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# UPPER BOUNDS FOR THE DOMINATION SUBDIVISION AND BONDAGE NUMBERS OF GRAPHS ON TOPOLOGICAL SURFACES

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*Abstract.* For a graph property $\mathcal{P}$ and a graph $G$, we define the domination subdivision number with respect to the property $\mathcal{P}$ to be the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to change the domination number with respect to the property $\mathcal{P}$. In this paper we obtain upper bounds in terms of maximum degree and orientable/non-orientable genus for the domination subdivision number with respect to an induced-hereditary property, total domination subdivision number, bondage number with respect to an induced-hereditary property, and Roman bondage number of a graph on topological surfaces.

*Keywords*: domination subdivision number, graph property, bondage number, Roman bondage number, induced-hereditary property, orientable genus, non-orientable genus

*MSC 2010*: 05C69

## 1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. For a vertex $x$ of $G$, $N(x)$ denotes the set of all neighbors of $x$ in $G$, $N[x] = N(x) \cup \{x\}$ and the degree of $x$ is $\deg(x) = |N(x)|$. The maximum and minimum degrees of vertices in the graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$ respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a graph $G$, let $x \in X \subseteq V(G)$. A vertex $y \in V(G)$ is a *private neighbor of $x$ with respect to $X$* if $N[y] \cap X = \{x\}$. The *private neighbor set of $x$ with respect to $X$* is $\mathrm{pn}[x, X] = \{y\colon N[y] \cap X = \{x\}\}$. A *perfect matching $M$* in $G$ is a set of independent edges in $G$ such that every vertex of $G$ is incident to an edge of $M$. For every edge $e = xy \in E(G)$ we define $\xi(e) = |N(x) \cup N(y)| = \deg(x) + \deg(y) - |N(x) \cap N(y)|$ and let $\xi(G) = \min\{\xi(e)\colon e \in E(G)\}$.

A surface is a connected compact Hausdorff space which is locally homeomorphic to an open disc in the plane. If a surface $\Sigma$ is obtained from the sphere by adding some number $g \geqslant 0$ of handles or some number $\overline{g} > 0$ of crosscaps, $\Sigma$ is said to be, respectively, orientable of genus $g = g(\Sigma)$ or non-orientable of genus $\overline{g} = \overline{g}(\Sigma)$. We shall follow the usual convention of denoting the surface of orientable genus $g$ or non-orientable genus $\overline{g}$, respectively, by $S_g$ or by $N_{\overline{g}}$. Any topological surface is homeomorphically equivalent either to $S_h$ $(h \geqslant 0)$, or to $N_k$ $(k \geqslant 1)$. For example, $S_1$, $N_1$, $N_2$ are the *torus*, the *projective plane*, and the *Klein bottle*, respectively. A graph $G$ is embeddable on a topological surface $S$ if it admits a drawing on the surface with no crossing edges. Such a drawing of $G$ on the surface $S$ is called an embedding of $G$ on $S$. An embedding of a graph $G$ on an orientable surface or non-orientable surface $\Sigma$ is minimal if $G$ cannot be embedded on any orientable or non-orientable surface $\Sigma'$ with $g(\Sigma') < g(\Sigma)$ or $\overline{g}(\Sigma') < \overline{g}(\Sigma)$, respectively. Graph $G$ is said to have orientable genus $g$ (non-orientable genus $\overline{g}$) if $G$ minimally embeds on a surface of orientable genus $g$ (non-orientable genus $\overline{g}$). An embedding of a graph $G$ on a surface $\Sigma$ is said to be 2-*cell* if every face of the embedding is homeomorphic to a disc. The set of faces of a particular embedding of $G$ on $S$ is denoted by $F(G)$. If every face of a graph embedding is three-sided, then the embedding is *triangular*. In a *quadrilateral embedding*, every face is four-sided.

A *Roman dominating function* (RDF) on a graph $G$ is defined in [19], [22] as a function $f\colon V(G) \to \{0,1,2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The *weight* of a RDF is the value $f(V(G)) = \sum\limits_{u \in V(G)} f(u)$. The *Roman domination number*, $\gamma_R(G)$, of $G$ is the minimum weight of a RDF on $G$. Following Jafari Rad and Volkmann [11], the *Roman bondage number* $b_R(G)$ of a graph $G$ with maximum degree at least two is the cardinality of a smallest set of edges $E_1 \subseteq E(G)$ for which $\gamma_R(G - E_1) > \gamma_R(G)$.

Let $\mathcal{I}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any non-empty subset of $\mathcal{I}$. We say that a *graph $G$ has the property $\mathcal{P}$* whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. For example, we list some graph properties:

- ▷ $\mathcal{O} = \{H \in \mathcal{I}\colon\ H \text{ is totally disconnected}\}$;
- ▷ $\mathcal{C} = \{H \in \mathcal{I}\colon\ H \text{ is connected}\}$;
- ▷ $\mathcal{M} = \{H \in \mathcal{I}\colon\ H \text{ has a perfect matching }\}$;
- ▷ $\mathcal{T} = \{H \in \mathcal{I}\colon\ \delta(H) \geqslant 1\ \}$.

A graph property $\mathcal{P}$ is called: (a) *induced-hereditary*, if from the fact that a graph $G$ has property $\mathcal{P}$, it follows that all induced subgraphs of $G$ also belong to $\mathcal{P}$, and (b) *nondegenerate* if $\mathcal{O} \subseteq \mathcal{P}$. Any set $S \subseteq V(G)$ such that the induced subgraph $\langle S, G\rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-*set*. A set of vertices $D \subseteq V(G)$ is a *dominating set* of $G$ if every vertex not in $D$ is adjacent to a vertex in $D$. The *domination number with respect to the property* $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of $G$. A dominating $\mathcal{P}$-set of $G$ with cardinality $\gamma_{\mathcal{P}}(G)$ is called a $\gamma_{\mathcal{P}}(G)$-*set*. If a property $\mathcal{P}$ is nondegenerate, then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. Note that $\gamma_{\mathcal{I}}(G)$ and $\gamma_{\mathcal{T}}(G)$ are well known as the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$, respectively. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [7] and has been studied, for example, in [15], [20], [21] and elsewhere.

For every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, the *plus bondage number with respect to the property* $\mathcal{P}$, denoted $b^+_{\mathcal{P}}(G)$, is the cardinality of a smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}}(G-U) > \gamma_{\mathcal{P}}(G)$. This concept was introduced by the present author in [21]. Since $\gamma_{\mathcal{P}}(G-E(G)) = |V(G)| > \gamma_{\mathcal{P}}(G)$ for every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, it follows that $b^+_{\mathcal{P}}(G)$ always exists.

For every graph $G$ with $\Delta(G) \geqslant 2$ and for each property $\mathcal{P} \subseteq \mathcal{I}$, we define the *domination (minus domination, plus domination*, respectively) *subdivision number with respect to the property* $\mathcal{P}$, denoted $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$ ($\mathrm{sd}^{-}_{\gamma_{\mathcal{P}}}(G)$, $\mathrm{sd}^{+}_{\gamma_{\mathcal{P}}}(G)$) to be the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to change (decrease, increase, respectively) $\gamma_{\mathcal{P}}(G)$. The following special cases for $\mathrm{sd}^{+}_{\gamma_{\mathcal{P}}}(G)$ have been investigated up to now: (a) $\mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)$—the domination subdivision number defined by Velammal [25] (note that $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{I}}}(G) = \mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)$), (b) $\mathrm{sd}^{+}_{\gamma_{\mathcal{T}}}(G)$—the total domination subdivision number introduced by Haynes et al. in [8], (c) $\mathrm{sd}^{+}_{\gamma_{\mathcal{M}}}(G)$—the paired domination subdivision number introduced by Favaron et al. in [5], and (d) $\mathrm{sd}^{+}_{\gamma_{\mathcal{C}}}(G)$—the connected domination subdivision number introduced by Favaron et al. in [4].

The rest of the paper is organized as follows. In Section 2 we begin the investigation of $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$ in case when $\mathcal{P} \subseteq \mathcal{I}$ is induced-hereditary and closed under union with $K_1$ graph property. We show that $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$ is well defined whenever $\Delta(G) \geqslant 2$ and we present upper bounds for $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$ in terms of degrees. In Section 3 for graphs with nonnegative Euler characteristic we obtain tight upper bounds for $\xi(G)$ in terms of maximum degree. In Section 4 we find upper bounds in terms of orientable/non-orientable genus and maximum degree for $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$, $\mathrm{sd}^{+}_{\gamma_{\mathcal{T}}}(G)$, $b_R(G)$ and $b^+_{\mathcal{P}}(G)$.

Note that each induced-hereditary and closed under union with $K_1$ property $\mathcal{P} \subseteq \mathcal{I}$ is, clearly, nondegenerate and hence $\gamma_{\mathcal{P}}(G)$ exists. For a graph $G$ and a set $U \subseteq E(G)$, by $S(G, U)$ we denote the graph obtained from $G$ by subdividing all edges belonging to $U$.

**Theorem 2.1.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. Let $G$ be a graph which contains an edge $uv$ such that $\deg(u) \geqslant 2$, $\deg(v) \geqslant 2$ and let $F \subseteq E(G)$ be the union of the set of all edges incident to $v$ and the set of all edges joining $u$ to a vertex in $N(u) - N[v]$. Then there is a set $U \subsetneq F$ with $\gamma_{\mathcal{H}}(S(G, U)) < \gamma_{\mathcal{H}}(S(G, F))$. In particular (Favaron et al. [3] when $\mathcal{H} = \mathcal{I}$), $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) \leqslant \xi(uv) - 1$.*

Proof. We denote shortly $G_1 = S(G, F)$. Let $N(v, G) = \{u = z_0, z_1, \ldots, z_p\}$, $p \geqslant 1$, and let $v_i \in V(G_1)$ be the subdivision vertex for $vz_i$, $i = 0, 1, \ldots, p$. Let $N(u, G) - N(v, G) = \{v = w_0, w_1, \ldots, w_q\}$, $q \geqslant 0$, $u_0 = v_0$ and if $q \geqslant 1$, then let $u_i \in V(G_1)$ be the subdivision vertex for $uw_i$, $i = 1, \ldots, q$. Among all $\gamma_{\mathcal{H}}(G_1)$-sets let $D_1$ be the one which contains a minimum number of subdivision vertices. Denote by $S$ the set of all subdivision vertices which belong to $D_1$. First assume $S$ is empty. Then $v \in D_1$. If $u \in D_1$, then $D_1 - \{v\}$ is a dominating $\mathcal{H}$-set of a graph $G'$ obtained from $G$ by subdividing all edges joining $u$ to a vertex in $N(u) - N[v]$ (it is possible that $G' = G$). If $u \notin D_1$, then there is $z_i \in D_1$ with $z_i u \in E(G)$. But then $D_1 - \{v\}$ is a dominating $\mathcal{H}$-set of $G$ ($\mathcal{H}$ is induced-hereditary). So, assume $S$ is not empty.

*Case 1*: $S = \{v_0\}$. If $u, v \notin D_1$, then all neighbors of $u$ and $v$ in $G$, except for $u$ and $v$, are in $D_1$; this implies $D_1 - \{v_0\}$ is a dominating $\mathcal{H}$-set of $G$. If exactly one of $u$ and $v$ is in $D_1$, then $D_1 - \{v_0\}$ is a dominating $\mathcal{H}$-set of $S(G, F - \{uv\})$. There are no other possibilities because $\mathcal{H}$ is induced-hereditary.

*Case 2*: $S = \{v_1\}$. If $z_1 \notin \mathrm{pn}[v_1, D_1]$, then the set $D_2 = (D_1 - \{v_1\}) \cup \{v\}$ is a dominating $\mathcal{H}$-set of $G_1$ ($\mathcal{H}$ is induced-hereditary and closed under union with $K_1$) of cardinality at most $\gamma_{\mathcal{H}}(G_1)$ and $D_2$ contains no subdivision vertices, a contradiction. If $v \in D_1$, then the set $D_3 = (D_1 - \{v_1\}) \cup \{z_1\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set without subdivision vertices, a contradiction. Since $v, v_0 \notin D_1$ it follows that $u \in D_1$ and if $p \geqslant 2$, then $z_2, \ldots, z_p \in D_1$. But then the set $(D_1 - \{v_1, u\}) \cup \{v\}$ is a dominating $\mathcal{H}$-set of a graph $G_2$ defined as follows: (a) $G_2 = G$ when $p = 1$, and (b) $G_2 = S(G, \{vz_2, \ldots, vz_p\})$ when $p \geqslant 2$.

*Case 3*: At least two subdivision vertices which are adjacent to $v$ are in $D_1$. Say, without loss of generality, $S_v = S \cap N(v, G_1) = \{v_r, v_{r+1}, \ldots, v_{r+s}\}$, $r \geqslant 0$, $s \geqslant 1$. Let $r \leqslant i \leqslant r + s$. Then $z_i \notin D_1$. Moreover, $z_i \notin \mathrm{pn}[v_i, D_1]$—otherwise the set $(D_1 - \{v_i\}) \cup \{z_i\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D_1$, a contradiction. But then the set $(D_1 - S_v) \cup \{v\}$ is a dominating $\mathcal{H}$-set of a graph $G_3$ obtained from $G_1$ by deleting $S_v$ and adding $vz_r, \ldots, vz_{r+s}$.

*Case 4*: $S = \{v_1, u_1\}$. Assume $v \in D_1$. This implies $z_1 \in \mathrm{pn}[v_1, D_1]$ and then the set $(D_1 - \{v_1\}) \cup \{z_1\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D_1$, a contradiction. Hence $v \notin D_1$. Now, assume $u \in D_1$. But then $w_1 \in \mathrm{pn}[u_1, D_1]$, which leads to $(D_1 - \{u_1\}) \cup \{w_1\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D_1$, a contradiction. Therefore there is no vertex in $D_1$ which dominates $v_0$, a contradiction.

*Case 5*: $S = \{u_1\}$. If $u \in D_1$, then $w_1 \in \mathrm{pn}[u_1, D_1]$, which leads to $D - \{u_1\}$ being a dominating $\mathcal{H}$-set of $S(G, F - \{uw_1\})$. So, let $u \notin D_1$. Hence $v \in D_1$. If $w_1 \notin \mathrm{pn}[u_1, D_1]$, then $D_1 - \{u_1\}$ is a dominating $\mathcal{H}$-set of $S(G, F - \{uw_1, uv\})$. Assume $w_1 \in \mathrm{pn}[u_1, D_1]$. If $u \notin \mathrm{pn}[u_1, D_1]$, then $(D_1 - \{u_1\}) \cup \{w_1\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D_1$, a contradiction. If $u \in \mathrm{pn}[u_1, D_1]$, then $(D_1 - \{u_1, v\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of a graph $G_4$ defined as follows: (a) $G_4 = G$ for $q = 1$, and (b) $G_4 = S(G, \{uw_2, \ldots, uw_q\})$ for $q \geqslant 2$.

*Case 6*: At least two subdivision vertices which are adjacent to $u$ are in $D_1$. Say, without loss of generality, $S_u = S \cap N(u, G_1) = \{u_r, u_{r+1}, \ldots, u_{r+s}\}$ where $0 \leqslant r$ and $s \geqslant 1$. Let $r \leqslant i \leqslant s + r$. Then $w_i \notin D_1$. If $w_i \in \mathrm{pn}[u_i, D_1]$, then the set $(D_1 - \{u_i\}) \cup \{w_i\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D_1$, a contradiction. Thus $w_i \notin \mathrm{pn}[u_i, D_1]$, $i = r, \ldots, r + s$. If there is no $z_j \in D_1$, $j \geqslant 1$, with $z_j u \in E(G)$, then the set $(D_1 - S_u) \cup \{u\}$ is a dominating $\mathcal{H}$-set of a graph $G_1$, a contradiction. If there is $z_j \in D_1$, $j \geqslant 1$ with $z_j u \in E(G)$, then $D_1 - S_u$ is a dominating $\mathcal{H}$-set of a graph $G_5$ obtained from $G_1$ by deleting $S_u$ and adding $uw_r, \ldots, uw_{r+s}$. □

**Observation 2.2.** *Let $\mathcal{H}$ be a nondegenerate graph property. If $G$ is a graph with $\Delta(G) \geqslant 2$ and $\gamma_{\mathcal{H}}(G) = 1$, then $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) = sd^{+}_{\gamma_{\mathcal{H}}}(G) = 1$.*

By Theorem 2.1 and Observation 2.2 it immediately follows that $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G)$ is well-defined for every graph $G$ with $\Delta(G) \geqslant 2$ provided $\mathcal{H} \subseteq \mathcal{I}$ is an induced-hereditary and closed under union with $K_1$ graph property.

**Observation 2.3.** *Let $\mathcal{H}$ be a nondegenerate graph property. Then*

(i) $\gamma_{\mathcal{H}}(C_n) = \lceil \frac{1}{3} n \rceil$, *where* $n \geqslant 3$;

(ii) $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(C_{3k}) = sd^{+}_{\gamma_{\mathcal{H}}}(C_{3k}) = 1$, $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(C_{3k+1}) = sd^{+}_{\gamma_{\mathcal{H}}}(C_{3k+1}) = 3$, *and* $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(C_{3k+2}) = sd^{+}_{\gamma_{\mathcal{H}}}(C_{3k+2}) = 2$, *where* $k \geqslant 1$.

By Observation 2.3(ii) it immediately follows that the bound stated in Theorem 2.1 is attainable when $G = C_{3k+1}$, $k \geqslant 1$.

Define $\mathbf{V}^-_{\mathcal{H}}(G) = \{v \in V(G)\colon\ \gamma_{\mathcal{H}}(G - v) < \gamma_{\mathcal{H}}(G)\}$. The next results in this section show that the set $\mathbf{V}^-_{\mathcal{H}}(G)$ plays an important role in studying the subdivision numbers with respect to a graph property.

**Observation 2.4.** *Let $\mathcal{H}$ be a nondegenerate and closed under union with $K_1$ graph property. Let $G$ be a graph.*

(i) [20] $\mathbf{V}^-_{\mathcal{H}}(G) = \{v \in V(G)\colon\ \gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1\}$.

(ii) *If $v \in \mathbf{V}^-_{\mathcal{H}}(G)$, then $\gamma_{\mathcal{H}}(G') \leqslant \gamma_{\mathcal{H}}(G)$, where $G'$ is a graph which results from subdividing at least one edge incident to $v$.*

Proof. (ii) Let $M$ be a $\gamma_{\mathcal{H}}(G-v)$-set. Since $v \in \mathbf{V}^-_{\mathcal{H}}(G)$, $M$ is not a dominating $\mathcal{H}$-set of $G$. Since $\mathcal{H}$ is closed under union with $K_1$, $M \cup \{v\}$ is a dominating $\mathcal{H}$-set of both $G'$ and $G$. Hence $M \cup \{v\}$ is a $\gamma_{\mathcal{H}}(G)$-set and the result follows. □

In special cases where a graph has some structural property we can obtain better upper bounds for $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G)$ than that stated in Theorem 2.1.

**Theorem 2.5.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. Let $G$ be a graph, $v \in V(G)$, $\deg(v) \geqslant 2$ and let $F \subseteq E(G)$ consist of all edges incident to $v$. Then at least one of the following assertions holds:*

(i) *there is $U \subseteq F$ with $\gamma_{\mathcal{H}}(G) \neq \gamma_{\mathcal{H}}(S(G,U))$ (in particular $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) \leqslant \deg(v)$);*

(ii) *$v \in \mathbf{V}^-_{\mathcal{H}}(G)$;*

(iii) *there exist $u \in N(v,G) \cap \mathbf{V}^-_{\mathcal{H}}(G)$ and a $\gamma_{\mathcal{H}}(G)$-set $D_u$ such that $N(v,G) \subseteq D_u$, $v \notin D$ and $\mathrm{pn}[u, D_u] = \{u\}$.*

Proof. Denote shortly $G_1 = S(G,F)$. Assume (i) does not hold. Hence $\gamma_{\mathcal{H}}(G_1) = \gamma_{\mathcal{H}}(G)$. Among all $\gamma_{\mathcal{H}}(G_1)$-sets let $D$ be the one which contains a minimum number of subdivision vertices. Let all neighbors of $v$ in $G$ be $w_1, \ldots, w_r$ and let $v_i \in V(G_1)$ be the subdivision vertex for $vw_i$, $i = 1, 2, \ldots, r$. Let $S$ be the set of all subdivision vertices which belong to $D$ and if $S$ is not empty let $S = \{v_1, \ldots, v_k\}$. If $S$ is empty, then $v \in D$ and $D - \{v\}$ is a dominating $\mathcal{H}$-set of $G - v$ ($\mathcal{H}$ is induced-hereditary). Hence $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_1) = |D| \geqslant 1 + \gamma_{\mathcal{H}}(G - v)$ and by the definition of $\mathbf{V}^-_{\mathcal{H}}(G)$ it follows that (ii) holds. Now assume $k \geqslant 1$. We distinguish two cases according to $k$.

*Case 1*: $k = 1$. If $v \in D$, then since $\mathcal{H}$ is induced-hereditary, $w_1 \in \mathrm{pn}[v_1, D]$. But then $D - \{v_1\}$ is a dominating $\mathcal{H}$-set of the graph $G_2$ obtained from $G_1$ by deleting $v_1$ and adding $vw_1$, a contradiction. So $v \notin D$ which immediately implies $w_2, \ldots, w_r \in D$. If $w_1 \in D$, then $D - \{v_1\}$ is a dominating $\mathcal{H}$-set of $G_2$, a contradiction. If $w_1 \notin D$ and $w_1 \notin \mathrm{pn}[v_1, D]$, then $(D - \{v_1\}) \cup \{v\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set without subdivision vertices—a contradiction. So, let $w_1 \in \mathrm{pn}[v_1, D]$. But then $D_{w_1} = (D - \{v_1\}) \cup \{w_1\}$

is a $\gamma_{\mathcal{H}}(G)$-set with $\mathrm{pn}[w_1, D_{w_1}] = \{w_1\}$. This implies $w_1 \in \mathbf{V}^{-}_{\mathcal{H}}(G)$ and then (iii) holds (with $u \equiv w_1$).

*Case 2*: $k \geqslant 2$. By the choice of $D$ it follows that $w_i \notin D$ for all $i = 1, \ldots, k$ (otherwise $D - \{v_i\}$ would be a dominating $\mathcal{H}$-set of $G_1$, a contradiction). If $w_i \in \mathrm{pn}[v_i, D]$ for some $i \in \{1, \ldots, k\}$, then $(D - \{v_i\}) \cup \{w_i\}$ is a $\gamma_{\mathcal{H}}(G_1)$-set with fewer subdivision vertices than $D$, a contradiction. Hence $w_i \notin \mathrm{pn}[v_i, D]$ for all $i = 1, \ldots, k$. But then $(D - S) \cup \{v\}$ is a dominating $\mathcal{H}$-set of $G_1$, a contradiction. □

The next two corollaries follow immediately from Theorem 2.5.

**Corollary 2.6.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. Let $G$ be a graph, $v \in V(G)$ and $\deg(v) \geqslant 2$. Then there is a subset $U$ of the set of all edges incident to $v$ with $\gamma_{\mathcal{H}}(G) \neq \gamma_{\mathcal{H}}(S(G, U))$ (in particular $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) \leqslant \deg(v)$) provided one of the following holds:*

(i) *$v$ and none of the isolated vertices of the graph $\langle N(v), G\rangle$ belong to $\mathbf{V}^{-}_{\mathcal{H}}(G)$;*

(ii) *$v \notin \mathbf{V}^{-}_{\mathcal{H}}(G)$ and $\langle N(v), G\rangle \notin \mathcal{H}$.*

**Corollary 2.7.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. If a graph $G$ has $\Delta(G) \geqslant 2$ and $\mathbf{V}^{-}_{\mathcal{H}}(G) = \emptyset$, then $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) \leqslant \min\{\deg(x)\colon x \in V(G) \text{ and } \deg(x) \geqslant 2\}$.*

**Corollary 2.8.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. If a graph $G$ has $\Delta(G) \geqslant 2$ and $\gamma_{\mathcal{H}}(G) < (|V(G)| + \Delta(G))/(\Delta(G) + 1)$, then $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G) \leqslant \min\{\deg(x)\colon x \in V(G) \text{ and } \deg(x) \geqslant 2\}$.*

Proof. Assume $x \in \mathbf{V}^{-}_{\mathcal{H}}(G)$. Then $|V(G)| \leqslant (\gamma_{\mathcal{H}}(G) - 1)(\Delta(G) + 1) + 1$, which implies $\gamma_{\mathcal{H}}(G) \geqslant (|V(G)| + \Delta(G))/(\Delta(G) + 1)$, a contradiction. The result now follows by Corollary 2.7. □

**Corollary 2.9.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. Let $G$ be a graph and let $2 \leqslant \delta(G) \leqslant \Delta(G) < \mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G)$. Then $\mathbf{V}^{-}_{\mathcal{H}}(G)$ is a dominating set of $G$.*

## 3. Upper bounds for $\xi(G)$

For 2-cell embeddings, we have the important result known as *generalized Euler's formula*.

**Theorem 3.1** (Thomassen [24]). *If $G$ is 2-cell embedded on surface $\Sigma$ having genus $g$ or non-orientable genus $\overline{g}$ and if the embedded $G$ has $|V(G)| = p$ vertices, $|E(G)| = q$ edges and $|F(G)| = f$ faces, then $p - q + f = 2 - 2g$ or $p - q + f = 2 - \overline{g}$, respectively.*

The following two results are of paramount importance when working with minimal embeddings. The former is due to J. W. T. Youngs [26] and the latter to Parsons, Pica, Pisanski and Ventre [18].

**Theorem 3.2.** *Every minimal orientable embedding of a graph $G$ is 2-cell.*

**Theorem 3.3.** *Every graph $G$ has a minimal non-orientable embedding which is 2-cell.*

The *Euler characteristic* of a surface is equal to $|V(G)| + |F(G)| - |E(G)|$ for any graph $G$ that is 2-cell embedded on that surface. The Euclidean plane, the projective plane, the torus, and the Klein bottle are all the surfaces of nonnegative Euler characteristic.

Let $G$ be a graph 2-cell embedded on a surface $S$. For each edge $e = xy \in E(G)$ we define

$$D_e = D_{xy} = \frac{1}{d(x)} + \frac{1}{d(y)} + \frac{1}{r_e^1} + \frac{1}{r_e^2} - 1,$$

where $r_e^1$ is the number of edges on the boundary of a face on one side of $e$, and $r_e^2$ is the number of edges on the boundary of the face on the other side of $e$. In case when an edge $e$ is on the boundary of exactly one face, say $f$, let $r_e^1 = r_e^2 = 2r_e$, where $r_e$ is the number of edges on the boundary of $f$. We observe that $\sum\limits_{e\in E(G)} (1/d(x)+1/d(y)) = |V(G)|$ and $\sum\limits_{e\in E(G)} (1/r_e^1 + 1/r_e^2) = |F(G)|$, and therefore

$$\sum_{e\in E(G)} D_e = |V(G)| + |F(G)| - |E(G)|. \tag{3.1}$$

**Theorem 3.4.** *Let $G$ be a connected graph and let at least one of $g(G) = 0$ and $\overline{g}(G) = 1$ hold. Then $\xi(G) \leqslant \Delta(G) + 3$. Moreover, $\xi(G) \leqslant \Delta(G) + 2$ provided one of the following assertions holds:*

($P_1$) *$\Delta(G) \notin \{3, 4, 5, 6, 7\}$;*

($P_2$) *$\Delta(G) \in \{6, 7\}$ and every edge $e = xy \in E(G)$ with $d(x) = 5$ and $d(y) = \Delta(G)$ is contained in at most one triangle.*

P r o o f. Suppose $G$ is 2-cell embedded on at least one of $S_0$ and $N_1$. Let $e = xy \in E(G)$, $d(x) \leqslant d(y)$ and $r_e^1 \leqslant r_e^2$.

*Case 1*: One of ($P_1$) and ($P_2$) holds. Assume to the contrary that $\xi(G) \geqslant \Delta(G)+3$. Hence $\Delta(G) \geqslant 6$. If $d(x) \leqslant 3$, then $d(x) = 3$, $d(y) = \Delta(G)$ and $r_e^1 \geqslant 4$; hence $D_e \leqslant \frac{1}{3}+\frac{1}{\Delta(G)}+\frac{1}{4}+\frac{1}{4}-1 \leqslant 0$. If $d(x) = 4$, then $d(y) \geqslant \Delta(G)-1+|N(x)\cap N(y)|$, which implies either $r_e^1 \geqslant 4$ and $d(y) \in \{\Delta(G)-1, \Delta(G)\}$ or $r_e^1 = 3, r_e^2 \geqslant 4$ and $d(y) = \Delta(G)$; hence either $D_e \leqslant \frac{1}{4} + \frac{1}{\Delta(G)-1} + \frac{1}{4} + \frac{1}{4} - 1 < 0$ or $D_e \leqslant \frac{1}{4} + \frac{1}{\Delta(G)} + \frac{1}{3} + \frac{1}{4} - 1 \leqslant 0$.

Let $d(x) = 5$. Then $d(y) \geqslant \Delta(G) - 2 + |N(x) \cap N(y)|$, which leads to $5 \leqslant d(y) \in \{\Delta(G) - 2, \Delta(G) - 1, \Delta(G)\}$. If $d(y) = \Delta(G) - 2$, then $r_e^1 \geqslant 4$; hence $D_e \leqslant \frac{1}{5} + \frac{1}{\Delta(G)-2} + \frac{1}{4} + \frac{1}{4} - 1 < 0$. If $d(y) = \Delta(G) - 1$, then $r_e^2 \geqslant 4$; hence $D_e \leqslant \frac{1}{5}+\frac{1}{\Delta(G)-1}+\frac{1}{3}+\frac{1}{4}-1 < 0$. If $d(y) = \Delta(G)$, then (a) $D_e \leqslant \frac{1}{5}+\frac{1}{\Delta(G)}+\frac{1}{3}+\frac{1}{3}-1 < 0$ when $\Delta(G) \geqslant 8$, and (b) $D_e \leqslant \frac{1}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{4} - 1 < 0$ when $\Delta(G) \in \{6, 7\}$.

Finally, if $d(x) \geqslant 6$, then $D_e \leqslant \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 = 0$.

Therefore $1 \leqslant |V(G)| + |F(G)| - |E(G)| = \sum_{e\in E(G)} D_e \leqslant 0$, a contradiction.

*Case 2*: $\Delta(G) \in \{6, 7\}$ and there is an edge $e = xy \in E(G)$ with $d(x) = 5$ and $d(y) = \Delta(G)$ which belongs to at least 2 triangles. Clearly $\xi(e) \leqslant \Delta(G) + 3$.

*Case 3*: $\Delta(G) = 5$. Assume to the contrary that $\xi(G) \geqslant \Delta(G)+4$. Then one of the following conditions holds: (a) $d(x) = 4$, $d(y) = 5$ and $r_e^1 \geqslant 4$, (b) $d(x) = d(y) = 5$ and $r_e^2 \geqslant 4$. Hence $D_e < 0$ and we obtain a contradiction as in Case 1.

*Case 4*: $\Delta(G) = 4$. Assume $G$ is regular. Then $G$ contains a triangle-otherwise $D_e \leqslant 0$ for each edge $e \in E(G)$, a contradiction.

*Case 5*: $\Delta(G) \leqslant 3$. Obviously $\xi(G) \leqslant \Delta(G) + 3$. □

The equality $\xi(G) = \Delta(G) + 3$ holds at least for triangle-free cubic planar (projective) graphs. For example, such graphs are: (a) a prism graph $CL_n$, $n \geqslant 4$, which is a graph corresponding to the skeleton of an $n$-prism, and (b) the Petersen graph which is nonplanar and can be embedded without crossings in the projective plane.

**Theorem 3.5.** *Let $G$ be a connected graph and let at least one of the identities $g(G) = 1$ and $\overline{g}(G) = 2$ hold. Then $\xi(G) \leqslant \Delta(G) + 4$ with equality if and only if one of the following conditions is valid:*

($P_3$) *$G$ is 4-regular without triangles;*

($P_4$) *$G$ is 6-regular and no edge of $G$ belongs to at least 3 triangles.*

Proof. Suppose $G$ is 2-cell embedded on at least one of $S_1$ and $N_2$. Let $e = xy \in E(G)$, $d(x) \leqslant d(y)$ and $r_e^1 \leqslant r_e^2$.

Assume that $\xi(G) \geqslant \Delta(G) + 4$. Hence $\delta(G) \geqslant 4$. First let $d(x) = 4$. Then $d(y) = \Delta(G)$ and $r_e^1 \geqslant 4$, which leads to $D_e \leqslant \frac{1}{4} + \frac{1}{\Delta(G)} + \frac{1}{4} + \frac{1}{4} - 1 \leqslant 0$ with equality when $d(x) = d(y) = \Delta(G) = 4$ and $r_e^1 = r_e^2 = 4$. If $d(x) = 5$, then either $d(y) = \Delta(G)$ and $r_e^2 \geqslant 4$, or $d(y) = \Delta(G)-1$ and $r_e^1 \geqslant 4$; hence either $D_e \leqslant \frac{1}{5}+\frac{1}{\Delta(G)}+\frac{1}{3}+\frac{1}{4}-1 < 0$ or $D_e \leqslant \frac{1}{5} + \frac{1}{\Delta(G)-1} + \frac{1}{4} + \frac{1}{4} - 1 < 0$. Now, let $d(x) = 6$. Then either $d(y) = \Delta(G)$

and $r_e^1 \geqslant 3$, $r_e^2 \geqslant 3$ or $d(y) = \Delta(G) - 1$, $r_e^1 \geqslant 3$ and $r_e^2 \geqslant 4$ or $d(y) = \Delta(G) - 2$ and $r_e^1 \geqslant 4$. Hence either $D_e \leqslant \frac{1}{6} + \frac{1}{\Delta(G)} + \frac{1}{3} + \frac{1}{3} - 1 \leqslant 0$ with equality when $d(x) = d(y) = \Delta(G) = 6$ and $r_e^1 = r_e^2 = 3$, or $D_e \leqslant \frac{1}{6} + \frac{1}{\Delta(G)-1} + \frac{1}{3} + \frac{1}{4} - 1 < 0$ or $D_e \leqslant \frac{1}{6} + \frac{1}{\Delta(G)-2} + \frac{1}{4} + \frac{1}{4} - 1 < 0$, respectively. Finally, if $d(x) \geqslant 7$, then $D_e \leqslant \frac{1}{7} + \frac{1}{7} + \frac{1}{3} + \frac{1}{3} - 1 < 0$.

Therefore $0 = |V(G)| + |F(G)| - |E(G)| = \sum_{e \in E(G)} D_e \leqslant 0$ with equality if and only if one of the following conditions is valid:

(a) $G$ is 4-regular and $r_e^1 = r_e^2 = 4$ for each $e \in E(G)$;

(b) $G$ is 6-regular and $r_e^1 = r_e^2 = 3$ for each $e \in E(G)$.

Thus $\xi(G) = \Delta(G) + 4$ and one of ($P_3$) and ($P_4$) holds.

It remains to note that (i) if ($P_3$) holds, then clearly $\xi(G) = \Delta(G) + 4$, and (ii) if $G$ is 6-regular, then Theorem 3.1 implies $r_e^1 = r_e^2 = 3$ for each edge $e \in E(G)$; therefore $\xi(G) = \Delta(G) + 4$ when ($P_4$) is satisfied. □

It follows from Theorem 3.1 that a 4-regular graph without triangles has a quadrilateral embedding. A classification of 4-regular graphs with quadrilateral emdedding on the torus and the Klein bottle was given by Altshuler [1] and Nakamoto and Negami [16], respectively. Theorem 3.1 also implies that a graph with minimum degree 6 embedded in the torus or the Klein bottle is a 6-regular triangulation. Altshuler [1] found a characterization of 6-regular toroidal maps and Negami [17] characterized 6-regular graphs which embed in the Klein bottle.

## 4. Upper bounds for the domination subdivision and bondage numbers

We will need the following results.

**Theorem 4.1** (Haynes et al. [9]). *For any connected graph $G$ with adjacent vertices $u$ and $v$, each of them of degree at least two, we have* $\mathrm{sd}^+_{\gamma_\mathcal{T}}(G) \leqslant \xi(uv) - 1$.

**Theorem 4.2** (Samodivkin [21]). *Let $\mathcal{H}$ be a nondegenerate and induced-hereditary graph property. For any connected graph $G$ with adjacent vertices $u$ and $v$,* $b^+_{\mathcal{H}}(G) \leqslant \xi(uv) - 1$.

**Theorem 4.3** (Jafari Rad and Volkmann [11]). *Let $G$ be a graph and $xy, yz \in E(G)$. Then $b_R(G) \leqslant \xi(xy) + d(z) - 3$. If $xz \in E(G)$, then $b_R(G) \leqslant \xi(xy) + d(z) - 4$.*

If $\xi(xy) = \xi(G)$, then by Theorem 4.3 we obtain the next result immediately.

**Corollary 4.4.** *Let $G$ be a connected graph of order at least 3. Then $b_R(G) \leqslant \xi(G)+\Delta(G)-3$. If every edge of $G$ lies in a triangle, then $b_R(G) \leqslant \xi(G)+\Delta(G)-4$.*

First we concentrate on graphs with nonnegative Euler characteristic. Combining Theorem 3.4 and Theorem 3.5 with Theorem 2.1 and Theorem 4.1 yields:

**Theorem 4.5.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property and let $G$ be a connected graph with $\delta(G) \geqslant 2$. Let at least one of the equalities $g(G) = i$ and $\overline{g}(G) = 1 + i$ be valid for some $i \in \{0,1\}$. Then $\max\{\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G), \mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)\} \leqslant \Delta(G) + 2 + i$. Moreover:* (a) $\max\{\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G), \mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)\} \leqslant \Delta(G) + 1$ *provided $i = 0$ and one of* $(\mathrm{P}_1)$ *and* $(\mathrm{P}_2)$ *holds, and* (b) $\max\{\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G), \mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)\} \leqslant \Delta(G) + 2$ *provided $i = 1$ and neither* $(\mathrm{P}_3)$ *nor* $(\mathrm{P}_4)$ *holds.*

Combining Theorem 3.4 and Theorem 3.5 with Theorem 4.2 we obtain

**Theorem 4.6.** *Let $\mathcal{H}$ be a nondegenerate and induced-hereditary graph property. Let $G$ be a nontrivial connected graph and let at least one of the equalities $g(G) = i$ and $\overline{g}(G) = 1 + i$ be valid for some $i \in \{0,1\}$. Then $b^{+}_{\mathcal{H}}(G) \leqslant \Delta(G) + 2 + i$. Moreover:* (a) $b^{+}_{\mathcal{H}}(G) \leqslant \Delta(G) + 1$ *provided $i = 0$ and one of* $(\mathrm{P}_1)$ *and* $(\mathrm{P}_2)$ *holds, and* (b) $b^{+}_{\mathcal{H}}(G) \leqslant \Delta(G) + 2$ *provided $i = 1$ and neither* $(\mathrm{P}_3)$ *nor* $(\mathrm{P}_4)$ *holds.*

The inequality $b^{+}_{\mathcal{H}}(G) \leqslant \Delta(G) + 2 + i$ stated in Theorem 4.6 was proven by (a) Kang and Yuan [14] for $g(G) = 0$ and $\mathcal{H} = \mathcal{I}$, (b) Samodivkin [21] when $g(G) = 0$ and $\mathcal{H}$ is additive and induced-hereditary, (c) Carlson and Develin [2] for $g(G) = 1$ and $\mathcal{H} = \mathcal{I}$, and (d) Gagarin and Zverovich [6] for $g(G) \in \{0,1\}$, $\overline{g}(G) \in \{1,2\}$ and $\mathcal{H} = \mathcal{I}$.

As we already know a 6-regular graph embedded in the torus or the Klein bottle is a triangulation. Combining Theorem 3.4 and Theorem 3.5 with Corollary 4.4 we obtain the following result.

**Theorem 4.7.** *Let $G$ be a connected graph of order at least 3 and let at least one of the equalities $g(G) = i$ and $\overline{g}(G) = 1 + i$ be valid for some $i \in \{0,1\}$. Then (Jafari Rad and Volkmann* [12] *when $g(G) = 0$) $b_R(G) \leqslant 2\Delta(G) + i$. Moreover:*

(a) $b_R(G) \leqslant 2\Delta(G) - 1$ *provided $i = 0$ and one of* $(\mathrm{P}_1)$ *and* $(\mathrm{P}_2)$ *holds, and*

(b) $b_R(G) \leqslant 2\Delta(G)$ *provided $i = 1$ and* $(\mathrm{P}_3)$ *does not hold.*

Next we find upper bounds in terms of orientable/non-orientable genus for $\mathrm{sd}^{\neq}_{\gamma_{\mathcal{P}}}(G)$, $\mathrm{sd}^{+}_{\gamma_{\mathcal{I}}}(G)$, $b_R(G)$ and $b^{+}_{\mathcal{P}}(G)$. We need the following notation and results.

Let

$$h_3(x) = \begin{cases} 2x+13 & \text{for } 0 \leqslant x \leqslant 3, \\ 4x+7 & \text{for } x \geqslant 3, \end{cases} \qquad h_4(x) = \begin{cases} 8 & \text{for } x = 0, \\ 4x+5 & \text{for } x \geqslant 1, \end{cases}$$

$$k_3(x) = \begin{cases} 2x+11 & \text{for } 1 \leqslant x \leqslant 2, \\ 2x+9 & \text{for } 3 \leqslant x \leqslant 5, \\ 2x+7 & \text{for } x \geqslant 6, \end{cases} \quad \text{and} \quad k_4(x) = \begin{cases} 8 & \text{for } x = 1, \\ 2x+5 & \text{for } x \geqslant 2. \end{cases}$$

**Theorem 4.8** (Ivančo [10]). *If $G$ is a connected graph of orientable genus $g$ and minimum degree at least 3, then $G$ contains an edge $e = xy$ such that $\deg(x) + \deg(y) \leqslant h_3(g)$. Furthermore, if $G$ does not contain 3-cycles, then $\deg(x)+\deg(y) \leqslant h_4(g)$. Moreover, all bounds are the best possible.*

**Theorem 4.9** (Jendrol′ and Tuhársky [13]). *If $G$ is a connected graph of minimum degree at least 3 on a nonorientable surface of genus $\overline{g} \geqslant 1$, then $G$ contains an edge $e = xy$ such that $\deg(x) + \deg(y) \leqslant k_3(\overline{g})$. Furthermore, if $G$ does not contain 3-cycles, then $\deg(x) + \deg(y) \leqslant k_4(\overline{g})$. Moreover, all bounds are the best possible.*

The next theorem follows by combining Theorem 2.1 and Theorem 4.1 with Theorem 4.8 and Theorem 4.9.

**Theorem 4.10.** *Let $\mathcal{H}$ be an induced-hereditary and closed under union with $K_1$ graph property. For a connected graph $G$ of orientable genus $g$, non-orientable genus $\overline{g}$ and minimum degree at least 3, we have $\max\{\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G), \mathrm{sd}^{+}_{\gamma_{\mathcal{T}}}(G)\} \leqslant \min\{h_3(g), k_3(\overline{g})\} - 1$. Furthermore, if $G$ does not contain 3-cycles, then*

$$\max\{\mathrm{sd}^{\neq}_{\gamma_{\mathcal{H}}}(G), \mathrm{sd}^{+}_{\gamma_{\mathcal{T}}}(G)\} \leqslant \min\{h_4(g), k_4(\overline{g})\} - 1.$$

Corollary 4.4, Theorem 4.8 and Theorem 4.9 together lead to

**Theorem 4.11.** *Let $G$ be a connected graph of minimum degree at least 3, orientable genus $g$ and non-orientable genus $\overline{g}$. Then $b_R(G) \leqslant \min\{h_3(g), k_3(\overline{g})\} + \Delta(G) - 3$. If every edge of $G$ lies in a triangle, then $b_R(G) \leqslant \min\{h_3(g), k_3(\overline{g})\} + \Delta(G) - 4$. If $G$ does not contain triangles, then $b_R(G) \leqslant \min\{h_4(g), k_4(\overline{g})\} + \Delta(G) - 3$.*

Gagarin and Zverovich [6] have recently proposed the following conjecture.

**Conjecture 4.12.** *For a connected graph $G$ of orientable genus $g$ and non-orientable genus $\overline{g}$ we have, $b(G) \leqslant \min\{c_g, c'_{\overline{g}}\}$, where $c_g$ and $c_{\overline{g}}$ are constants depending, respectively, on the orientable and non-orientable genera of $G$.*

In this connection, combining Theorem 4.2 with Theorem 4.8 and Theorem 4.9 we have the following result.

**Theorem 4.13.** *Let $\mathcal{H}$ be a nondegenerate and induced-hereditary graph property. For a nontrivial connected graph $G$ of orientable genus $g$, non-orientable genus $\overline{g}$ and minimum degree at least 3 we have $b^+_{\mathcal{H}}(G) \leqslant \min\{h_3(g), k_3(\overline{g})\} - 1$. Furthermore, if $G$ does not contain 3-cycles, then $b^+_{\mathcal{H}}(G) \leqslant \min\{h_4(g), k_4(\overline{g})\} - 1$.*

The next conjecture in the case provided $\mathcal{P} = \mathcal{I}$ is the main outstanding conjecture on ordinary bondage number.

**Conjecture 4.14** (Teschner [23] when $\mathcal{P} = \mathcal{I}$). *Let $\mathcal{P}$ be a nondegenerate and induced-hereditary graph property. Then for any graph $G$, $b^+_{\mathcal{P}}(G) \leqslant 1.5\Delta(G)$.*

Theorem 4.13 gives particular support for this conjecture. Namely, Conjecture 4.14 is true when $\min\{h_3(g), k_3(\overline{g})\} - 1 \leqslant 1.5\Delta(G)$ and $\delta(G) \geqslant 3$.

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