Alejandro Illanes
Pseudo-homotopies of the pseudo-arc

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 53 (2012), No. 4, 629--635

Persistent URL: http://dml.cz/dmlcz/143195

# Pseudo-homotopies of the pseudo-arc

Alejandro Illanes

*Abstract.* Let $X$ be a continuum. Two maps $g, h : X \to X$ are said to be pseudo-homotopic provided that there exist a continuum $C$, points $s, t \in C$ and a continuous function $H : X \times C \to X$ such that for each $x \in X$, $H(x, s) = g(x)$ and $H(x, t) = h(x)$. In this paper we prove that if $P$ is the pseudo-arc, $g$ is one-to-one and $h$ is pseudo-homotopic to $g$, then $g = h$. This theorem generalizes previous results by W. Lewis and M. Sobolewski.

*Keywords:* pseudo-arc, pseudo-contractible, pseudo-homotopy

*Classification:* Primary 54F15; Secondary 54B10, 54F50

## 1. Introduction

A *continuum* is a nondegenerate compact connected metric space. The letter $P$ will denote the pseudo-arc. We will use the definition of the pseudo-arc as it is given in [7, 1.7]. A *map* is a continuous function. Two maps $h, g : P \to P$ are *pseudo-homotopic* provided that there exist a continuum $C$, points $s_0, t_0 \in C$ and a map $H : P \times C \to P$ such that $H(p, s_0) = g(p)$ and $H(p, t_0) = h(p)$ for each $p \in P$. In this case, we say that $H$ is a *pseudo-homotopy* between $g$ and $h$. The continuum $X$ is *pseudo-contractible*, provided that the identity in $X$ is pseudo-homotopic to a constant map. An *$\varepsilon$-map* between continua is a map $f : X \to Y$ such that diameter$(f^{-1}(y)) < \varepsilon$ for each $y \in f(X)$. A continuum $X$ is *chainable* provided that for each $\varepsilon > 0$, there exists an $\varepsilon$-map from $X$ into $[0, 1]$. Another way to define a chainable continuum [9, Theorem 12.11] is the following: a *chain* in a continuum $X$ is a nonempty, finite, indexed collection $\mathcal{C} = \{U_1, \ldots, U_n\}$ of open subsets $U_i$ of $X$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The continuum $X$ is *chainable* provided that for each $\varepsilon > 0$ there exists a chain $\mathcal{C} = \{U_1, \ldots, U_n\}$ in $X$ such that $X = U_1 \cup \ldots \cup U_n$ and diameter$(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$.

The concepts of pseudo-homotopy between maps of a continuum and of a pseudo-contractible continuum were introduced by W. Kuperberg [5] and the first example of a pseudo-contractible continuum which is not contractible was also given by him. This example appears in page 2983 of [10].

This paper was partially supported by the project “Hiperespacios topológicos (0128584)” of Consejo Nacional de Ciencia y Tecnología (CONACYT), 2009.

Answering a question by W. Kuperberg, in 2007 [10], M. Sobolewski proved that the pseudo-arc is not pseudo-contractible. In fact, he proved that the only pseudo-contractible chainable continuum is the arc.

There are only two known types of pseudo-homotopies for maps into the pseudo-arc, namely those pseudo-homotopies $H : P\times C \to P$ satisfying $H(P\times\{c\})$ is degenerate for each $c \in C$ or those for which there exists a map $f : P \to P$ such that $H(x, c) = f(x)$ for each $(x, c) \in X \times C$. So the following problem arises naturally.

**Problem 1.** *Do there exist pseudo-homotopies on the pseudo-arc different from the ones described in the paragraph above?*

In [6], W. Lewis proved that if $g$ is a homeomorphism from the pseudo-arc onto itself and $h$ is pseudo-homotopic to $g$, then $h = g$. From here, he deduced that in the space of homeomorphisms $\mathcal{H}(P)$ of the pseudo-arc there are not nondegenerate continua. It is still an open problem to determine if $\mathcal{H}(P)$ is totally disconnected [8, Question 21].

In this paper we use the technique developed by Sobolewski in [10] to prove that if $g : P \to P$ is one-to-one and $h$ is pseudo-homotopic to $g$, then $g = h$.

## 2. Results

Given a continuum $X$, let $C(X)$ be the hyperspace of subcontinua of $X$ endowed with the Hausdorff metric [2, Definition 2.1]. Given subcontinua $A$ and $B$ of a continuum $X$ such that $A \subsetneq B$, an *order arc* from $A$ to $B$ is a map $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and, if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. The existence of order arcs is proved in [2, Theorem 14.6].

**Lemma 2.** *Let $g_1, h_1 : P \to P$ be pseudo-homotopic maps such that $g_1$ is not constant and $g_1 \neq h_1$. Then there exist a pseudo-arc $P_1$ and pseudo-homotopic maps $h, g : P_1 \to P$ such that $\operatorname{Im} g \cap \operatorname{Im} h = \emptyset$, $g$ is not constant and, if $g_1$ is one-to-one, then $g$ is one-to-one.*

Proof: Let $H : P \times C \to P$ be a pseudo-homotopy between $g_1$ and $h_1$ and let $s_0, t_0 \in C$ be such that $H(p, s_0) = g_1(p)$ and $H(p, t_0) = h_1(p)$ for each $p \in P$. Let $p_0 \in P$ be such that $g_1(p_0) \neq h_1(p_0)$. Let $D$ (resp., $E$) be the component of $g_1^{-1}(g_1(p_0))$ (resp., $h_1^{-1}(h_1(p_0))$) containing $p_0$. Then $D \subset E$ or $E \subset D$. Let $D_1 = D \cap E$. Since $g_1$ is not constant, $D_1$ is a proper subcontinuum of $P$. Let $\alpha : [0, 1] \to C(P)$ be an order arc from $D_1$ to $P$. Since $g_1(\alpha(0)) = g_1(D_1) = \{g_1(p_0)\}$ and $h_1(\alpha(0)) = h_1(D_1) = \{h_1(p_0)\}$, we have that there exists $t > 0$ such that $g_1(\alpha(t)) \cap h_1(\alpha(t)) = \emptyset$. Let $P_1 = \alpha(t)$. Then $P_1$ is homeomorphic to $P$ and either $P_1 \nsubseteq g_1^{-1}(g_1(p_0))$ or $P_1 \nsubseteq h_1^{-1}(h_1(p_0))$. This implies that $g_1|P_1$ or $h_1|P_1$ is not constant. We may assume that $g_1|P_1$ is not constant. In the case that $g_1$ is one-to-one, we have indeed that $g_1|P_1$ is not constant. Let $H_1 = H|(P_1\times C) : P_1\times C \to P$. Then $H_1$ is a pseudo-homotopy between $g_1|P_1$ and $h_1|P_1$. Define $g = g_1|P_1$ and $h = h_1|P_1$. □

We consider $P$ constructed in the plane $\mathbb{R}^2$ [7, 1.7] by using a sequence of chains $\mathcal{C}_n$, where for each $n \in \mathbb{N}$, $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$, the mesh of $\mathcal{C}_n$ is less than $\frac{1}{n}$, $\mathcal{C}_{n+1}$ is crooked in $\mathcal{C}_n$, $\mathcal{C}_n = \{U_1^{(n)}, \ldots, U_{m_n}^{(n)}\}$, the sets $U_1^{(n)}, \ldots, U_{m_n}^{(n)}$ are open in $P$ and they cover $P$, and $\mathrm{cl}_P(U_i^{(n)}) \cap \mathrm{cl}_P(U_j^{(n)}) \neq \emptyset$ if and only if $|i-j| \leq 1$. Given $n \in \mathbb{N}$ and $1 \leq i \leq j \leq m_n$, let $W(i,j,n) = U_i^{(n)} \cup \ldots \cup U_j^{(n)}$. In the case that $0 \leq j < i \leq m_n$, we define $W(i,j,n) = \emptyset$.

**Theorem 3.** *Let $g, h : P \to P$ be pseudo-homotopic maps such that $g$ is one-to-one. Then $g = h$.*

Proof: Let $d$ be a metric for $P$. Suppose to the contrary that $g \neq h$. We are going to get a contradiction. By Lemma 2, we may assume that $\mathrm{Im}\, g \cap \mathrm{Im}\, h = \emptyset$. Let $H : P \times C \to P$ be a pseudo-homotopy between $g$ and $h$ and let $s_0, t_0 \in C$ be such that $H(p, s_0) = g(p)$ and $H(p, t_0) = h(p)$ for each $p \in P$.

Let $B = \mathrm{Im}\, g$. Then $B$ is a nondegenerate subcontinuum of $P$. Let $\varepsilon =$ diameter$(B)$. Let $N \in \mathbb{N}$ be such that $\frac{20}{N} < \varepsilon$ and $N$ has the following properties: (a) if $d(g(p), g(q)) < \frac{3}{N}$, then $d(H(p,c), H(q,c)) < \frac{\varepsilon}{20}$ for each $c \in C$ (recall that $g$ is one-to-one); and (b) $\frac{1}{N} < \min\{d(p,q) : p \in \mathrm{Im}\, g \text{ and } q \in \mathrm{Im}\, h\}$. Let $p_0, q_0 \in P$ be such that diameter$(B) = d(g(p_0), g(q_0))$. Let $i_0, j_0 \in \{1, \ldots, m_N\}$ be such that $g(p_0) \in U_{i_0}^{(N)}$ and $g(q_0) \in U_{j_0}^{(N)}$. We may assume that $i_0 < j_0$. Notice that $19 < j_0 - i_0$. Let $i_1, j_1 \in \{1, \ldots, m_{N+1}\}$ be such that $g(p_0) \in U_{i_1}^{(N+1)}$ and $g(q_0) \in U_{j_1}^{(N+1)}$. Then there exist $u_0, v_0 \in \{1, \ldots, m_{N+1}\}$ such that $u_0, v_0 \in \{\min\{i_1, j_1\}, \ldots, \max\{i_1, j_1\}\}$, $U_{u_0}^{(N+1)} \cap U_{i_0}^{(N)} \neq \emptyset$, $U_{v_0}^{(N+1)} \cap U_{j_0}^{(N)} \neq \emptyset$ and $W(u_0, v_0, N+1) \subset W(i_0, j_0, N)$. We may assume that $u_0 < v_0$. Since $U_{i_1}^{(N+1)} \cap B \neq \emptyset$ and $U_{j_1}^{(N+1)} \cap B \neq \emptyset$, we have that $U_i^{(N+1)} \cap B \neq \emptyset$ for each $u_0 \leq i \leq v_0$.

By the choice of $N$, $\mathrm{Im}\, h$ does not intersect $W(u_0, v_0, N+1)$. Therefore, $\mathrm{Im}\, h \subset W(1, u_0 - 1, N+1)$ or $\mathrm{Im}\, h \subset W(v_0 + 1, m_{N+1}, N+1)$.

Since $\mathcal{C}_{N+1}$ is crooked in $\mathcal{C}_N$, there exist $k_0, l_0 \in \{1, \ldots, m_{N+1}\}$ such that $u_0 < k_0 < l_0 < v_0$, $U_{k_0}^{(N+1)} \cap U_{j_0-1}^{(N)} \neq \emptyset$ and $U_{l_0}^{(N+1)} \cap U_{i_0+1}^{(N)} \neq \emptyset$.

An appropriate use of Urysohn's lemma for metric continua allows us to construct a map $f_0 : P \to [-\frac{1}{2}, \frac{3}{2}]$ such that: $\mathrm{cl}_P(W(1, i_0 - 1, N)) \subset f_0^{-1}([-\frac{1}{2}, 0])$, $\mathrm{cl}_P(W(j_0 + 1, m_N, N)) \subset f_0^{-1}([1, \frac{3}{2}])$, $f_0^{-1}(0) = \mathrm{cl}_P(W(i_0, i_0 + 2, N))$, $f_0^{-1}(1) = \mathrm{cl}_P(W(j_0 - 2, j_0, N))$, $f_0^{-1}([0,1]) = \mathrm{cl}_P(W(i_0, j_0, N))$ and $f_0$ is a $\frac{3}{N}$-map. Again, by Urysohn's lemma, it is possible to construct a $\frac{3}{N}$-map $f : \mathrm{cl}_P\ (W(1, u_0, N+1)) \cup \mathrm{cl}_P\ (W(v_0, m_{N+1}, N+1)) \to [-\frac{1}{2}, 0] \cup [1, \frac{3}{2}]$ such that $\mathrm{cl}_P(W(1, u_0, N+1)) = f^{-1}([-\frac{1}{2}, 0])$, $\mathrm{cl}_P\ (W(v_0, m_{N+1}, N+1)) = f^{-1}([1, \frac{3}{2}])$, $\mathrm{cl}_P(U_{u_0}^{(N+1)}) = f^{-1}(0)$ and $\mathrm{cl}_P(U_{v_0}^{(N+1)}) = f^{-1}(1)$. We extend $f$ to $P$, defining $f(p) = f_0(p)$ for each $p \in \mathrm{cl}_P$ $(W(u_0, v_0, N+1))$. Given $p \in \mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0, i_0 + 1, N))$, we have $f_0(p) = 0$, and given $p \in \mathrm{cl}_P(U_{v_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0 - 1, j_0, N))$, we have $f_0(p) = 1$. This implies that $f$ is a well-defined map from $P$ into $[-\frac{1}{2}, \frac{3}{2}]$. It is easy to check that $f$ is a $\frac{3}{N}$-map.

Let $\varphi : P \to [\frac{1}{2}, \frac{9}{2}]$ be given by

$$\varphi(p) = \begin{cases} f(p)+1, & \text{if } \ p \in \mathrm{cl}_P(W(1,k_0,N+1)), \\ 3-f(p), & \text{if } \ p \in \mathrm{cl}_P(W(k_0,l_0,N+1)), \\ 3+f(p), & \text{if } \ p \in \mathrm{cl}_P(W(l_0,m_{N+1},N+1)). \end{cases}$$

If $p \in \mathrm{cl}_P(W(1,k_0,N+1)) \subset \mathrm{cl}_P(W(1,u_0,N+1) \cup \mathrm{cl}_P(W(i_0,j_0,N)$, then $f(p) \in [-\frac{1}{2}, 1]$ and $\varphi(p) \in [\frac{1}{2}, 2]$. If $p \in \mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0,i_0+1,N))$, then $f(p)=0$ and $\varphi(p)=1$. If $p \in \mathrm{cl}_P(U_{k_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0-2,j_0,N))$, then $f(p)=1$ and $\varphi(p)=2$. If $p \in \mathrm{cl}_P(U_{l_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0,i_0+2,N))$, then $f(p)=0$ and $\varphi(p)=3$. If $p \in \mathrm{cl}_P(U_{v_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0-1,j_0,N))$, then $f(p)=1$ and $\varphi(p)=4$. If $p \in \mathrm{cl}_P\ (W(k_0,l_0,N+1)) \subset \mathrm{cl}_P(W(i_0,j_0,N))$, then $f(p) \in [0,1]$ and $\varphi(p) \in [2,3]$. If $p \in \mathrm{cl}_P(W(l_0,m_{N+1},N+1)) \subset \mathrm{cl}_P(W(i_0,j_0,N) \cup \mathrm{cl}_P(W(v_0,m_{N+1},N+1)$, then $f(p) \in [0,\frac{3}{2}]$, so $\varphi(p) \in [3,\frac{9}{2}]$. These relations in particular imply that $\varphi$ is well-defined and continuous.

Since $U_{u_0}^{(N+1)} \cap B \neq \emptyset$ and $\mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset f^{-1}(0)$, we have that $0 \in f(B)$ and $1 \in \varphi(B)$, similarly, $4 \in \varphi(B)$. Thus, $\varphi(g(P))$ is a closed interval containing $[1,4]$. Consider the map $\eta = (\varphi \times \varphi) \circ (g \times g) : P \times P \to [\frac{1}{2}, \frac{9}{2}]^2$ and let $D = \mathrm{Im}\,\eta = \varphi(g(P)) \times \varphi(g(P))$. Then $D$ is a 2-cell containing $[1,4]^2$. By [3] and [4, Proposition 1.2], $\eta$ is a *universal* map and hence essential. Recall that a map between continua $\gamma : X \to Y$ is universal provided that for each map $\lambda : X \to Y$, there exists a point $x \in X$ such that $\gamma(x) = \lambda(x)$. Moreover, a map $\gamma : X \to D$, where $D$ is a 2-cell is *essential* provided that each map $\lambda : X \to D$ such that $\gamma(x) = \lambda(x)$ for each $x \in \gamma^{-1}(\partial D)$ is surjective.

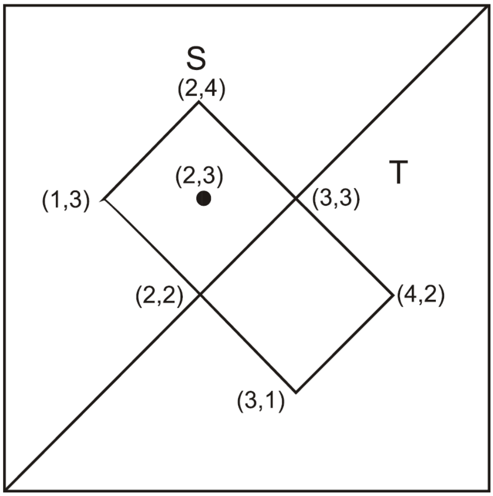
Let $\psi : [0,5] \to [-1,2]$ be given by

$$\psi(x) = \begin{cases} x-1, & \text{if } \ x \in [0,2], \\ 3-x, & \text{if } \ x \in [2,3], \\ x-3, & \text{if } \ x \in [3,5]. \end{cases}$$

Clearly, $\psi$ is a continuous function such that for each $p \in P$, $\psi(\varphi(p)) = f(p)$. Let $T = \{(x,y) \in [0,5]^2 : \psi(x) = \psi(y)\}$. Then $T$ is the union of the diagonal of $[0,5]^2$ and a rectangle (see Figure 1). Let $S$ be the rhombus in the plane with vertices $(1,3)$, $(2,4)$, $(3,3)$ and $(2,2)$ and let $r : [0,5]^2 \backslash \{(2,3)\} \to S$ be the radial retraction with the point $(2,3)$ as the center. Let $A = \eta^{-1}(S)$. Since $\eta$ is essential, by [3, p. 225], $\eta|A : A \to S$ is not homotopic to a constant map.

Given $(p,q) \in A$, we have $\psi(\varphi(g(p))) = \psi(\varphi(g(q)))$, so $f(g(p)) = f(g(q))$. Hence, $d(g(p),g(q)) < \frac{3}{N}$. Let $c \in C$. By the choice of $N$, $d(H(p,c),H(q,c)) < \frac{\varepsilon}{20}$. We claim that $(\varphi(H(p,c)),\varphi(H(q,c))) \neq (2,3)$. Suppose to the contrary that $\varphi(H(p,c)) = 2$ and $\varphi(H(q,c)) = 3$. Considering the options in the definition of $\varphi$, we obtain that $f(H(p,c)) = 1$ and $f(H(q,c)) = 0$. Thus, $H(p,c) \in \mathrm{cl}_P(W(j_0-2,j_0,N))$ and $H(q,c) \in \mathrm{cl}_P(W(i_0,i_0+2,N))$. This implies that

$\max\{d(H(p,c),g(q_0)),d(H(q,c),g(p_0))\} < \frac{3}{N}$. Hence, $d(g(p_0),g(q_0)) < \frac{\varepsilon}{20} + \frac{6}{N} < \frac{\varepsilon}{20} + \frac{3\varepsilon}{10} < \varepsilon$, a contradiction. We have shown that for every $(p,q) \in A$ and $c \in C$, $(\varphi(H(p,c)),\varphi(H(q,c))) \neq (2,3)$.




Notice that $S \subset T$. By the paragraph above, the map $\sigma : A \times C \to S$ given by $\sigma((p,q),c) = r(\varphi(H(p,c)),\varphi(H(q,c)))$ is well-defined. Since for each $(p,q) \in A$, $\sigma((p,q),s_0) = r(\varphi(H(p,s_0)),\varphi(H(q,s_0))) = r(\varphi(g(p),\varphi(g(q)))$ and $\sigma((p,q),t_0) = r(\varphi(h(p),\varphi(h(q)))$, the maps $\sigma_0, \sigma_1 : A \to S$ given by $\sigma_0(p,q) = r(\varphi(g(p),\varphi(g(q)))$ and $\sigma_1(p,q) = r(\varphi(h(p),\varphi(h(q)))$ are pseudo-homotopic. Since $S$ is an ANR, $\sigma_0$ and $\sigma_1$ are homotopic (see Claim 1 in [10]).

Notice that $\operatorname{Im} h \subset \varphi^{-1}([\frac{1}{2},1])$ or $\operatorname{Im} h \subset \varphi^{-1}([4,\frac{9}{2}])$. In the first case, for each $(p,q) \in A$, $(\varphi(h(p),\varphi(h(q))) \in [\frac{1}{2},1]^2$, so $\sigma_1(p,q)$ lies on the side of $S$ that joins the points $(2,2)$ and $(1,3)$. This implies that $\sigma_1$ is homotopic to a constant map. The second case is similar. We conclude that $\sigma_0$ is homotopic to a constant map.

Given $(p,q) \in A$, $\eta(p,q) = (\varphi(g(p)),\varphi(g(q))) \in S$, so $\eta(p,q) = r(\eta(p,q)) = \sigma_0(p,q)$. Hence, $\sigma_0 = \eta|A$ is not homotopic to a constant map. This contradiction completes the proof of the theorem. □

## 3. Conclusions

**Corollary 4.** *Let $g,h : P \to P$ be pseudo-homotopic maps. Suppose that $A$ is a nondegenerate subcontinuum of $P$ such that $g|A : A \to P$ is one-to-one. Then $g|A = h|A$.*

**Corollary 5.** *Let $H : P \times C \to P$ be a pseudo-homotopy between the maps $g$ and $h$. If $g \neq h$, then for each $c \in C$, $\bigcup\{A \in C(P) : A$ is nondegenerate and $H|A \times \{c\}$ is one-to-one$\}$ is not dense in $P$.*

Corollary 5 shows that if there is a pseudo-homotopy between two different non-constant maps, all the "levels" of the pseudo-homotopy must have a complicated behavior. On the other hand, a negative answer to Problem 1 would lead to answer other open problems on the pseudo-arc. Next we recall some of them.

**Problem 6** ([1, Problem 6]). *Let $e : P_1 \times \ldots \times P_m \to P_1 \times \ldots \times P_m$ be an embedding of a finite product of pseudo-arcs into itself. Must $e$ be a product of embeddings composed with a permutation of coordinates? Recently in* [1] *this problem has been solved for the product of two pseudo-arcs.*

**Problem 7** ([8, Question 14]). *Does there exist a continuum $X$ with the fixed point property such that $X \times P$ does not have the fixed point property?*

**Problem 8** ([8, Question 20]). *Assume that $r : P \times P \to \Delta = \{(x, x) \in P \times P : x \in P\}$ is a continuous retraction. Must $r$ be of the form $r(x, y) = (x, x)$ for all $(x, y)$ or $r(x, y) = (y, y)$ for all $(x, y)$?*

**Problem 9.** *Does $E(P)$, the space of all continuous functions from the pseudo-arc into itself, contain any nondegenerate compact connected sets other than collections of constant maps?*

Notice that Theorem 3 implies that if $\mathcal{A}$ is a nondegenerate continuum contained in $E(P)$, then $\mathcal{A}$ does not contain a one-to-one element of $E(P)$. This extends the result in [6] that says that in the space of homeomorphisms $\mathcal{H}(P)$ of the pseudo-arc there are not nondegenerate continua. Notice also that Theorem 3 implies that $P$ is not pseudo-contractible. Related to Problem 9, we can mention the following two important problems.

**Problem 10** ([8, Question 22]). *Does $E(P)$, the space of all continuous functions from the pseudo-arc into itself, contain any nondegenerate connected sets other than collections of constant maps?*

**Problem 11** ([8, Question 21]). *Is $\mathcal{H}(P)$, the topological group of all self-homeomorphisms of the pseudo-arc $P$, totally disconnected?*

**Acknowledgment.** The author wishes to thank the participants in the Fifth Workshop on Continua and Hyperspaces, celebrated in México City during the Summer of 2011, particularly Wayne Lewis and Enrique Castañeda for useful discussions.

## References

[1] Chacón-Tirado M.E., Illanes A., Leonel R., *Factorwise rigidity of embeddings of the products of pseudo-arcs*, Colloq. Math. **128** (2012), 7–14.

[2] Illanes A., Nadler S.B., Jr., *Hyperspaces Fundamentals and Recent Advances*, Monographs and Textbooks in Pure and Applied Mathematics, 216, Marcel Dekker, Inc., New York, Basel, 1999.

[3] Holsztyński W., *Universal mappings and fixed point theorems*, Bull. Acad. Pol. **15** (1967), 433–438.

[4] Holsztyński W., *Universality of the product mappings into products of $I^n$ and snake-like spaces*, Fund. Math. **64** (1969), 147–155.

[5] Kuperberg W., *Continua with the Houston Problem Book*, H. Cook, W.T. Ingram, K.T. Kuperberg, A. Lelek and P. Minc (Eds.), Lecture Notes in Pure and Applied Mathematics, 170, Marcel Dekker, New York, 1995, pp. 372–373.
[6] Lewis W., *Pseudo-arcs and connectedness in homeomorphism groups*, Proc. Amer. Math. Soc. **87** (1983), no. 4, 745–748.
[7] Lewis W., *The pseudo-arc*, Bol. Soc. Mat. Mexicana (3) **5** (1999), 25–77.
[8] Lewis W., *Indecomposable Continua*, Open Problems in Topology II, 304–318, edited by E. Pearl, Elsevier, 2007.
[9] Nadler S.B., Jr., *Continuum Theory. An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 158, Marcel Dekker, New York, 1992.
[10] Sobolewski M., *Pseudo-contractibility of chainable continua*, Topology Appl. **154** (2007), 2983–2987.

Instituto de Matematicas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México 04510, D.F.

*E-mail:* illanes@matem.unam.mx

(*Received* December 9, 2011, *revised* September 15, 2012)