [5]. In this paper [5], the infinite groups whose all subgroups are subnormal or contranormal have been also considered. Following J. Rose [21], we call a subgroup H of a group G *contranormal* in G if $H^G = G$. L. A. Kurdachenko and H. Smith [10] described the groups in which all subgroups are either subnormal or selfnormalizing.

Abnormal subgroups are a partial case of pronormal subgroups. A subgroup H of a group G is called *pronormal* in G if for every element g of G the subgroups H and H^g conjugate in $\langle H, H^g \rangle$ (see for example, [24, Definition 17.1]). Observe that there is no such significant distinction between subnormal and pronormal subgroups as we have noted for subnormal and abnormal subgroups. A pronormal subgroup can be subnormal, and in this case, it is normal. In the articles of P. Legovini [15], [16], the author obtained some results about the finite groups with only subnormal and pronormal subgroups. In the paper [11], some infinite groups of this kind have been considered.

Recall that a subgroup H of a group G is called *permutable* if $HK = KH$ for each subgroup K of G. The study of permutable subgroups has been continued for considerable time, and many interesting results have been obtained here (see, for example, the book [23]). Observe that a permutable subgroup is ascendant [25]. It is logical to investigate the groups whose subgroups are either permutable or pronormal. Thus the case where groups whose finitely generated subgroups are either permutable or pronormal seems to be an interesting subject for research. Note that in the groups whose finitely generated subgroups are permutable every subgroup is permutable. The groups whose finitely generated subgroups are pronormal have been studied by I. Ya. Subbotin and Kuzennyi [14].

In the paper [12], the study of groups whose finitely generated subgroups are either permutable or pronormal was initiated. More concretely, the authors described the locally finite groups whose finitely generated subgroups are either permutable or pronormal. Note that this result cannot be extended to arbitrary periodic groups.

A. Yu. Olshanskij [18, Theorem 28.2] constructed an example of an infinite p-group G, where p is a big enough prime, whose proper subgroups have order p. Clearly, every subgroup of G is pronormal. The current article is a continuation of [12]. Here we consider some infinite groups whose finitely generated subgroups are either permutable or pronormal. In the non-periodic case we need additional restrictions. Indeed, A. Yu. Olshanskij [18, Theorem 28.3] constructed an example of a simple infinite torsion-free group G whose proper subgroups are cyclic. Every subgroup of this group is pronormal. Therefore in the current article, we employ the following quite weak restriction. We recall that a group G is called *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group G has either an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathbf{Lnr}(G)$ of G is non-identity. In the second case, G includes a non-identity normal locally finite subgroup. Clearly, in every group G the subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups (the *locally finite radical*) is the largest normal locally finite subgroup. Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. Observe also that the class of locally generalized radical groups is quite wide. For instance, it includes all locally finite, locally soluble, and almost locally soluble groups. Observe that a periodic generalized radical group is locally finite, and hence a periodic locally generalized radical group is locally finite.

Our main result in this paper is the following

Theorem. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable. If G is non-periodic then every subgroup of G is permutable.*

1. Preliminary results

We will need the following preliminary results about the groups whose finitely generated subgroups are either pronormal or permutable. Note at once, that if G is a group whose subgroups (finitely generated subgroups) are either pronormal or permutable, then the same property holds for every subgroup H of G and every factor-group of G, and therefore for every section of G.

If G is a group, then put $\mathbf{Gr}(G) = \langle g \in G\colon \ \langle g\rangle \text{ is an ascendant subgroup of } G\rangle$.

A subgroup $\mathbf{Gr}(G)$ is called the *Gruenberg radical of group G*.

Note that a subgroup $\mathbf{Gr}(G)$ is locally nilpotent (see, [6]). In particular, the locally nilpotent radical includes $\mathbf{Gr}(G)$. If $G = \mathbf{Gr}(G)$, then G is called a *Gruenberg group.*

Suppose that G is a group whose finitely generated subgroups are either pronormal or subnormal. If a cyclic subgroup $\langle g\rangle$ of G is not pronormal, then $\langle g\rangle$ is permutable in G. It follows that $\langle g\rangle$ is ascendant in G [25, Theorem 1]. Then $\mathbf{Gr}(G) \neq \langle 1\rangle$, in particular, the locally nilpotent radical of G is non-identity.

Lemma 1.1. *Let G be a group whose finitely generated subgroups are either pronormal or subnormal. If L is a locally nilpotent radical of G, then every cyclic subgroup of G/L is pronormal in G/L.*

Proof. Suppose that $x \notin L$. If we suppose that $\langle x\rangle$ is permutable in G, then $\langle x\rangle$ is ascendant in G [25, Theorem 1], in particular, $x \in L$. This contradiction shows that a cyclic subgroup $\langle x\rangle$ is pronormal. Then $\langle xL\rangle$ is pronormal in G/L. □

The next result, which has been proved in [13], is very useful.

Lemma 1.2. *Let G be a locally nilpotent group. If H is a pronormal subgroup of G, then H is normal in G.*

We will need some results about the groups whose cyclic subgroups are pronormal. Finite groups with this property have been studied by T. A. Peng [19]. Locally soluble groups whose cyclic subgroups are pronormal have been studied by N. F. Kuzenny and I. Ya. Subbotin [14]. Now we want to extend the main result of the paper [14].

Proposition 1.3. *Let G be a locally generalized radical group whose cyclic subgroups are pronormal.*

(i) *If G is not periodic, then G is abelian.*

(ii) *If G is periodic, then G includes an abelian normal subgroup L satisfying the following conditions:*
every subgroup of L is G-invariant,
G/L is a Dedekind group,
$2 \notin \mathbf{\Pi}(L)$,
$\mathbf{\Pi}(L) \cap \mathbf{\Pi}(G/L) = \emptyset$.

Proof. Let K be an arbitrary finitely generated subgroup of G. Being a generalized radical group, K has an ascending series whose factors are locally nilpotent or locally finite. Let U, V be normal subgroups of K such that $U \leqslant V$ and V/U is locally finite. It is not hard to see that every cyclic subgroup of V/U is pronormal in V/U. Let F/U be a finite subgroup of V/U. Then F/U is metabelian [19]. It follows that V/U is metabelian. Hence K is the radical. Let L be the locally nilpotent radical of K. By Lemma 1.2, every cyclic subgroup of L is normal in L. It follows that every cyclic subgroup of L is subnormal in K. Being pronormal, every cyclic subgroup of L is normal in K. Then $K/C_K(L)$ is abelian (see, for example, [23, Theorem 1.5.1]). Since K is the radical, $C_K(L) \leqslant L$ [20, Lemma 4], so that K/L is abelian. In particular, K is soluble and hence G is locally soluble. Now we can apply the main result of [14]. $\square$

Corollary 1.4. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable and let B be a locally nilpotent radical of G.*

(i) *If G/B is not periodic, then G/B is abelian.*

(ii) *If G/B is periodic, then G/B includes an abelian normal subgroup L/B such that every subgroup of L/B is G-invariant, G/L is a Dedekind group, $2 \notin (L/B)$, $\mathbf{\Pi}(L/B) \cap \mathbf{\Pi}(G/L) = \emptyset$.*

P r o o f. Indeed, by Lemma 1.1, every cyclic subgroup of G/B is pronormal, and we may apply Proposition 1.3. $\square$

Lemma 1.5. *Let G be a group whose finitely generated subgroups are either pronormal or permutable. If H is a locally nilpotent subgroup of G, then every subgroup of H is permutable in H. In particular, H is metabelian.*

P r o o f. Indeed, consider an arbitrary cyclic subgroup $\langle x \rangle$ of H. Suppose that $\langle x \rangle$ is not permutable in G. It follows that $\langle x \rangle$ is pronormal in G. Consider an arbitrary finitely generated subgroup F of H containing x. Then F is nilpotent, and hence $\langle x \rangle$ is subnormal in F. Being pronormal and subnormal, $\langle x \rangle$ is normal in F. Since this is true for each finitely generated subgroup F of H, $\langle x \rangle$ is normal in H. Thus in any case, $\langle x \rangle$ is permutable in H. Hence every cyclic subgroup of H is permutable in H. It follows that every subgroup of H is permutable in H. The last assertion follows from the description of groups whose subgroups are permutable (see, for example, [23, Lemma 2.4.10, Theorem 2.4.11 and Theorem 2.4.12]). $\square$

Corollary 1.6. *Let G be a group whose finitely generated subgroups are either pronormal or permutable. If H is an ascendant locally nilpotent subgroup of G, then every subgroup of H is permutable in G. In particular, every subgroup of the locally nilpotent radical of G is permutable in G.*

P r o o f. Indeed, consider an arbitrary cyclic subgroup $\langle x \rangle$ of H. Then $\langle x \rangle$ is either permutable or pronormal in G. If $\langle x \rangle$ is pronormal in G, then $\langle x \rangle$ is pronormal in H and, by Lemma 1.4, $\langle x \rangle$ is normal in H. It follows that $\langle x \rangle$ is ascendant in G. But an ascendant pronormal subgroup is normal, so that $\langle x \rangle$ is normal in G. Thus every cyclic subgroup of H is permutable in G. It follows that every subgroup of H is permutable in G. $\square$

Lemma 1.7. *Let G be a group and g an element of G. Suppose that A is a $\langle g\rangle$-invariant subgroup of G such that $A \cap \langle g\rangle = \langle 1\rangle$ and $a \in A$. If $\langle g\rangle\langle a\rangle = \langle a\rangle\langle g\rangle$, then $\langle a\rangle^{\langle g\rangle} = \langle a\rangle$.*

Proof. Put $L = \langle a\rangle^{\langle g\rangle}$. Then it is not hard to see that $K = \langle g, a\rangle = L\langle g\rangle$. The equation $\langle g\rangle\langle a\rangle = \langle a\rangle\langle g\rangle$ implies

$$L = L \cap (\langle a\rangle\langle g\rangle) = \langle a\rangle(L \cap \langle g\rangle) = \langle a\rangle.$$

□

2. Proof of the main theorem

Lemma 2.1. *Let G be a group whose finitely generated subgroups are either pronormal or permutable and let g be an element of G such that $\langle g\rangle$ is pronormal in G (in particular, it happens if g does not belong to the locally nilpotent radical of G). Suppose that G contains a $\langle g\rangle$-invariant torsion-free abelian subgroup B. If the element $gC_G(B)$ has finite order, then $g \in C_G(B)$.*

Proof. We can suppose that $gC_G(B)$ is a p-element for some prime p. Then $g^k \in C_G(B)$ for some $k = p^m$. Put $C = C_B(g)$. Suppose the contrary and let $C \neq B$. Observe that the subgroup C is $\langle g\rangle$-invariant. Let b be an element of B such that $b^m \in C$ for some $m \in \mathbb{N}$. Put $b_1 = b^g$, then

$$b_1^m = (b^g)^m = (b^g)^m = (b^m)^g = b^m.$$

Since a subgroup B is torsion-free abelian, it follows that $b^g = b_1 = b$, that is $b \in C$. In other words, the factor-group B/C is torsion-free. Let $u \in B \setminus C$, $U = \langle u\rangle^{\langle g\rangle}$. Clearly U is a finitely generated subgroup and by the above $U/(U \cap C)$ is torsion-free. Being finitely generated, $U/(U \cap C)$ is free abelian. In particular, $U \cap C$ has a complement in U. Then U includes a $\langle g\rangle$-invariant subgroup V such that $V \cap (U \cap C) = \langle 1\rangle$, and $V(U \cap C)$ has finite index in U [9, Corollary 5.10]. In particular, V is non-identity. Since $g \in D$, a subgroup $\langle g\rangle$ is pronormal. Thus $\langle g\rangle$ is pronormal in $V\langle g\rangle$. Put $V_1 = V^p$, $V_2 = V_1^p$, $V_{n+1} = V_n^p$, $n \in \mathbb{N}$. Then V/V_n is a finite p-group and $(gV_n)^k \in \zeta(\langle V/V_n, gV_n\rangle)$. It follows that $\langle V/V_n, gV_n\rangle/\zeta(\langle V/V_n, gV_n\rangle)$ is a p-group. Therefore $\langle V/V_n, gV_n\rangle$ is nilpotent. Being pronormal in $\langle V/V_n, gV_n\rangle$, $\langle gV_n\rangle$ is normal in $\langle V/V_n, gV_n\rangle$. It follows that $[V, g] \leqslant \langle g\rangle V_n$. Since this is valid for all $n \in \mathbb{N}$, $[V, g] \leqslant \bigcap_{n\in\mathbb{N}} \langle g\rangle V_n$. The equality $\langle 1\rangle = V \cap (U \cap C)$ implies that $V \cap \langle g\rangle = \langle 1\rangle$. Since $\bigcap_{n\in\mathbb{N}} V_n = \langle 1\rangle$, it follows that $\bigcap_{n\in\mathbb{N}} \langle g\rangle V_n = \langle g\rangle$. Hence $[V, g] \leqslant \langle g\rangle$.

On the other hand, V is $\langle g\rangle$-invariant, so that $[V, g] \leqslant V$. Thus $[V, g] \leqslant V \cap \langle g\rangle = \langle 1\rangle$. In turn, it follows that $V \leqslant C$, and we obtain a contradiction. This contradiction proves the equality $A = C_A(g)$, which is required. $\square$

If G is a group, then by $\mathbf{Tor}(G)$ we denote the *maximal normal periodic subgroup of* G. We recall that if a group G is locally nilpotent, then $\mathbf{Tor}(G)$ contains all elements of finite order, so that $G/\mathbf{Tor}(G)$ is torsion-free in this case.

Corollary 2.2. *Let G be a group whose finitely generated subgroups are either pronormal or permutable, and let D be the locally nilpotent radical of G. Suppose that $D \neq \mathbf{Tor}(D)$ and G/D is locally finite. Then $G/\mathbf{Tor}(D)$ is abelian.*

P r o o f. Put $T = \mathbf{Tor}(D)$. Then D/T is abelian (see, for example, [23, Lemma 2.4.10 and Theorem 2.4.11]). Suppose that $\zeta(G/T)$ does not contain D/T. Then there exist elements $x \in D \setminus T$ and $g \in G$ such that $gT \notin C_{G/T}(xT)$. Since D/T is abelian, $D/T \leqslant C_{G/T}(xT)$, which shows that $g \notin D$. Put $X/T = \langle xT\rangle^{\langle gT\rangle}$. Then X/T is torsion-free and $\langle g\rangle$-invariant. Since G/D is locally finite, Lemma 2.1 shows that $gT \in C_{G/T}(X/T)$, in particular, $gT \in C_{G/T}(xT)$. This contradiction proves that $D/T \leqslant \zeta(G/T)$. By Corollary 1.4, G/D contains a normal abelian subgroup L/D such that G/L is a Dedekind group. The inclusion $D/T \leqslant \zeta(G/T)$ implies that L/T is nilpotent. Let yT be an arbitrary element of L/T having infinite order. If $yT \in D/T$, then by the above, $yT \in \zeta(G/T)$. Assume that $y \in L \setminus D$. Then $\langle y\rangle$ is pronormal in G. Hence $\langle yT\rangle$ is pronormal in G/T. The fact that L/T is nilpotent implies that $\langle yT\rangle$ is subnormal in L/T, and hence in G/T. Being simultaneously subnormal and pronormal, $\langle yT\rangle$ is normal in G/T. Using the above argument, we again obtain that $yT \in \zeta(G/T)$. Now we observe that every nilpotent non-periodic group is generated by its elements having infinite order. It follows that $L/T \leqslant \zeta(G/T)$. Since G/L is a Dedekind group, G/T is nilpotent. Now we can repeat the above argument and obtain that G/T is abelian $\square$

Corollary 2.3. *Let G be a group whose finitely generated subgroups are either pronormal or permutable and let D be the locally nilpotent radical of G. Suppose that $D \neq \mathbf{Tor}(D)$ and G/D is abelian. Then $G/\mathbf{Tor}(D)$ is abelian.*

P r o o f. Put $T = \mathbf{Tor}(D)$. Then D/T is abelian (see, for example, [23, Lemma 2.4.10 and Theorem 2.4.11]). Let $R/D = \mathbf{Tor}(G/D)$. If $R = G$, then the result follows from Corollary 2.2. Therefore suppose that $G \neq R$. Lemma 2.1 shows that $D/T \leqslant \zeta(R/T)$. By Corollary 1.6, every subgroup of D/T is permutable in G/T. If $g \in G \setminus R$, then gR has infinite order, in particular, gD has infinite order. It follows that $\langle gT\rangle \cap D/T = \langle 1\rangle$. If $d \in D \setminus T$, then by Lemma 1.7, subgroup $\langle dT\rangle$ is $\langle gT\rangle$-invariant. It follows that $(xT)^2 \in C_{G/T}(D/T)$ for each element $x \in G$. Then

$(G/T)/C_{G/T}(D/T)$ is periodic and Lemma 2.1 implies the inclusion $D/T \leqslant \zeta(G/T)$. By repeating the argument of the proof of Corollary 2.2 we can obtain that G/T is abelian. □

Corollary 2.4. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable and let D is the locally nilpotent radical of G. If $D \neq \mathbf{Tor}(D)$, then $G/\mathbf{Tor}(D)$ is abelian.*

Proof. Put $T = \mathbf{Tor}(D)$. By Corollary 1.6, G/D is either locally finite or abelian. If G/D be locally finite, then the result follows from Corollary 2.2. If G/D is abelian, then the result follows from Corollary 2.3. □

Lemma 2.5. *Let G be a group and suppose that G includes a normal periodic abelian subgroup T such that G/T is abelian and non-periodic. If every finitely generated subgroup of G is either pronormal or permutable, then every subgroup of T is G-invariant.*

Proof. By Corollary 1.4, every subgroup of T is permutable in G. Let x be an arbitrary element of T and g an element having infinite order. Then $\langle g\rangle \cap T = \langle 1\rangle$, and Lemma 1.7 implies that $\langle x\rangle$ is a $\langle g\rangle$-invariant subgroup. Since G/T is non-periodic, it is generated by all its elements having infinite order. It follows that $\langle x\rangle$ is a G-invariant subgroup. Since every cyclic subgroup of T is G-invariant, every subgroup of T is G-invariant. □

Lemma 2.6. *Let G be a group and suppose that G includes a normal abelian p-subgroup T, and p is a prime such that G/T is abelian and non-periodic. Suppose that every finitely generated subgroup of G is either pronormal or permutable. If L is the locally nilpotent radical of G, then G/L is a finite cyclic group of order dividing $p-1$.*

Proof. By Lemma 2.5, every subgroup of T is G-invariant.

Put $L = \mathbf{\Omega}_1(T) = \{a \in T;\ a^p = 1\}$. Let $C = C_G(L)$. Then G/C is a cyclic group of order $p-1$ (see, for example, [23, Theorem 1.5.6]). We observe that the mapping $z \mapsto z^p$, $z \in T$, is a G-endomorphism of T, and therefore the factor $\mathbf{\Omega}_2(T)/\mathbf{\Omega}_1(T)$ is C-central. Similarly, the factor $\mathbf{\Omega}_{n+1}(T)/\mathbf{\Omega}_n(T)$ is also C-central for each $n \in \mathbb{N}$. Since G/T is abelian, it follows that C is hypercentral. Let K be the locally nilpotent radical of G. Then $C \leqslant K$, and hence G/K is finite of order dividing $p-1$. □

Corollary 2.7. *Let G be a group and suppose that G includes a normal abelian 2-subgroup T such that G/T is abelian and non-periodic. If every finitely generated subgroup of G is either pronormal or permutable, then G is locally nilpotent.*

Lemma 2.8. *Let G be a group and L a locally nilpotent radical of G. Suppose that L includes a G-invariant abelian p-subgroup T, where p is a prime satisfying the following conditions:*

G/T is abelian and non-periodic;

every subgroup of $\mathbf{Tor}(L)$ *is G -invariant;*

$\mathbf{r}_0(G/T) \geqslant 2$.

If every finitely generated subgroup of G is either pronormal or permutable, then G is abelian.

Proof. By Lemma 2.6, G/L is a finite cyclic group of order dividing $p - 1$. By Corollary 1.6, every subgroup of L is permutable in G. Then the condition $\mathbf{r}_0(G/T) \geqslant 2$ implies that L is abelian (see, for example, [23, Lemma 2.4.10]).

Let a be an arbitrary element of L. If a has finite order, then by our conditions the subgroup $\langle a \rangle$ is G-invariant. Suppose now that a has infinite order. Let g be such an element that $G/L = \langle gL \rangle$. We remark that if g has finite order, then because the order of gL divides $p-1$, we can assume that p does not divide the order of g. Hence $A = \langle a \rangle^G = \langle a \rangle^{\langle g \rangle}$. Since G/T is abelian, $\langle a \rangle^{\langle g \rangle} = \langle a \rangle E$ for some finite subgroup E of T. Since T is a p-subgroup, E is a finite p-subgroup. It follows that there is a positive integer k such that $B = A^s \leqslant \langle a \rangle$ where $s = pk$. Clearly the subgroup B is G-invariant. The factor A/B is a finite p-group. By Corollary 1.6, every subgroup of A/B is permutable in G/B. The choice of g implies that $\langle gB \rangle \cap A/B = \langle 1 \rangle$. Lemma 1.7 shows that the subgroup $\langle aB \rangle$ is $\langle gB \rangle$-invariant. The inclusion $B \leqslant \langle a \rangle$ proves that the subgroup $\langle a \rangle$ is $\langle g \rangle$-invariant. We noted above that L is abelian, so that $\langle a \rangle$ is G-invariant. Consequently, every subgroup of L is G-invariant.

Since L is not periodic, $G/C_G(L)$ is a group of order 2 (see, for example, [23, Theorem 1.5.7]). A group G is soluble, therefore $C_G(L) \leqslant L$ [20, Lemma 4], so that G/L has order at most 2. Suppose that $G \neq L$. Then $x^g = x^{-1}$ for each element $x \in L$ (see, for example, [23, Theorem 1.5.7]). Being non-periodic, L contains an element d of infinite order. Then $d \notin T$ and the equation $d^g = d^{-1}$ implies that $(dT)^{gT} = (dT)^{-1}$. On the other hand, a factor-group G/T is abelian, which implies $(dT)^{gT} = dT$. This contradiction shows that $G = L$. □

Proposition 2.9. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable and let D be the locally nilpotent radical of G. If $D \neq \mathbf{Tor}(D)$ and $\mathbf{r}_0(G) \geqslant 2$, then G is abelian.*

Proof. Put $T = \mathbf{Tor}(D)$. Then by Corollary 2.4, G/T is abelian, in particular, if $T = \langle 1 \rangle$, then G is abelian. Therefore suppose that T is non-identity. By Lemma 2.5, every subgroup of T is G-invariant.

We have $T = \underset{p\in\mathbf{\Pi}(T)}{\mathbf{Dr}} T_p$ where T_p is a Sylow p-subgroup of T. Put $Q_p = \underset{q\in\mathbf{\Pi}(T), q\neq p}{\mathbf{Dr}} T_q$. We remark that $\mathbf{Tor}(D/Q_p) = T/Q_p \cong T_pQ_p/O_p \cong_G T_p$. It follows that every subgroup of T/Q_p is G-invariant.

Let

$$L_p/Q_p = \mathbf{\Omega}_1(T/Q_p) = \{yQ_p \in T/Q_p\colon\ (yQ_p)^p = 1\},\ p \in \mathbf{\Pi}(T).$$

By Corollary 2.7, the factor-group G/Q_2 is locally nilpotent. Corollary 1.6 shows that every subgroup of G/Q_2 is permutable. The condition $\mathbf{r}_0(G/Q_2) \geqslant 2$ implies that G/Q_2 is abelian (see, for example, [23, Lemma 2.4.10]).

Let now $p \neq 2$. Every finitely generated subgroup of G/Q_p is either pronormal or permutable. Let K_p/Q_p be the locally nilpotent radical of G/Q_p. Let aQ_p be an arbitrary element of K_p/Q_p having finite order. If $aQ_p \in T/Q_p$, then by the above $\langle aQ_p\rangle$ is G-invariant. If $aQ_p \notin T/Q_p$, then $a \notin D$. It follows that the subgroup $\langle a\rangle$ is not permutable, so that $\langle a\rangle$ is pronormal in G. Then $\langle aQ_p\rangle$ is pronormal in G/Q_p. On the other hand, Corollary 1.6 shows that every subgroup of K_p/Q_p is permutable in G/Q_p. Being permutable, $\langle aQ_p\rangle$ is ascendant in G/Q_p. Now we recall that every ascendant pronormal subgroup is normal. Thus again $\langle aQ_p\rangle$ is G-invariant. Now we can apply Lemma 2.8 to the factor-group G/Q_p. By this Lemma, G/Q_p is abelian, so that $[G,G] \leqslant Q_p$ for each prime p. Clearly, $\langle 1\rangle = \bigcap_{p\in\Pi(T)} G_p$, which implies that $[G,G] = \langle 1\rangle$, i.e. G is abelian. □

Lemma 2.10. *Let G be a finite group whose subgroups are either pronormal or permutable. Then permutability is a transitive relation in G.*

Proof. Let K be a permutable subgroup of L and L be a permutable subgroup of G. Suppose that K is not permutable in G. We remark that every permutable subgroup of finite group is subnormal (see, for example, [23, Theorem 5.1.1]). Since K is not permutable, it is pronormal in G. Being pronormal and subnormal, K is normal. This contradiction shows that it permutable in G. □

Proposition 2.11. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable and let D be the locally nilpotent radical of G. If $D \neq \mathbf{Tor}(D)$ and $\mathbf{r}_0(G) = 1$, then every subgroup of G is permutable.*

Proof. Put $T = \mathbf{Tor}(D)$. Since $D \neq \mathbf{Tor}(D)$ and $\mathbf{r}_0(G) = 1$, we have $\mathbf{r}_0(D) = 1$. By Corollary 1.6, every subgroup of D is permutable in G. Then either D is abelian, or D/T is locally cyclic, T is abelian and every subgroup of T is D-invariant (see, for example, [23, Theorem 2.4.11]). By Corollary 2.4, the factor-group

G/T is abelian, in particular, if $T = \langle 1 \rangle$, then G is abelian. Therefore suppose that T is non-identity. By Lemma 2.5, every subgroup of T is G-invariant.

We have $T = \underset{p \in \mathbf{\Pi}(T)}{\mathbf{Dr}} T_p$ where T_p is a Sylow p-subgroup of T. Put $Q_p = \underset{q \in \mathbf{\Pi}(T), q \neq p}{\mathbf{Dr}} T_q$. We remark that $\mathbf{Tor}(D/Q_p) = T/Q_p \cong T_pQ_p/O_p \cong_G T_p$. It follows that every subgroup of T/Q_p is G-invariant.

Let

$$\begin{aligned} L_p/Q_p &= \mathbf{\Omega}_1(T/Q_p) = \{yQ_p \in T/Q_p\colon\ (yQ_p)^p = 1\} \\ &= \Omega_1(T/Q_p) = \mathbf{\Omega}_1(T_p)Q_p/Q_p,\ p \in \mathbf{\Pi}(T). \end{aligned}$$

By Corollary 2.7, the factor-group G/Q_2 is locally nilpotent. Corollary 1.6 shows that every subgroup of G/Q_2 is permutable. Then either G/Q_2 is abelian or hypercentral (see, for example, [23, Lemma 2.4.10]). In any case, the upper hypercenter of G/Q_2 contains T/Q_2. The isomorphism $T/Q_2 \cong T_2Q_2/O_2 \cong_G T_2$ implies that the upper hypercenter of G includes T_2.

Let now $p \neq 2$. Every finitely generated subgroup of G/Q_p is either pronormal or permutable. Suppose that the center of G/Q_p does not include L_p/Q_p. By the above every subgroup of L_p/Q_p is G-invariant. Let $C/Q_p = C_{G/Q_p}(L_p/Q_p)$, then G/C is a cyclic group of order $p-1$ (see, for example, [23, Theorem 1.5.6]). If g is the element such that $G/C = \langle gC \rangle$, then by our assumption there exists an element $aQ_p \in L_p/Q_p$ such that $\langle gQ_p, aQ_p \rangle$ is not nilpotent. Let also bQ_p be an element of C/Q_p having infinite order. Since $(G/Q_p)/(T/Q_p)$ is abelian, $\langle bQ_p, T/Q_p \rangle$ is normal. It follows that $U_p/Q_p = \langle bQ_p, aQ_p \rangle^{\langle gQ_p \rangle} = (P/Q_p)\langle bQ_p \rangle$ where P/Q_p is a finite p-subgroup. Moreover, this subgroup is nilpotent. So there exists a number $k = p^s$ such that $V_p/Q_p = (Up/Qp)^k$ is torsion-free and has finite index in U_p/Q_p; moreover, $(U_p/Q_p)/(V_p/Q_p)$ is a finite p-group. Without loss of generality we may suppose that $(V_p/Q_p)(P/Q_p) \neq U_p/Q_p$. Denote $K_p/Q_p = \langle gQ_p, aQ_p, bQ_p \rangle$ and consider the factor-group $(K_p/Q_p)/(V_p/Q_p)$. Every finitely generated subgroup of $(K_p/Q_p)/(V_p/Q_p)$ is either pronormal or permutable. Since $\mathbf{r}_0(G) = 1$, $(K_p/Q_p)/(V_p/Q_p)$ is finite, every subgroup of $(K_p/Q_p)/(V_p/Q_p) \cong (K_pV_p)/V_p$ is either pronormal or permutable. By Lemma 2.10, $(K_pV_p)/V_p$ is a PT-group. As we remarked above, G/T is abelian, therefore PV_p/V_p includes the nilpotent residual R/V_p of $(K_pV_p)/V_p$. We remark that a nilpotent residual of a finite soluble PT-group is its Hall subgroup [26]. On the other hand, the choice of Vp/Q_p implies that PV_p/V_p is not a maximal p-subgroup of $(K_pV_p)/V_p$. This contradiction shows that the center of G/Q_p includes L_p/Q_p. In turn, it follows that the upper hypercenter of G/Q_p includes T/Q_p. The isomorphism $T/Q_p \cong T_pQ_p/O_p \cong_G T_p$ implies that the upper hypercenter of G includes T_p. Since this is true for each $p \in \mathbf{\Pi}(T)$, the upper hypercenter of G includes T. As remarked above, G/T is abelian, which implies that G is hypercentral. By Corollary 1.6, the lemma is proved. □

Now we can prove the main result of this paper.

Theorem 2.12. *Let G be a locally generalized radical group whose finitely generated subgroups are either pronormal or permutable. If G is non-periodic then every subgroup of G is permutable.*

P r o o f. Let D be a locally nilpotent radical of G, and put $T = \mathbf{Tor}(D)$. Suppose first that $D \neq \mathbf{Tor}(D)$. If $\mathbf{r}_0(G) \geqslant 2$, then G is abelian by Proposition 2.9. If $\mathbf{r}_0(G) = 1$, then our assertion follows from Proposition 2.11. Suppose now that D is periodic. Lemma 1.5 and Corollary 1.4 imply that G is soluble. Since G is not periodic, G/D is not periodic. Corollary 1.4 shows that G/D is abelian.

We have $D = \mathop{\mathbf{Dr}}\limits_{p \in \mathbf{\Pi}(T)} D_p$ where D_p is a Sylow p-subgroup of D. Suppose that D is abelian. Let $p \in \mathbf{\Pi}(D)$ and put $Q_p = \mathop{\mathbf{Dr}}\limits_{p \in \mathbf{\Pi}(T), q \neq p} D_q$. We remark that $D/Q_p \cong D_pQ_p/O_p \cong_G D_p$. Consider further the factor-group G/Q_p. Then D/Q_p is an abelian p-group. Every finitely generated subgroup of G/Q_p is either pronormal or permutable. Let L/Q_p be the locally nilpotent radical of G/Q_p. Then Lemma 2.6 shows that G/L is a finite cyclic group of order dividing $p-1$. Let aQ_p be an arbitrary element of L/Q_p. Lemma 2.5 shows that every subgroup of D/Q_p is G-invariant, so if $a \in D$, then $\langle aQ_p \rangle$ is G-invariant. If $a \notin D$, then the subgroup $\langle a \rangle$ is pronormal in G. It follows that $\langle aQ_p \rangle$ is pronormal in G/Q_p and hence in L/Q_p. Using Lemma 1.2, we obtain that $\langle aQ_p \rangle$ is normal in L/Q_p, in particular, $\langle aQ_p \rangle$ is subnormal in G/Q_p. Being simultaneously subnormal and pronormal, $\langle aQ_p \rangle$ is normal in G/Q_p. Consequently, every cyclic subgroup (and hence every subgroup) of L/Q_p is G-invariant.

Since G/Q_p is not periodic and G/L is finite, L/Q_p is not periodic. The group L/Q_p is generated by its elements of infinite order. Using the argument from the proof of Lemma 2.1, we can obtain that these elements belong to the center of G/Q_p. On the other hand, since G/Q_p is soluble, L/Q_p includes its centralizer [20, Lemma 4], so that $G = L$. In other words, G/Q_p is locally nilpotent and non-periodic. As we have seen above, every subgroup of G/Q_p is normal. Being a non-periodic Dedekind group, G/Q_p is abelian [1]. Since this is valid for each prime p, the equality $\langle 1 \rangle = \bigcap\limits_{p \in \Pi(T)} G_p$ implies that G is abelian.

Suppose that D is not abelian. Then there is a prime p such that D_p is non-abelian. By Corollary 1.6, every subgroup of D is permutable in G. It follows that D_p is nilpotent and bounded. Put $C_p = [D_p, D_p]$, $R_p = Q_pC_p$. Then D/R_p is an abelian p-group. Repeating word by word the above argument, we obtain that G/R_p is abelian. We have

$$\begin{aligned}[D/Q_p, D/Q_p] &= [D_pQ_p/Q_p, D_pQ_p/Q_p] \\ &= [D_p, D_p]Q_p/Q_p] = C_pQ_p/Q_p = R_p/Q_p.\end{aligned}$$

Thus $(G/Q_p)/[D/Q_p, D/Q_p]$ is abelian, and the fact that D/Q_p is nilpotent implies that G/Q_p is likewise nilpotent [8, Theorem 7]. By Corollary 1.6, every subgroup of G/Q_p is permutable. Since G/Q_p is not periodic, $\mathbf{Tor}(G/Q_p)$ is abelian (see, for example, [23, Lemma 2.4.10 and Theorem 2.4.11]), in particular, $D_p \cong D_pQ_p/O_p = D/Q_p$ is abelian. This final contradiction completes the proof. $\square$

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