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FROM INFINITESIMAL HARMONIC TRANSFORMATIONS TO RICCI SOLITONS

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Abstract. The concept of the Ricci soliton was introduced by R. S. Hamilton. The Ricci soliton is defined by a vector field and it is a natural generalization of the Einstein metric. We have shown earlier that the vector field of the Ricci soliton is an infinitesimal harmonic transformation. In our paper, we survey Ricci solitons geometry as an application of the theory of infinitesimal harmonic transformations.

Keywords: Ricci soliton, infinitesimal harmonic transformation, Riemannian manifold

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1. Harmonic diffeomorphisms and infinitesimal harmonic transformations

A smooth mapping $f\colon (M,g) \to (M',g')$ between two Riemannian manifolds is called *harmonic* (see [4]) if f provides an extremum of the Dirichlet functional $E_\Omega(f) = \frac{1}{2}\int_\Omega \|\mathrm{d}f\|^2\,\mathrm{d}V$ with respect to the variations of f that are compactly supported in a relatively compact open subset $\Omega \subset M$. (Here, $\mathrm{d}V$ is the volume element of the metric g.) The following theorem is true (see [4]).

Theorem 1.1. *A smooth mapping $f\colon (M,g) \to (M',g')$ is harmonic if and only if it satisfies the Euler-Lagrange equations*

$$g^{ij}(\partial_i\partial_j f^\beta - \Gamma^k_{ij}\partial_k f^\beta + \partial_i f^\beta \partial_j f^\gamma ({\Gamma'}^\alpha_{\beta\gamma} \circ f)) = 0 \tag{1.1}$$

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where $y^\alpha = f^\alpha(x^1, \dots, x^n)$ is the local representation of f; g^{ij} are local contravariant components of the metric tensor g; Γ^k_{ij} and ${\Gamma'}^\alpha_{\beta\gamma}$ are the Christoffel symbols of (M, g) and (M', g'), respectively; $i, j, k = 1, \dots, n = \dim M$ and $\alpha, \beta, \gamma = 1, \dots, n' = \dim M'$.

If we suppose that $\dim M = \dim M' = n$ and $f\colon\ (M, g) \to (M', g')$ is a diffeomorphism then f is locally represented by the equations $y^i = x^i$ for $i, j, k, \dots = 1, 2, \dots, n$ and therefore the Euler-Lagrange equations (1.1) take the form

$$g^{ij}(({\Gamma'}^k_{ij} \circ f) - \Gamma^k_{ij}) = 0 \tag{1.2}$$

where Γ^k_{ij} and ${\Gamma'}^k_{ij}$ are the Christoffel symbols of the Levi-Civita connection ∇ on (M, g) and ∇' on (M', g'), respectively.

Suppose that we have a local one-parameter group of infinitesimal point transformations $f_t(x) = x'(x^k + t\xi^k)$ generated by a vector field $\xi = \xi^k \partial_k$ on (M, g) for the so-called canonical parameter t such that $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$ for $\varepsilon > 0$. In this case the Lie derivative of the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection ∇ has the form (see [22], pp. 8–9)

$$(L_\xi \Gamma^k_{ij})t = {\Gamma'}^k_{ij} - \Gamma^k_{ij} = \nabla_i \nabla_j \xi^k - R^k_{ijl}\xi^l \tag{1.3}$$

where ${\Gamma'}^k_{ij}(x) = f_t^*(\Gamma^k_{ij}(x'))$.

Definition 1.1 (see [15]; [20]). A vector field ξ on (M, g) is called an *infinitesimal harmonic transformation* if the one-parameter group of local transformations of (M, g) generated by ξ consists of local harmonic diffeomorphisms.

By the definition and (1.3) we deduce the equation

$$\Delta\theta = 2\,\mathrm{Ric}^*\,\xi \tag{1.4}$$

where ξ is an infinitesimal harmonic transformation and $\theta = g(\xi, \cdot)$ is its dual 1-form; $\Delta := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ is the Hodge Laplacian on the space 1-forms $\Omega^1(M)$; Ric^* is the linear Ricci operator defined by the identity $g(\mathrm{Ric}^*\,X, \cdot) = \mathrm{Ric}(X, \cdot)$ for the tensor Ricci Ric and an arbitrary vector field X on M.

Theorem 1.2 (see [15], [20]). *The equality $\Delta\theta = 2\,\mathrm{Ric}^*\,\xi$ is a necessary and sufficient condition for a vector field ξ to be an infinitesimal harmonic transformation on a Riemannian manifold (M, g).*

2. Examples of infinitesimal harmonic transformations

In this section we will give four examples of infinitesimal harmonic transformations on Riemannian, nearly Kählerian and Kählerian manifolds.

E x a m p l e 2.1. An infinitesimal isometric transformation on a Riemannian manifold is an infinitesimal harmonic transformation.

A vector field ξ on an n-dimensional Riemannian manifold (M, g) is an *infinitesimal isometric transformation* if $L_\xi g = 0$ where L_ξ is the Lie derivative in the direction of ξ. By direct computation, we can deduce the equalities $\Delta\theta = 2\,\mathrm{Ric}^*\,\xi$ and $\mathrm{d}^*\theta = 0$ for $\theta = g(\xi, \cdot)$. Moreover, these equalities are a necessary and sufficient condition for a vector field ξ to be an infinitesimal isometric transformation on a compact Riemannian manifold (M, g) (see [22], p. 221). By Theorem 1.2, an arbitrary infinitesimal isometric transformation on a compact Riemannian manifold must be an infinitesimal harmonic transformation.

E x a m p l e 2.2. An infinitesimal conformal transformation on a two-dimensional Riemannian manifold is a harmonic transformation.

Recall that a vector field ξ is an *infinitesimal conformal transformation* if $L_\xi g = -2n^{-1}(\mathrm{d}^*\theta)g$ for $\theta = g(\xi, \cdot)$. By direct computation, we can deduce the equality $\Delta\theta + (1 - 2/n)\mathrm{d}\mathrm{d}^*\theta = 2\,\mathrm{Ric}^*\,\xi$. Moreover, by virtue of the Lichnerowicz theorem (see [14]) this equality is a necessary and sufficient condition for a vector field ξ to be an infinitesimal conformal transformation on a compact Riemannian manifold (M, g). In particular, for $n = 2$ we have the equality $\Delta\theta = 2\,\mathrm{Ric}^*\,\xi$. Therefore, any infinitesimal harmonic transformation on a two-dimension compact Riemannian manifold is an infinitesimal conformal transformation.

E x a m p l e 2.3 (see [20]). A holomorphic vector field on a nearly Kählerian manifold is an infinitesimal harmonic transformation.

Let a triplet (M, g, J) be a *nearly Kählerian manifold* (see [7]) where $\mathrm{J} \in T^*M \otimes TM$ is such that $\mathrm{J}^2 = -\,\mathrm{id}_M$, $g(\mathrm{J}, \mathrm{J}) = g$ and $(\nabla_X \mathrm{J})Y + (\nabla_Y \mathrm{J})X = 0$ for any $X, Y \in TM$ and let ξ be a *holomorphic vector field* on (M, g, J), i.e. $L_\xi \mathrm{J} = 0$. In this case, as we have proved in [20], the equality $\Delta\theta = 2\,\mathrm{Ric}^*\,\xi$ holds.

R e m a r k 2.1. On a compact Kählerian manifold (M, g, J), where it is well known that $\nabla \mathrm{J} = 0$, a vector field ξ is holomorphic if and only if $\Delta\theta = 2\,\mathrm{Ric}^*\,\xi$ (see [22], p. 280). Therefore, in particular, a vector field ξ on a compact Kählerian manifold is an infinitesimal harmonic transformation if and only if ξ is holomorphic.

E x a m p l e 2.4 (see [21]). A vector field ξ that makes a Riemannian metric g into a Ricci soliton metric is necessarily an infinitesimal harmonic transformation.

Definition 2.1 (see [2], pp. 22–23). Let M be a smooth manifold. A *Ricci soliton* (g, ξ, λ) is a Riemannian metric g together with a vector field ξ on M and a constant λ that satisfies the equation $-2\operatorname{Ric} = L_\xi g + 2\lambda g$.

The Lie derivative of ∇ has the form (see [22], p. 52)

$$L_\xi \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i L_\xi g_{jl} + \nabla_j L_\xi g_{il} - \nabla_l L_\xi g_{ij}). \tag{2.1}$$

Substituting the identity $L_\xi g = -2(\operatorname{Ric} + \lambda g)$ in (2.1) we find $L_\xi \Gamma^k_{ij} = g^{kl}(-\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij})$ for the local components R_{ij} of the Ricci tensor Ric. From the last equation we have $g^{ij}(L_\xi \Gamma^k_{ij}) = g^{kl}(-2\nabla_j R^j_l + \nabla_l s) = 0$ for the scalar curvature $s = g^{ij} R_{ij}$. Here we have taken advantage of Schur's lemma $2\nabla_j R^j_l = \nabla_l s$.

Remark 2.2. If $\theta = \mathrm{d}F$ for a smooth function $F\colon M \to \mathbb{R}$ then the equation of an infinitesimal harmonic transformation $\Delta\theta = 2\operatorname{Ric}^* \xi$ can be written as $\Delta(\nabla_k F) = 2R^j_k \nabla_j F$ where $\Delta(\nabla_k F) = \nabla_k(\Delta F)$. On the other hand, if we put $\xi = \operatorname{grad} F$ then from the equation of a Ricci soliton we conclude $\Delta F = s + n\lambda$ and hence the equation $\nabla_k(\Delta F) = 2R^j_k \nabla_j F$ is equivalent to $\nabla_k s = 2R^j_k \nabla_j F$. The last equation was proved by Hamilton for a gradient Ricci soliton (see [9]).

3. The Yano Laplacian

Let (M, g) be a compact Riemannian manifold. We may also assume that (M, g) is orientable; if (M, g) is not orientable, we have only to take an orientable twofold covering space of (M, g). Denote by $S^p M$ the bundle of symmetric bilinear forms on (M, g), by δ^* the symmetric differentiation operator $\delta^*\colon C^\infty S^p M \to C^\infty S^{p+1} M$ and by δ the linear differential operator $\delta\colon C^\infty S^{p+1} M \to C^\infty S^p M$ as the adjoint operator to δ^* with respect to the global scalar product on $S^p M$

$$\langle \varphi, \varphi' \rangle = \int_M \frac{1}{p!}\, g(\varphi, \varphi')\, \mathrm{d}V,$$

which we get by integrating the pointwise inner product $g(\varphi, \varphi')$ for all $\varphi, \varphi' \in C^\infty S^p M$.

Definition 3.1 (see [18]; [19]). A differential operator $\Box\colon C^\infty S^p M \to C^\infty S^p M$ is called the *Yano differential operator* if $\Box = \delta\delta^* - \delta^*\delta$.

The Yano operator $\Box$ and the Bochner Laplacian $\nabla^*\nabla$ are connected by the Weitzenbock formula $\Box = \nabla^*\nabla + \Re_p$ for the symmetric endomorphism $\Re_p$ of the bundle $S^p M$ such that $\Re_p$ can be algebraically (even linearly) expressed through the curvature and the Ricci tensors of (M, g) (see [18]; [19]). In particular, for $p = 1$ we have $\Re_1 = -\operatorname{Ric}^*$ and hence $\Box = \Delta - 2\operatorname{Ric}^*$ (see [19]).

R e m a r k 3.1. This form of the operator $\square$ was used by K. Yano (see [23], p. 40) for the investigation of local isometric transformations of (M, g). Therefore we have named $\square$ the Yano operator. Moreover, Yano has named a vector field ξ *geodesic* if $\square\,\xi = 0$ (see [24]).

In view of what we have told above we can formulate the following theorem.

Theorem 3.1 (see [20]). *A necessary and sufficient condition for a vector field ξ on a Riemannian manifold (M, g) to be an infinitesimal harmonic transformation is that $\xi \in \operatorname{Ker}\square$ for the Yano operator $\square$.*

From the equality $\langle \square\,\varphi, \varphi' \rangle = \langle \varphi, \square\,\varphi' \rangle$ we conclude that $\square$ is a self-adjoint differential operator (see [18]). In addition, the symbol σ of the Yano operator $\square$ satisfies (see [18]) the condition $\sigma(\square)(\vartheta, x)\varphi_x = -g(\vartheta, \vartheta)\varphi_x$ for an arbitrary $x \in M$ and $\vartheta \in T_x^*M - \{0\}$. Hence the Yano operator $\square$ is a self-adjoint Laplacian operator and its kernel is a finite-dimensional vector space on the compact (M, g). In addition, we recall that the vector spaces $\operatorname{Ker}\square$ and $\operatorname{Im}\square$ are orthogonal complements of each other with respect to the global scalar product defined on the compact (M, g), i.e. the vector space $\Omega^p(M)$ of smooth section of S^pM has an orthogonal decomposition $\Omega^p(M) = \operatorname{Ker}\square \oplus \operatorname{Im}\square$. In particular, for $p = 1$ we can formulate the following

Theorem 3.2. *The vector space $\operatorname{Ker}\square$ of all infinitesimal harmonic transformations on the compact Riemannian manifold (M, g) is a finite-dimensional vector space and the orthogonal decomposition $\Omega^1(M) = \operatorname{Ker}\square \oplus \operatorname{Im}\square$ holds.*

For any conformal Killing vector field ζ and its dual 1-form ω on the compact smooth manifold (M, g) we have $\langle \Delta\omega + (1 - 2n^{-1})\mathrm{d}\mathrm{d}^*\omega - 2\operatorname{Ric}^*\zeta, \omega \rangle \geqslant 0$ (see [10]). From this inequality we conclude that $\langle \delta^*\omega, \delta^*\omega \rangle \geqslant 2n^{-1}\langle \delta\omega, \delta\omega \rangle \geqslant 0$ and hence $\langle \square\,\omega, \omega \rangle \geqslant 0$ for $n \geqslant 2$.

4. Two decomposition theorems

In this section we will consider the vector space $\operatorname{Ker}\square$ of all infinitesimal harmonic transformations on a compact Riemannian manifold. The following theorem is true.

Theorem 4.1. *If the vector field ξ is an infinitesimal harmonic transformation on a compact Riemannian manifold (M, g) then ξ is decomposed in the form $\xi = \xi' + \xi''$ where ξ' is an infinitesimal isometric transformation and ξ'' is a gradient infinitesimal harmonic transformation on (M, g). This decomposition is necessarily orthogonal with respect to the global scalar product defined on (M, g).*

P r o o f. The vector space $\operatorname{Ker}\Box\cap\operatorname{Ker}\mathrm{d}^*$ of all infinitesimal isometric transformations on a compact Riemannian manifold (M,g) is a subspace of the finite-dimensional Euclidean vector space $\operatorname{Ker}\Box$ (see Exp. 2.1). On the other hand, it is well known (see [17], p. 205) that by virtue of the Fredholm alternative the vector spaces $\operatorname{Im}\mathrm{d}$ and $\operatorname{Ker}\mathrm{d}^*$ are orthogonal complements of each other with respect to the global scalar product on the compact Riemannian manifold (M,g), i.e. $\Omega^1(M)=\operatorname{Ker}\mathrm{d}^*\oplus\operatorname{Im}\mathrm{d}$. Therefore the vector space $\operatorname{Ker}\Box\cap\operatorname{Ker}\mathrm{d}$ of all infinitesimal gradient harmonic transformations must be an orthogonal complement of $\operatorname{Ker}\Box\cap\operatorname{Ker}\mathrm{d}^*$ with respect to the whole space $\operatorname{Ker}\Box$. This vector subspace consists of all gradient vector fields ∇F such that $\nabla_i(\Delta F)=2R_i^j\nabla_j F$ for smooth scalar functions $F\colon M\to\mathbb{R}$. $\Box$

R e m a r k 4.1. The last result has been known (see [24]) in the case of a compact Einstein n-dimensional ($n\geqslant 2$) manifold (M,g) with constant scalar curvature s.

In view of Example 2.2 we can formulate the following corollary.

Corollary 4.1. *On a compact Riemannian manifold (M,g) of dimension 2 an arbitrary infinitesimal conformal transformation ξ has the form $\xi=\xi'+\operatorname{grad}F$ where ξ' is an infinitesimal isometric transformation and F is a smooth scalar function on (M,g) such that the vector field $\operatorname{grad}F$ is an infinitesimal harmonic and a conformal transformation simultaneously. Moreover, if $L_{\operatorname{grad}F}s=0$ then the manifold (M,g) is isometric to the sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{R}^3$.*

P r o o f. By virtue of Example 2.3 an arbitrary infinitesimal conformal transformation ξ is an infinitesimal harmonic transformation on a two-dimensional compact Riemannian manifold (M,g) and therefore the following decomposition is true: $\xi=\xi'+\operatorname{grad}F$ where ξ' is an infinitesimal isometric transformation and F is a smooth scalar function on (M,g) such that the vector field $\xi''=\operatorname{grad}F$ with local coordinates $g^{ik}\nabla_k F$ is an infinitesimal harmonic transformation. Then by direct computation, we obtain $L_\xi g=L_{\xi'}g+L_{\operatorname{grad}F}g=L_{\operatorname{grad}F}g=2\nabla\nabla F$ and $\operatorname{div}\xi=-\Delta F$. As a result we obtain the equality $L_{\operatorname{grad}F}g=2\nabla\nabla F=-(\Delta F)g$ from which we can conclude that the vector field $\xi''=\operatorname{grad}F$ is an infinitesimal conformal transformation as well.

It as well known, if a compact Riemannian manifold (M,g) of dimension $n\geqslant 2$ admits a nonconstant scalar function F such that $\nabla\nabla F=n^{-1}(-\Delta F)g$ then $(M,\,g)$ is conformal to the sphere $\mathbb{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$ (see [12]). Therefore if we suppose that $L_{\operatorname{grad}F}s=0$ then our (M,g) must be isometric to the sphere $\mathbb{S}^2$. $\Box$

R e m a r k 4.2. The vector space of infinitesimal conformal transformations on $(\mathbb{S}^2,\overline{g})$ has dimension equal to 6 and admits decomposition into the sum of two subspaces (see [5]). Three of the dimensions arise from $\overline{\nabla}F$ where F is a spherical harmonic. The other three dimensions come from the infinitesimal isometric transformations for the standard metric $\overline{g}$ on $\mathbb{S}^2$. Therefore our decomposition of the vector space of infinitesimal conformal transformations on a compact Riemannian manifold is an analog of the above decomposition on the sphere $\mathbb{S}^2$.

Now we shall formulate the decomposition theorem for an arbitrary infinitesimal harmonic transformation on a compact Kählerian manifold.

Theorem 4.2. *If ξ is a holomorphic vector field on a compact Kählerian manifold (M, g, J) then ξ is decomposed in the form $\xi = \xi' + \mathrm{J}\xi''$ where both ξ' and ξ'' are infinitesimal isometric transformations. This decomposition is necessarily orthogonal with respect to the global scalar product defined on (M, g, J).*

P r o o f. On a compact Kählerian manifold (M, g, J), where it is well known that $\nabla \mathrm{J} = 0$, a vector field ξ on a compact Kählerian manifold is an infinitesimal harmonic transformation if and only if ξ is a holomorphic vector field (see Ex. 5). Therefore, by virtue of Theorem 4.1 we have the orthogonal decomposition $\xi = \xi' + \operatorname{grad} F$ where ξ' is an infinitesimal isometric transformation and $\operatorname{grad} F$ is a holomorphic vector field for some smooth scalar function F on (M, g). On the other hand, it is well known (see Theorem 6.8 of Chapter IV in [25]) that $\mathrm{J}X$ is an infinitesimal isometric transformation if a holomorphic vector field X is closed. Therefore we can state that $\xi = \xi' + \operatorname{grad} F = \xi' + \mathrm{J}\xi''$ where ξ'' is an infinitesimal isometric transformation. $\square$

R e m a r k 4.3. Lichnerowicz proved the following theorem (see [26]): A holomorphic vector field ξ on a compact Kählerian manifold (M, g, J) with constant scalar curvature is decomposed in the form $\xi = \xi' + \mathrm{J}\xi''$ where both ξ' and ξ'' are infinitesimal isometric transformations. Theorem 4.2 is a generalization of this theorem.

5. Ricci solitons

Let (g, ξ, λ) be a *Ricci soliton* on a smooth n-dimensional manifold M (see Ex. 2.3), where g is a Riemannian metric and ξ is a smooth vector field on M such that the identity

$$-2\,\mathrm{Ric} = L_\xi g + 2\lambda g \tag{5.1}$$

holds for some constant λ (see [2], p. 22; [3], p. 353). A Ricci soliton is called *steady* if $\lambda = 0$, *shrinking* if $\lambda < 0$, and, finally, *expanding* if $\lambda > 0$.

In the case $\xi = \operatorname{grad} F$ for some smooth function $F\colon\ M \to \mathbb{R}$ the equation can be rewritten as

$$-\,\mathrm{Ric} = \nabla\nabla F + \lambda g \tag{5.2}$$

and (g, ξ, λ) is called a *gradient Ricci soliton* (see [2], p. 22; [3], p. 353). Moreover, (M, g) is called a *trivial Ricci soliton* if $F = \text{const}$ and hence (M, g) is an Einstein manifold.

By Example 2.3 a vector field ξ that makes a Riemannian metric g into a metric of a Ricci soliton is necessarily an infinitesimal harmonic transformation. In addition, by the first decomposition theorem a harmonic transformation ξ on a compact Riemannian manifold (M, g) has the form $\xi = \xi' + \xi''$ where ξ' is an infinitesimal isometric transformation and ξ'' is a gradient infinitesimal harmonic transformation on (M, g). By these propositions we can rewrite the identity (5.1) as

$$-2\,\mathrm{Ric} = L_\xi g + 2\lambda g = L_{\xi'+\xi''} g + 2\lambda g = 2\nabla\nabla F + 2\lambda g$$

where $\xi'' = \operatorname{grad} F$ for some smooth scalar function F. Now we can formulate the following result:

Theorem 5.1. *Every Ricci soliton on a compact smooth manifold M is a gradient Ricci soliton.*

R e m a r k 5.1. By means of Perelman's work [16] and some previous ones by others authors, see Hamilton [8] for dimension two and Ivey [11] for dimension 3 we know that every compact Ricci soliton is a gradient Ricci soliton. And hence the Perelman-Hamilton-Ivey proposition is a corollary of our theorem about infinitesimal harmonic transformations on a compact smooth manifold.

For the vector field ξ of the Ricci soliton (g, ξ, λ), by using the equation (5.2) we get

$$\int_M L_\xi s\,\mathrm{d}V = \langle \xi,\, \mathrm{d}s \rangle = \langle \xi,\, \mathrm{d}(s + n\lambda) \rangle = \langle \xi,\, \mathrm{d}\delta\xi \rangle = \langle \delta\xi, \delta\xi \rangle \geqslant 0$$

which is equivalent to $\int_M L_\xi s\,\mathrm{d}V = \int_M (\Delta F)^2\,\mathrm{d}V \geqslant 0$.

By means of this inequality we can formulate the following theorem:

Theorem 5.2. *If a shrinking Ricci soliton (g, ξ, λ) on a compact smooth manifold M satisfies the condition $L_\xi s \leqslant 0$ then this soliton is trivial.*

R e m a r k 5.2. It is well known that a compact steady or expanding Ricci soliton (g, ξ, λ) is a gradient soliton (see [16]) and, on the other hand, a compact gradient steady or expanding Ricci soliton is a trivial soliton (see [9]). On the other hand, every shrinking compact Ricci soliton when $n > 3$ and the Weyl tensor is zero is trivial (see [6] and [27]). But there is an open problem (see [6], p. 11): Are the special conditions in dimension $n \geqslant 4$ ensuring that a shrinking compact Ricci soliton is trivial? Our Theorem 5.2 may be one of possible answers to this question.

6. Infinitesimal harmonic transformations and Ricci solitons in negative Ricci curvature

We define the function $f = \|\xi\|^2 := 2^{-1} g(\xi, \xi)$ for any smooth vector field ξ on M and by direct calculation we find the equality

$$\Delta f := \sum_{i=1}^{n} \nabla^2 f(X_i, X_i) = -g(\Box\,\xi, \xi) - \operatorname{Ric}(\xi, \xi) + \|\nabla\xi\|^2 \tag{6.1}$$

where $X_1, \ldots, X_n$ is an orthonormal basis in $T_x M$ for any point $x \in M$.

Theorem 6.1. *Suppose (M, g) is compact and has $\operatorname{Ric} \leqslant 0$. Then every infinitesimal harmonic transformation is parallel. Furthermore, if $\operatorname{Ric} < 0$, then there are no nontrivial infinitesimal harmonic transformations.*

P r o o f. If we define $f = 2^{-1}\|\xi\|^2$ for an infinitesimal harmonic transformation ξ, then using Stokes's theorem and the condition $\operatorname{Ric} \leqslant 0$ yields

$$0 = \int_M \Delta f \, \mathrm{d}V = \int_M (-\operatorname{Ric}(\xi, \xi) + \|\nabla\xi\|^2) \, \mathrm{d}V \geqslant \int_M \|\nabla\xi\|^2 \, \mathrm{d}V \geqslant 0.$$

Thus $\|\nabla\xi\| \equiv 0$ holds and ξ must be parallel. In addition, $\operatorname{Ric}(\xi, \xi) \leqslant 0$ and $\int_M \operatorname{Ric}(\xi, \xi) \, \mathrm{d}V = 0$, so $\operatorname{Ric}(\xi, \xi) \equiv 0$. If $\operatorname{Ric} < 0$ this implies that $\xi \equiv 0$.

R e m a r k 6.1. Theorem 6.1 is a generalization of Bochner's result about infinitesimal isometric transformations (see [1]) and was formulated in [18].

Let ξ be a parallel vector field of the Ricci soliton (g, ξ, λ). This condition follows from (5.1) and the Ricci identities that $\lambda\xi = 0$. Hence $\lambda = 0$ or $\lambda \neq 0$ and $\xi \equiv 0$. If, in addition, assume that (g, ξ, λ) is a non trivial Ricci soliton then from the equation $\lambda\xi = 0$ we can conclude that $\lambda = 0$ and as a corollary of this equation that $\operatorname{Ric} = 0$. Therefore Theorem 6.1 allows us to formulate the following corollary.

Corollary 6.1. *A Riemannian metric g on a compact smooth manifold M cannot be the metric of a Ricci soliton (g, ξ, λ) such that $\operatorname{Ric}(\xi, \xi) < 0$. If $\operatorname{Ric}(\xi, \xi) \leqslant 0$ then one of the following two conditions holds: either (g, ξ, λ) is a trivial Ricci soliton or (g, ξ, λ) is a steady Ricci soliton with a Ricci flat metric g.*

The result about the non existence of infinitesimal harmonic transformations can be slightly improved to yield

Theorem 6.2. *Suppose (M, g) is a compact manifold with quasi-negative Ricci curvature, i.e.,* $\mathrm{Ric} \leqslant 0$ *and* $\mathrm{Ric}(X, X) < 0$ *for all* $X \in T_xM - \{0\}$ *for some* $x \in M$. *Then (M, g) admits no nontrivial infinitesimal harmonic transformations.*

P r o o f. We already know that any infinitesimal harmonic transformation is parallel. Thus an infinitesimal harmonic transformation is either always zero or never zero. If the latter holds, then $\mathrm{Ric}(\xi, \xi) < 0$, but this contradicts

$$0 = \Delta f(x) = -\mathrm{Ric}(\xi, \xi)(x) > 0.$$

□

In addition, we can formulate the following result as a corollary of Theorem 6.2.

Corollary 6.2. *Suppose (M, g) is a compact manifold with quasi-negative Ricci curvature, i.e.* $\mathrm{Ric} \leqslant 0$ *and* $\mathrm{Ric}(X, X) < 0$ *for all* $X \in T_xM - \{0\}$ *for some* $x \in M$. *Then (M, g) admits no nontrivial Ricci soliton (g, ξ, λ).*

Let $x \in M$ be a local maximum of the function $f = 2^{-1}\|\xi\|^2$ where ξ is an infinitesimal harmonic trasformation. Then

$$\Delta f(x) = -\mathrm{Ric}(\xi, \xi)(x) + \|\nabla\xi\|^2(x) \leqslant 0.$$

If this point is non-zero for ξ and $\mathrm{Ric} < 0$ everywhere on a neighborhood U_x of $x \in M$ then (see also the proof of Theorem 4.6 in [13]) there is a neighborhood V_x of $x \in M$ such that $V_x \subset U_x$ and $\xi = 0$ everywhere on V_x. The following theorem is true.

Theorem 6.3. *Let (M, g) be a noncompact Riemannian manifold. If the function $f = 2^{-1}\|\xi\|^2$ for an infinitesimal harmonic transformation ξ has a local maximum point $x \in M$ and* $\mathrm{Ric} < 0$ *everywhere on a neighborhood U_x of $x \in M$ then there is a neighborhood V_x of $x \in M$ such that $V_x \subset U_x$ and $\xi \equiv 0$ everywhere on V_x.*

This theorem allows us to formulate a corollary about Ricci solitons.

Corollary 6.3. *Let (g, ξ, λ) be a Ricci soliton on a noncompact smooth manifold M. If the function $f = 2^{-1}\|\xi\|^2$ has a local maximum point $x \in M$ and* $\mathrm{Ric} < 0$ *everywhere on a neighborhood U_x of $x \in M$ then there is a neighborhood V_x of $x \in M$ such that $V_x \subset U_x$ and (g, ξ, λ) is a trivial Ricci soliton everywhere on V_x.*

R e m a r k 6.2. Yau formulated and proved a generalized version of the Stokes theorem for an $(n-1)$-form on an n-dimensional complete noncompact Riemannian manifold (M, g) (see [25]). As applications of this result, he obtained a series of propositions. In particular, he proved that each non negative smooth subharmonic function f on an n-dimensional complete noncompact Riemannian manifold (M, g) is nonconstant if $\int_M f^p \,\mathrm{d}V < \infty$ for all $p > 1$.

Now we assume that $\mathrm{Ric}(\xi, \xi) \leqslant 0$ everywhere on (M, g) for an infinitesimal harmonic transformation ξ. Then $\Delta f \geqslant 0$ and hence $f = 2^{-1}|\xi\|^2$ is a subharmonic nonnegative function. If, in addition, $\int_M \|\xi\|^{2p} \,\mathrm{d}V < \infty$ for all $p > 1$, then by the above Yau result we have $f = \text{const}$. By virtue of (6.1), we conclude that $\mathrm{Ric}(\xi, \xi) = \|\nabla\xi\|^2 \geqslant 0$. Therefore necessarily $\mathrm{Ric}(\xi, \xi) = \|\nabla\xi\|^2 = 0$. If in addition we suppose that (M, g) is locally irreducible, then necessarily $\xi \equiv 0$. The following theorem is true.

Theorem 6.4. *Let ξ be an infinitesimal harmonic transformation on a complete noncompact Riemannian manifold (M, g). If $\mathrm{Ric}(\xi, \xi) \leqslant 0$ and $\int_M \|\xi\|^{2p} \,\mathrm{d}V < \infty$ for all $p > 1$, then $\nabla\xi = 0$. If, in addition, g is a locally irreducible Riemannian metric, then $\xi \equiv 0$.*

In particular, let ξ be a parallel vector field of the Ricci soliton (g, ξ, λ). It follows from (5.1) and the Ricci identities that $\lambda\xi = 0$. Then either $\lambda = 0$ or $\lambda \neq 0$ and $\xi \equiv 0$. If, in addition, we assume that (g, ξ, λ) is a non trivial Ricci soliton then from the equation $\lambda\xi = 0$ we can conclude that $\lambda = 0$ and as a corollary of this equation, $\mathrm{Ric} = 0$. Therefore, as a corollary of Theorem 6.4 we can formulate the following result.

Corollary 6.4. *Let M be a non compact smooth manifold and (g, ξ, λ) a non trivial Ricci soliton on M with a complete Riemannian metric g. If $\mathrm{Ric}(\xi, \xi) \leqslant 0$ and $\int_M \|\xi\|^{2p} \,\mathrm{d}V < \infty$ for all $p > 1$, then (g, ξ, λ) is a steady Ricci soliton with a Ricci flat metric g.*

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