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INTEGER MATRICES RELATED TO LIOUVILLE'S FUNCTION

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Abstract. In this note, we construct some integer matrices with determinant equal to certain summation form of Liouville's function. Hence, it offers a possible alternative way to explore the Prime Number Theorem by means of inequalities related to matrices, provided a better estimate on the relation between the determinant of a matrix and other information such as its eigenvalues is known. Besides, we also provide some comparisons on the estimate of the lower bound of the smallest singular value. Such discussion may be extended to that of Riemann hypothesis.

Keywords: Liouville's function, determinant, LU decomposition

MSC 2010: 11A25, 11C20, 15A15, 15B36

1. Introduction

We shall use n to denote an arbitrary positive integer throughout this paper. Define $\Omega(n)$ as the number of prime factors of n and $\omega(n)$ as that of distinct prime factors. The Möbius function $\mu(n)$ is $(-1)^{\omega(n)}$ when n is square-free and 0 elsewhere.

Since the time of Landau (cf. [5] or [7]), we have known that $M(n) = \sum\limits_{k=1}^{n} \mu(k)$ is closely related to the Prime Number Theorem (PNT):

$$\text{PNT} \iff M(n) = \mathrm{o}(n).$$

Redheffer [6] in 1977 introduced the matrix $R_n = (r_{ij}) \in \mathscr{M}_n$ consisting of 0 or 1 defined by

$$r_{ij} = \begin{cases} 1 & \text{if } \ i \mid j \text{ or } j = 1, \\ 0 & \text{else,} \end{cases}$$

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and showed that its determinant $\det R_n = M(n)$. This suggests a new possibility of proving the Prime Number Theorem by means of inequalities estimation related to matrices.

The Liouville function λ is defined by $\lambda(n) = (-1)^{\Omega(n)}$. Put $L(n) = \sum_{k=1}^{n} \lambda(k)$, it is also noted in Landau's thesis that

$$\text{PNT} \iff L(n) = \mathrm{o}(n).$$

We suggest the following new matrix $S_n = (s_{ij}) \in \mathscr{M}_n$ which contains more zeros than R_n. Put

$$s_{ij} = \begin{cases} 1 & \text{if } \ j = 1 \text{ or } j/i \text{ is a square-free integer,} \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

We shall prove

Theorem 1.1. $\det S_n = L(n)$.

In particular, we have

$$\text{PNT} \iff \det S_n = \mathrm{o}(n).$$

Denote by $C_k(S_n)$ the k-th column of S_n. If we just apply Hadamard's inequality

$$|\det S_n| \leqslant \prod_{k=1}^{n} \|C_k(S_n)\|_2,$$

we get a worse result than the trivial bound $|\det S_n| \leqslant n$ as each column will contain at least two 1's (exactly two when the k-th column is such that $\omega(k) = 1$ for $k > 1$).

Recently, O. Bordellès and B. Cloître continued by establishing another matrix that relates its determinants to the Prime Number Theorem. In [2], they construct a $n \times n$ matrix Γ_n with determinant equal to

$$n! \sum_{k=1}^{n} \frac{\mu(k)}{k}.$$

Thus the Prime Number Theorem is equivalent to the fact that $\det \Gamma_n = \mathrm{o}(n!)$.

In this paper, we shall also prove a similar result. Denote $T_n = \sum_{k=1}^{n} \lambda(k)/k$. Landau also noticed that the Prime Number Theorem is equivalent to $T_n = \mathrm{o}(1)$.

Denote by $Q(n)$ the number of positive square-free integers not larger than n. We know that $Q(n) = 6n/\pi^2 + \mathcal{O}(\sqrt{n})$ (cf. for example [7]). Consider now the matrix $V_n = (v_{ij})$ also consisting of integers defined as follows:

$$v_{ij} = \begin{cases} 1 & \text{if } \ j = 1 \text{ and } i = 1 \text{ or } n, \\ Q(j) - 2Q(j/2) - 1 & \text{if } \ i = 1 \text{ and } 2 \leqslant j \leqslant n, \\ iQ(j/i) - (i+1)Q(j/(i+1)) & \text{if } \ 2 \leqslant i \leqslant n-1 \text{ and } i \leqslant j \leqslant n, \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

We shall prove

Theorem 1.2. $\det V_n = n! \sum\limits_{k=1}^{n} \lambda(k)/k.$

2. Some notation, identities and proofs

Let $x \in \mathbb{R}$. Consider any functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ with support on $[1, +\infty[$. Their *generalized convolution* is

$$(f \star g)(x) := \sum_{n \leqslant x} f(n) g\Big(\frac{x}{n}\Big).$$

Denote the characteristic functions $\mathbb{1} := \mathbb{1}_{\mathbb{N}^*}$ and $\chi := \mathbb{1}_{[1,+\infty[}$, then

$$(f \star \chi)(x) = \sum_{n \leqslant x} f(n).$$

We can write then $M = \mu \star \chi$ and $L = \lambda \star \chi$.

If $f, g|_{\mathbb{R}\setminus\mathbb{N}} = 0$, then the generalized convolution $\star$ becomes the Dirichlet convolution $*$, which is commutative. Moreover, for any function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$f \star (g \star h) = (f * g) \star h.$$

We shall denote, provided it exists, the Dirichlet convolution inverse of f by $\tilde{f}$. Hence, we write $\tilde{\mu} = \mathbb{1}$ as $\mu * \mathbb{1} = \delta$ or

$$\sum_{k|n} \mu(k) = 0 \text{ when } n > 1.$$

The characteristic function of positive square-free integers is $|\mu|$. We have $Q = |\mu| \star \chi$.

Denote by κ the characteristic function of positive square integers. It is easy to verify that $\lambda = \mu * \kappa$ and $\tilde{\lambda} = \tilde{\kappa} * \mathbb{1} = |\mu|$ as these are multiplicative functions (see also [1] or [7]).

Hence, for any real number $x \geqslant 1$ we have

$$(\sharp) \qquad \begin{aligned} |\mu| \star L(x) &= |\mu| \star (\lambda \star \chi)(x) \\ &= (|\mu| * \lambda) \star \chi(x) \\ &= \chi(x) \\ &= 1. \end{aligned}$$

We have also

$$\mathbb{1} \star L(x) = \kappa \star \chi(x)$$

or equivalently

$$(\natural) \qquad \sum_{k \leqslant x} \lambda(k) \left\lfloor \frac{x}{k} \right\rfloor = \lfloor \sqrt{x} \rfloor.$$

2.1. Proof of Theorem 1.1. The coefficients s_{ij} except for the first column of S_n are in fact equal to $|\mu|(j/i)$ when $i \mid j$.

Put $A = (a_{ij}), B = (b_{ij}) \in \mathscr{M}_n$ with

$$a_{ij} = \begin{cases} |\mu|(j/i) & \text{if } \; i \mid j, \\ 0 & \text{else.} \end{cases}$$

and

$$b_{ij} = \begin{cases} L(n/i) & \text{if } \; j = 1, \\ 1 & \text{if } \; i = j \geqslant 2, \\ 0 & \text{else.} \end{cases}$$

For any positive integer $i \leqslant n$,

$$\begin{aligned} \sum_{k=1}^{n} a_{ik} b_{k1} &= \sum_{\substack{k \leqslant n \\ i \mid k}} |\mu|\left(\frac{k}{i}\right) L(n/k) \\ &= \sum_{k=1}^{\lfloor n/i \rfloor} |\mu|(k) L\left(\frac{n/i}{k}\right) \\ &= (|\mu| \star L)(n/i) \\ &= 1 \qquad \text{(from } (\sharp)) \\ &= s_{i1}. \end{aligned}$$

When $2 \leqslant j \leqslant n$, then
$$\sum_{k=1}^{n} a_{ik} b_{kj} = a_{ij} = s_{ij}.$$
Hence $S_n = A_n B_n$ and
$$\det S_n = \det A_n \det B_n = \det B_n = L(n).$$
This proves our Theorem 1.1. $\square$

2.2. Proof of Theorem 1.2. First, we shall describe a more general situation.

Let $(c_{k,l})$ be an arbitrary double indices sequence. Using the Abel transformation, we can write

$$(\flat) \qquad \sum_{k=1}^{l} \frac{\lambda(k)}{k} c_{k,l} = \sum_{k=1}^{l} \left(\sum_{j=1}^{k} \frac{\lambda(j)}{j} - \sum_{j=1}^{k-1} \frac{\lambda(j)}{j} \right) c_{k,l}$$
$$= \sum_{j=1}^{l} \frac{\lambda(j)}{j} c_{l,l} + \sum_{k=1}^{l-1} \left(\sum_{j=1}^{k} \frac{\lambda(j)}{j} \right) (c_{k,l} - c_{k+1,l}).$$

We easily verify that for any real number $x \geqslant 1$,
$$\begin{aligned} \sum_{n \leqslant x} \lambda(n) Q\left(\frac{x}{n}\right) &= \lambda \star (|\mu| \star \chi)(x) \\ &= (\lambda * |\mu|) \star \chi(x) \\ &= 1. \end{aligned}$$

This suggests to take roughly $c_{k,l} = kQ(l/k)$.

Put $L = (l_{ij}) \in \mathscr{M}_n$, $U = (u_{ij}) \in \mathscr{M}_n$ with
$$l_{ij} = \begin{cases} 1 & \text{if } \ i = j < n, \\ \sum\limits_{k=1}^{j} \lambda(k)/k & \text{if } \ i = n, \\ 0 & \text{else,} \end{cases}$$
and
$$u_{ij} = \begin{cases} 1 & \text{if } \ i = j = 1, \\ Q(j) - 2Q(j/2) - 1 & \text{if } \ i = 1 \text{ and } 2 \leqslant j \leqslant n, \\ iQ(j/i) - (i+1)Q(j/(i+1)) & \text{else.} \end{cases}$$

Then for any positive integer $i < n$,

$$\sum_{k=1}^{n} l_{ik} u_{kj} = u_{ij} = v_{ij}$$

If $i = n$, then for any $2 \leqslant j \leqslant n$,

$$\begin{aligned}\sum_{k=1}^{n} l_{nk} u_{kj} &= -1 + \sum_{k=1}^{j} \left(\sum_{t=1}^{k} \frac{\lambda(t)}{t} \right) (kQ(j/k) - (k+1)Q(j/(k+1))) \\ &= -1 + \sum_{k=1}^{j} \lambda(k) Q(j/k) \\ &= 0 \\ &= v_{ij}.\end{aligned}$$

For the last case $i = n, j = 1$ we have

$$\sum_{k=1}^{n} l_{nk} u_{k1} = l_{n1} = 1 = v_{n1}.$$

Finally, $V_n = L_n U_n$ and

$$\det V_n = \det U_n \det L_n = n! \sum_{k=1}^{n} \frac{\lambda(k)}{k},$$

which is the conclusion of Theorem 1.2. □

3. Discussion

In particular, the Prime Number Theorem can be proved if we show that $\det V_n = o(n!)$ but this seems to remain open under the consideration of various known inequalities involving determinants.

In fact, it is easy to construct other matrices with a similar property. Using (♮) and (♭) we could think of putting $c_{1,l} = l - \lfloor \sqrt{l} \rfloor - 2 \lfloor \frac{1}{2} l \rfloor$ and for $2 \leqslant k \leqslant l$,

$$c_{k,l} = k \left\lfloor \frac{l}{k} \right\rfloor - (k+1) \left\lfloor \frac{l}{k+1} \right\rfloor$$

in (♭).

This induces the consideration of the matrix $W = (w_{ij}) \in \mathscr{M}_n$ with

$$w_{ij} = \begin{cases} 1 & \text{if } j = 1 \text{ and } i = 1 \text{ or } n \\ j - \lfloor\sqrt{j}\rfloor - 2\lfloor j/2\rfloor & \text{if } i = 1 \text{ and } 2 \leqslant j \leqslant n \\ i\lfloor j/i\rfloor - (i+1)\lfloor j/(i+1)\rfloor & \text{if } 2 \leqslant i \leqslant n-1 \text{ and } i \leqslant j \leqslant n \\ 0 & \text{else,} \end{cases}$$

and we can prove that

$$\det W_n = n! \sum_{k=1}^{n} \frac{\lambda(k)}{k}.$$

In [2], O. Bordellès and B. Cloître construct an invertible matrix U_n with the smallest singular value σ_n such that $\left|\sum_{k=1}^{n} \mu(k)/k\right| \leqslant 1/n\sigma_n$. They cite a result in [3] that for any triangular matrix $A = (a_{ii})$ with dominant diagonal ($|a_{ii}| \geqslant |a_{ij}|$), we can have the estimate $\sigma_n \geqslant |\min a_{ii}|/2^{n-1}$. However, if we apply such estimate, we will only obtain

$$\left|\sum_{k=1}^{n} \frac{\mu(k)}{k}\right| \leqslant \frac{n-1}{n} 2^{n-1},$$

which is very far from the bound $\left|\sum_{k=1}^{n} \mu(k)/k\right| \leqslant 1$ obtained by further simple manipulation of the Möbius inversion formula.

In the consideration of our U_n in the course of the proof of the computation of our $\det T_n$, we can also show that $\left|\sum_{k=1}^{n} \lambda(k)/k\right|$ appears in the top-right entry of U_n^{-1}, as the spectral norm is not smaller than max norm; we have then

$$\left|\sum_{k=1}^{n} \frac{\lambda(k)}{k}\right| \leqslant \|U_n^{-1}\|_2 \leqslant \frac{1}{\sigma_n} \tag{3.1}$$

where σ_n is the smallest singular value of our U_n.

Now, if we consider the lower bound of the smallest singular value by using the estimate of in [4]:

$$\sigma_n \geqslant \left(\frac{n-1}{n}\right)^{(n-1)/2} |\det U_n| \frac{C_{\min}}{\prod_{i=1}^{n} \|C_i(U_n)\|_2}$$

where $C_{\min}$ is the minimum of $\|C_i(U_n)\|_2$.

On the one hand, Hadamard's inequality ensures that $|\det U_n|/\prod_{i=1}^{n} \|C_i(U_n)\|_2 \leqslant 1$.

On the other hand, the quantity $\big((n-1)/n\big)^{(n-1)/2}$ is bounded.

It is not realistic to expect that such estimate could make $\sigma_n \to \infty$ but only a bounded estimate.

Hence it remains the same for $\left|\sum_{k=1}^{n} \lambda(k)/k\right|$ in (3.1).

We can also relate an other famous conjecture such as the Riemann hypothesis to our study by using matrices inequalities. However, we wonder if the actual methods in linear algebra could eventually lead to another proof of the Prime Number Theorem.

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