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TWO CONVERGENCE THEOREMS FOR HENSTOCK-KURZWEIL INTEGRALS AND THEIR APPLICATIONS TO MULTIPLE TRIGONOMETRIC SERIES

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Abstract. We establish two new norm convergence theorems for Henstock-Kurzweil integrals. In particular, we provide a unified approach for extending several results of R. P. Boas and P. Heywood from one-dimensional to multidimensional trigonometric series.

Keywords: Henstock-Kurzweil integral, regularly convergent multiple series

MSC 2010: 42B05, 26A39

1. Introduction

A classical result of Boas [2] asserts that if $\sum\limits_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then the improper Riemann integral

$$\lim_{\delta\to 0^+}\int_{\delta}^{\pi}\frac{1}{t}\sum_{k=1}^{\infty} b_k \sin kt\,\mathrm{d}t$$

exists. Since then, many authors have presented different proofs and generalisations of this result; see, for example, [3], [18], [28].

In 1991, Móricz [28] proved that if $\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}$ is an absolutely convergent double series of real numbers, then the improper Riemann integral

$$\lim_{\substack{(\varepsilon,\delta)\to(0,0)\\(\varepsilon,\delta)\in(0,\pi)^2}}\int_{[\varepsilon,\pi]\times[\delta,\pi]}\sum_{(j,k)\in\mathbb{N}^2}\frac{b_{j,k}\sin js\sin kt}{st}\,\mathrm{d}(s,t) \tag{1}$$

exists. The proof of (1) depends on the following multiplicative property of the two-dimensional improper Riemann integral: the function $(x,y)\mapsto \sin x\sin y/xy$ is a multiplier for this type of improper Riemann integral; see [28, p. 462, (3.20)] for details. Unfortunately, this crucial property does not hold for this kind of conditionally convergent integrals (see Example 2.11 below).

Our approach is based on the generalized Riemann integral defined by Henstock [9] and Kurzweil [13] nearly fifty years ago. This integral, which is now commonly known as the Henstock-Kurzweil integral, is equivalent to the classical Perron integral (cf. [31]). While this is a classical integral studied by various authors; consult, for instance [4], [5], [7], [10], [12], [14], [15], [16], [17], [20], [21], [22], [25], [31], [32], [33], [34], [35], applications to multiple Fourier series are largely unexplored.

In this paper we prove two new norm convergence theorems for the Henstock-Kurzweil integral (see Theorems 3.1 and 6.3 below); in particular, we sharpen assertion (1) and provide a unified approach for extending several improper Riemann integrability theorems of Boas [2], [3], Heywood [11] and Móricz [28], [29].

The paper is organized as follows. In Section 2 we state a number of useful results concerning the Henstock-Kurzweil integral, with proofs where necessary. In Section 3 we prove Theorems 3.1 and 3.2. In Section 4 we apply Theorems 3.1, 3.2 and 4.2 to sharpen several integrability theorems of Boas [3], Heywood [11] and Móricz [28] concerning the single or double Fourier series. The proof of Theorem 4.2 is given in Section 5. In Section 6 we employ summation by parts and the generalized Dirichlet test to prove another new convergence theorem for Henstock-Kurzweil integrals (Theorem 6.3). In Section 7 we use various multiple summation by parts formulas to generalize a result of Boas [2, Theorem 4]; see Theorems 7.1 and 7.4 for details. Consequently, we deduce a necessary and sufficient condition for a multiple sine series to be Henstock-Kurzweil integrable on $[-\pi, \pi]^m$ (Theorem 8.1). Finally, we use a double sine series (Example 8.2) to show that Theorem 6.3 is, in some sense, sharp.

2. Preliminaries

Let $m \geqslant 1$ be an integer and let $\mathbb{R}^m$ denote the m-dimensional Euclidean space equipped with the maximum norm $\|\!|\cdot|\!\|$. Points $(x_1, \dots, x_m), (y_1, \dots, y_m), \dots$ are denoted by their corresponding bold letters $\boldsymbol{x}, \boldsymbol{y}, \dots$ For $\boldsymbol{x} \in \mathbb{R}^m$ and $r > 0$, set $B(\boldsymbol{x}, r) := \{\boldsymbol{y} \in \mathbb{R}^m \colon \max\limits_{k=1,\dots,m} |x_k - y_k| < r\}$. An *interval* in $\mathbb{R}^m$ is a set of the form $[\boldsymbol{u}, \boldsymbol{v}] := \prod\limits_{i=1}^{m} [u_i, v_i]$, where $u_i, v_i \in \mathbb{R}$ and $u_i < v_i$ for $i = 1, \dots, m$. Unless stated otherwise, $[\boldsymbol{a}, \boldsymbol{b}] := \prod\limits_{i=1}^{m} [a_i, b_i]$ denotes a fixed interval and $\mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$ the family of all subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$.

A *partial partition* of $[\boldsymbol{a}, \boldsymbol{b}]$ is a finite collection $\{(I_1, \boldsymbol{t}_1), \dots, (I_p, \boldsymbol{t}_p)\}$, where $I_1, \dots, I_p$ are nonoverlapping subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{t}_i \in I_i$ for $i = 1, \dots, p$. If

δ is a gauge (i.e. a positive function) on a set $Z \subseteq [\boldsymbol{a}, \boldsymbol{b}]$, we say that a partial partition $\{(I_1, \boldsymbol{t}_1), \ldots, (I_p, \boldsymbol{t}_p)\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ is δ-fine whenever $\boldsymbol{t}_i \in Z$ and $\boldsymbol{t}_i \in I_i \subset B(\boldsymbol{t}_i, \delta(\boldsymbol{t}_i)) \cap [\boldsymbol{a}, \boldsymbol{b}]$ for $i = 1, \ldots, p$.

Lemma 2.1 ([14, Lemma 6.2.6])**.** *If δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$, then there exists a δ-fine partial partition $\{(I_1, \boldsymbol{t}_1), \ldots, (I_p, \boldsymbol{t}_p)\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ with $\bigcup_{i=1}^{p} I_i = [\boldsymbol{a}, \boldsymbol{b}]$.*

Let μ_m denote the Lebesgue measure on $\mathbb{R}^m$.

Definition 2.2. A function $f\colon [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on $[\boldsymbol{a}, \boldsymbol{b}]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\left| \sum_{i=1}^{p} f(\boldsymbol{t}_i)\mu_m(I_i) - A \right| < \varepsilon \tag{2}$$

for each δ-fine partial partition $\{(I_1, \boldsymbol{t}_1), \ldots, (I_p, \boldsymbol{t}_p)\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ with $\bigcup_{i=1}^{p} I_i = [\boldsymbol{a}, \boldsymbol{b}]$. Here the unique number A is called the Henstock-Kurzweil integral of f over $[\boldsymbol{a}, \boldsymbol{b}]$, and we write A as $(\mathrm{HK}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$.

Unless stated otherwise, all functions in this paper are real-valued. The collection of all functions that are Henstock-Kurzweil integrable on $[\boldsymbol{a}, \boldsymbol{b}]$ will be denoted by $\mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$. The following properties are known for the Henstock-Kurzweil integral; see [14] for the proofs, where the term "Kurzweil-Henstock integral" is used to describe this integral. The term "Generalized Riemann integral" is used to describe this integral in [10].

Theorem 2.3.

(i) *$\mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$ is a linear space.*
(ii) *If $f \in \mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$, then $f \in \mathrm{HK}(J)$ for each $J \in \mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$.*
(iii) *If $f \in \mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$, then the interval function $J \mapsto (\mathrm{HK}) \int_J f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ is additive on $\mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$. This interval function is known as the indefinite Henstock-Kurzweil integral of f.*
(iv) *If $f \in \mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$, then for each $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$\left| (\mathrm{HK}) \int_J f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right| < \varepsilon$$

whenever $J \in \mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$ with $\mu_m(J) < \eta$.

(v) *If* $f \in L^1[\boldsymbol{a},\boldsymbol{b}]$*, then* $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\mu_m(\boldsymbol{x}) = (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

(vi) *If* $f, |f| \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$*, then* $f \in L^1[\boldsymbol{a},\boldsymbol{b}]$.

For the rest of this paper the space $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ will be equipped with the seminorm $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$, where

$$\|f\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} := \sup\left\{\left|(\mathrm{HK})\int_I f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|:\ I \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])\right\}. \tag{3}$$

We have the following useful remark concerning $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$.

Remark 2.4. $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$ is equivalent to the seminorm $|||\cdot|||_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$, where

$$|||f|||_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} = \sup\left\{\left|(\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{x}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\right|:\ \boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}]\right\}. \tag{4}$$

It is known that the space $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ is not complete; see, for example, [25]. Thus it is necessary to obtain a simple characterisation of those additive interval functions which are indefinite Henstock-Kurzweil integrals. Let F be an interval function on $\mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ and X an arbitrary subset of $[\boldsymbol{a},\boldsymbol{b}]$. We set

$$V_{\mathrm{HK}}F(X) := \inf_{\delta}\sup_{P}\sum_{(I,\boldsymbol{t})\in P}|F(I)|,$$

where δ is a gauge on X and P is a δ-fine partial partition of $[\boldsymbol{a},\boldsymbol{b}]$ with $\{\boldsymbol{t}\colon (I,\boldsymbol{t}) \in P\} \subseteq X$.

Theorem 2.5 ([6, Proposition 3.3])**.** *Let* $F\colon \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}]) \to \mathbb{R}$*. Then* $V_{\mathrm{HK}}F$ *is a metric outer measure.*

Let F be given as in Theorem 2.5. We say that $V_{\mathrm{HK}}F$ is *absolutely continuous* with respect to μ_m, in symbol $V_{\mathrm{HK}}F \ll \mu_m$, if $V_{\mathrm{HK}}F(Z) = 0$ whenever $Z \subset [\boldsymbol{a},\boldsymbol{b}]$ and $\mu_m(Z) = 0$. The next theorem gives a simple characterisation of indefinite Henstock-Kurzweil integrals.

Theorem 2.6 ([15, Theorem 4.3])**.** *Let* $F\colon \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}]) \to \mathbb{R}$ *be an additive interval function. Then the following statements are equivalent:*

(i) *There exists* $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *such that* F *is the indefinite Henstock-Kurzweil integral of* f.

(ii) $V_{\mathrm{HK}}F \ll \mu_m$.

We are now ready to state and prove a useful Hake theorem for Henstock-Kurzweil integrals.

Theorem 2.7. *Let $f\colon [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$, let $F\colon \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}]) \to \mathbb{R}$ be an additive interval function, and let $X \subset [\boldsymbol{a},\boldsymbol{b}]$ be a closed set such that $V_{\mathrm{HK}}F(X) = 0$. If for each $I \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ satisfying $I \cap X = \emptyset$, $f \in \mathrm{HK}(I)$ and $(\mathrm{HK})\int_I f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = F(I)$, then $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and F is the indefinite Henstock-Kurzweil integral of f.*

Proof. In view of Theorem 2.6, it suffices to prove that $V_{\mathrm{HK}}F \ll \mu_m$. Let $Z \subset [\boldsymbol{a},\boldsymbol{b}]$ be such that $\mu_m(Z) = 0$ and let $J_1, J_2, \ldots$ be nonoverlapping intervals such that $[\boldsymbol{a},\boldsymbol{b}] \setminus X = \bigcup_{k=1}^{\infty} J_k$. Then the assumption $V_{\mathrm{HK}}F(X) = 0$, Theorems 2.5 and 2.6 yield the desired result:

$$0 \leqslant V_{\mathrm{HK}}F(Z) \leqslant V_{\mathrm{HK}}F(X \cap Z) + \sum_{k=1}^{\infty} V_{\mathrm{HK}}F(J_k \cap Z) = 0.$$

□

The following theorem is a special case of Theorem 2.7.

Theorem 2.8. *Let $f\colon [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ and suppose that $f \in \mathrm{HK}[\boldsymbol{c},\boldsymbol{d}]$ for every $[\boldsymbol{c},\boldsymbol{d}] \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ disjoint from $\{\boldsymbol{a}\}$. Then $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ if and only if*

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]}} (\mathrm{HK})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

exists. In either case,

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]}} (\mathrm{HK})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} = (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}.$$

Following the proof of [17, Theorem 3.2], we obtain the following result.

Theorem 2.9. *Let $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and let ν be a finite signed Borel measure on $[\boldsymbol{a},\boldsymbol{b}) := \prod_{i=1}^{m} [a_i, b_i)$. Then the function $\boldsymbol{x} \mapsto f(\boldsymbol{x})\nu([\boldsymbol{a},\boldsymbol{x}))$ belongs to $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$,*

$$(\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\nu([\boldsymbol{a},\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} = \int_{[\boldsymbol{a},\boldsymbol{b})} \left\{(\mathrm{HK})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\right\} \mathrm{d}\nu(\boldsymbol{x}) \tag{5}$$

and

$$\left|(\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\nu([\boldsymbol{a},\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right| \leqslant \|f\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}|\nu|([\boldsymbol{a},\boldsymbol{b})). \tag{6}$$

The next theorem is an easy consequence of Theorem 2.9.

Theorem 2.10. *Let* $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *and suppose that for each* $i \in \{1,\dots,m\}$ *the function* $g_i\colon [a_i,b_i] \to \mathbb{R}$ *is non-negative and non-decreasing on* $[a_i,b_i]$. *Then* $f \bigotimes_{i=1}^{m} g_i \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *and there exists* $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$ *such that*

$$(\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t}) \prod_{i=1}^{m} g_i(t_i)\,\mathrm{d}\boldsymbol{t} = \prod_{i=1}^{m} g_i(b_i)\Big\{(\mathrm{HK})\int_{[\boldsymbol{\xi},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\Big\}.$$

The following simple example shows that higher-dimensional improper Riemann integrals are not powerful enough for our applications; in particular, Theorem 2.9 fails to hold for such integrals.

Example 2.11. We define a function $u\colon [0,\pi]^2 \to \mathbb{R}$ by setting

$$u(x,y) = \begin{cases} \dfrac{\sin 4x}{y} & \text{if } (x,y) \in \Big[\dfrac{\pi}{2},\pi\Big] \times (0,\pi], \\ 0 & \text{otherwise.} \end{cases}$$

Then the improper Riemann integral

$$\lim_{\substack{(\varepsilon,\delta)\to(0,0)\\(\varepsilon,\delta)\in(0,\pi)^2}} \int_{[\varepsilon,\pi]\times[\delta,\pi]} u(x,y)\,\mathrm{d}(x,y)$$

exists. On the other hand, since the map $x \mapsto \sin x/x$ is continuous and strictly decreasing on $[\pi/2,\pi]$, we have $\int_{\pi/2}^{\pi}(\sin x \sin 4x/x)\,\mathrm{d}x > 0$. Consequently, the improper Riemann integral

$$\lim_{\substack{(\varepsilon,\delta)\to(0,0)\\(\varepsilon,\delta)\in(0,\pi)^2}} \int_{[\varepsilon,\pi]\times[\delta,\pi]} u(x,y)\frac{\sin x \sin y}{xy}\,\mathrm{d}(x,y)$$

does not exist.

3. A convergence theorem for Henstock-Kurzweil integrals

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $\mathbb{N}_0^m := \bigtimes_{i=1}^{m} \mathbb{N}_0$. Given two m-tuples $\boldsymbol{p},\boldsymbol{q} \in \mathbb{N}_0^m$, we write $\boldsymbol{p} \leqslant \boldsymbol{q}$ if and only if $p_i \leqslant q_i$ for $i = 1,\dots,m$. Moreover, for any multiple sequence

$\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m\}}$ in a normed space $(X, \|\cdot\|_X)$ we set

$$\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} u_{\boldsymbol{k}} := \sum_{k_1=p_1}^{q_1} \dots \sum_{k_m=p_m}^{q_m} u_{\boldsymbol{k}} \quad (\boldsymbol{p},\boldsymbol{q}\in\mathbb{N}_0^m),$$

where an empty sum is taken to be zero. We write $W' = \{1,\dots,m\}\setminus W$ ($W\subseteq\{1,\dots,m\}$), and the symbol $\subset$ will be used for *proper* inclusion.

We are now ready to state and prove the main result of this section.

Theorem 3.1. *Let $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ be a multiple sequence of real numbers, let $\{\varphi_{i,j}\}_{j=0}^{\infty}\subset L^1[a_i,b_i]$ ($i=1,\dots,m$) and let $\{C_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If*

$$\max_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} |c_{\boldsymbol{k}}| \prod_{j\in\Gamma'} \|\varphi_{j,k_j}\|_{\mathrm{HK}[a_j,b_j]} \times \prod_{i\in\Gamma} \|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]} \leqslant C_n \quad (n\in\mathbb{N}), \tag{7}$$

then there exists $\varphi\in\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty} \Big\| \sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\mathbf{n}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i,k_i} - \varphi \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} = 0. \tag{8}$$

P r o o f. By virtue of (7) and the completeness of $L^1[\boldsymbol{a},\boldsymbol{b}]$, there exists a function $\varphi\colon [\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ such that

$$\varphi\in L^1[\boldsymbol{x},\boldsymbol{b}] \quad \text{and} \quad \int_{[\boldsymbol{x},\boldsymbol{b}]} \varphi(\boldsymbol{t})\,\mathrm{d}\mu_m(\boldsymbol{t}) = \Phi([\boldsymbol{x},\boldsymbol{b}]) \quad \text{for every } \boldsymbol{x}\in\prod_{i=1}^{m}(a_i,b_i), \tag{9}$$

where

$$\Phi(I) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \int_I \prod_{i=1}^{m} \varphi_{i,k_i}(t_i)\,\mathrm{d}\mu_m(\boldsymbol{t}) \quad (I\in\mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}]));$$

the multiple series on the right being absolutely convergent.

We shall next prove that $\varphi\in\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and Φ is the indefinite Henstock-Kurzweil integral of φ. In view of assertion (9), Theorems 2.3 (v), 2.7, and 2.5, it suffices to prove that

$$V_{\mathrm{HK}}\Phi(Z_{\Gamma,n}) = 0 \quad (n\in\mathbb{N},\ \Gamma\subset\{1,\dots,m\}), \tag{10}$$

where

$$[\boldsymbol{a},\boldsymbol{b}]\setminus(\boldsymbol{a},\boldsymbol{b}]=\bigcup_{n\in\mathbb{N}}\bigcup_{\Gamma\subset\{1,\dots,m\}}Z_{\Gamma,n},$$

$$Z_{\Gamma,n}:=\prod_{k=1}^{m}Z_{\Gamma,n,k}\quad\text{and}\quad Z_{\Gamma,n,k}:=\begin{cases}\{a_k\} & \text{if } k\in\Gamma',\\ \Big[a_k+\dfrac{b_k-a_k}{n+1},b_k\Big] & \text{if } k\in\Gamma.\end{cases}$$

P r o o f of (10). Let $n\in\mathbb{N}$ and let $\Gamma\subset\{1,\dots,m\}$. Given $\varepsilon>0$ we use (7) to select a positive integer $K=K(\Gamma,n,\varepsilon)$ such that

$$\max_{l=1,\dots,m}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_l>K}}|c_{\boldsymbol{k}}|\prod_{j\in\Gamma'}\|\varphi_{j,k_j}\|_{\mathrm{HK}[a_j,b_j]}\prod_{i\in\Gamma}\|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]}<\frac{\varepsilon}{2m}. \tag{11}$$

Using uniform continuity of the indefinite Henstock-Kurzweil integral (cf. Theorem 2.3 (iv)), we choose $\eta(\Gamma,n)>0$ such that

$$\begin{aligned}\max_{j\in\Gamma'}\max_{0\leqslant k_j\leqslant K}\sup_{\substack{[u,v]\subseteq[a_j,b_j]\\ 0<v-u<\eta(\Gamma,n)}}&\|\varphi_{j,k_j}\|_{\mathrm{HK}[u,v]}\\ &<\frac{\varepsilon}{2}\Big(1+\max_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant(K,\dots,K)}|c_{\boldsymbol{k}}|\Big)^{-1}\Big(1+\prod_{i\in\Gamma}\|\varphi_{i,n}|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]}\Big)^{-1}.\end{aligned} \tag{12}$$

Define a gauge $\delta_{\Gamma,n}(\boldsymbol{x})$ on $Z_{\Gamma,n}$ by letting $\delta_{\Gamma,n}(\boldsymbol{x}):=\eta(\Gamma,n)$ and select any $\delta_{\Gamma,n}$-fine partial partition $P_{\Gamma,n}=\{(J_1,\boldsymbol{t_1}),\dots,(J_q,\boldsymbol{t_q})\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ with $\{\boldsymbol{t_1},\dots,\boldsymbol{t_q}\}\subset Z_{\Gamma,n}$. We claim that

$$\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}}|\Phi(I)|<\varepsilon. \tag{13}$$

Clearly, the obvious equality $\mathrm{card}(P_{\emptyset,n})=1$, (11), and (12) imply that

$$\sum_{(I,\boldsymbol{t})\in P_{\emptyset,n}}|\Phi(I)|<\varepsilon.$$

On the other hand, suppose that $\Gamma\subset\{1,\dots,m\}$ is non-empty. In this case,

$$\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}}|\Phi(I)|\leqslant\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}|c_{\boldsymbol{k}}|\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}}S_{\boldsymbol{k},\Gamma}(I)S_{\boldsymbol{k},\Gamma'}(I), \tag{14}$$

where $S_{\boldsymbol{k},W}(I):=\prod_{i\in W}\big|\int_{I_i}\varphi_{i,k_i}\,\mathrm{d}\mu_1\big|$ $(W\subseteq\{1,\dots,m\})$.

According to our choice of $P_{\Gamma,n}$, $\mu_{\mathrm{card}(\Gamma)}\Big(\prod_{i\in\Gamma} I_i \cap \prod_{i\in\Gamma} I'_i\Big) = 0$ whenever $\Big(\prod_{i=1}^{m} I_i, \boldsymbol{x}\Big)$ and $\Big(\prod_{i=1}^{m} I'_i, \boldsymbol{y}\Big)$ are two distinct elements of $P_{\Gamma,n}$. Hence

$$\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} S_{\boldsymbol{k},\Gamma}(I) \leqslant \prod_{i\in\Gamma} \|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]} \quad (\boldsymbol{k}\in\mathbb{N}_0^m)$$

and so

$$\begin{aligned}(15)\qquad &\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} S_{\boldsymbol{k},\Gamma}(I) S_{\boldsymbol{k},\Gamma'}(I)\\ &\qquad\leqslant \prod_{j\in\Gamma'} \|\varphi_{j,k_j}\|_{\mathrm{HK}[a_j,b_j]} \prod_{i\in\Gamma} \|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]} \quad (\boldsymbol{k}\in\mathbb{N}_0^m).\end{aligned}$$

Combining (14), (15), (12), and (11) yields (13):

$$\begin{aligned}&\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} |\Phi(I)|\\ &\quad\leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ 0\leqslant k_i\leqslant K\,\forall\, i\in\{1,\dots,m\}}} |c_{\boldsymbol{k}}| \prod_{j\in\Gamma'} \|\varphi_{j,k_j}\|_{\mathrm{HK}[a_j,b_j]} \prod_{i\in\Gamma} \|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]}\\ &\qquad+ \sum_{l=1}^{m} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_l>K}} |c_{\boldsymbol{k}}| \prod_{j\in\Gamma'} \|\varphi_{j,k_j}\|_{\mathrm{HK}[a_j,b_j]} \prod_{i\in\Gamma} \|\varphi_{i,k_i}\|_{L^1[a_i+(b_i-a_i)/(n+1),b_i]} < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (10) is proved. It is now clear that (7) implies (8). The proof is complete. □

The next theorem, together with Theorem 3.1, will be used to prove Theorem 4.3.

Theorem 3.2. *If the following conditions are satisfied:*

(i) $\{c_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}_0^m}$ *is a multiple sequence of non-negative numbers;*

(ii) *for each* $i\in\{1,\dots,m\}$ *the function* $h_i\colon [a_i,b_i]\to\mathbb{R}$ *is positive and decreasing on* $(a_i,b_i]$, $\{\varphi_{i,n}\}_{n=0}^{\infty} \cup \{\varphi_{i,n}h_i\}_{n=0}^{\infty} \subset L^1[a_i,b_i]$ *and*

$$(16)\qquad \inf_{n\in\mathbb{N}_0} \min\left\{\int_{a_i}^{b_i} \varphi_{i,n} h_i \,\mathrm{d}\mu_1,\ \min_{x_i\in[a_i,b_i]} \int_{a_i}^{x_i} \varphi_{i,n}\,\mathrm{d}\mu_1\right\} \geqslant 0;$$

(iii) *there exists* $\varphi\in\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *such that*

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty} \Big\| \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i,k_i} - \varphi \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} = 0;$$

(iv) $\varphi \bigotimes_{i=1}^{m} h_i \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, *then the multiple series*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{a_i}^{b_i} \varphi_{i,k_i} h_i \,\mathrm{d}\mu_1$$

converges.

P r o o f. In view of (i), (16), and (iv), it suffices to prove that

$$\sup_{\boldsymbol{N}\in\mathbb{N}_0^m} \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{a_i}^{b_i} \varphi_{i,k_i} h_i \,\mathrm{d}\mu_1 \leqslant 4^m \Big\| \varphi \bigotimes_{i=1}^{m} h_i \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}. \tag{17}$$

Let $\boldsymbol{N}\in\mathbb{N}_0^m$ be arbitrary and let us write

$$W(\Gamma,\boldsymbol{k},\boldsymbol{y}) = \prod_{j\in\Gamma} \frac{1}{h_j(y_j)} \int_{y_j}^{b_j} \varphi_{j,k_j} h_j \,\mathrm{d}\mu_1 \prod_{l\in\Gamma'} \int_{a_l}^{y_l} \varphi_{l,k_l} \,\mathrm{d}\mu_1$$
$$(\boldsymbol{k}\in\mathbb{N}_0^m,\ \boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}],\ \Gamma\subseteq\{1,\dots,m\}).$$

We will first prove that

$$\lim_{\substack{\boldsymbol{y}\to\boldsymbol{a}\\ \boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]}} \prod_{i=1}^{m} h_i(y_i) \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} W(\Gamma,\boldsymbol{k},\boldsymbol{y}) = \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{a_i}^{b_i} \varphi_{i,k_i} h_i \,\mathrm{d}\mu_1. \tag{18}$$

To prove (18) we consider two cases.
Case 1: $\Gamma=\{1,\dots,m\}$. A simple computation gives

$$\lim_{\substack{\boldsymbol{y}\to\boldsymbol{a}\\ \boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]}} \prod_{i=1}^{m} h_i(y_i) \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} W(\Gamma,\boldsymbol{k},\boldsymbol{y}) = \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{a_i}^{b_i} \varphi_{i,k_i} h_i \,\mathrm{d}\mu_1.$$

Case 2: $\Gamma\subset\{1,\dots,m\}$. In this case, we deduce from (ii) that

$$\begin{aligned}
&\lim_{\substack{\boldsymbol{y}\to\boldsymbol{a}\\ \boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]}} \prod_{i=1}^{m} h_i(y_i) \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} W(\Gamma,\boldsymbol{k},\boldsymbol{y})\\
&\quad= \lim_{\substack{\boldsymbol{y}\to\boldsymbol{a}\\ \boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]}} \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} c_{\boldsymbol{k}} \prod_{j\in\Gamma} \int_{y_j}^{b_j} \varphi_{j,k_j} h_j \,\mathrm{d}\mu_1 \prod_{i\in\Gamma'} \int_{\theta_i}^{y_i} \varphi_{i,k_i} h_i \,\mathrm{d}\mu_1\\
&\qquad\qquad (\text{for some } \theta_i\in[a_i,y_i]\ (i\in\Gamma'))\\
&\quad= 0.
\end{aligned}$$

Next, it is easy to see that

$$\inf_{\boldsymbol{k}\in\mathbb{N}_0^m} \sum_{\Gamma\subseteq\{1,\ldots,m\}} W(\Gamma,\boldsymbol{k},\boldsymbol{y}) \geqslant 0 \tag{19}$$

because $\boldsymbol{k}\in\mathbb{N}_0^m$, (ii), and (16) imply

$$\begin{aligned}\sum_{\Gamma\subseteq\{1,\ldots,m\}} W(\Gamma,\boldsymbol{k},\boldsymbol{y}) &= \sum_{\Gamma\subseteq\{1,\ldots,m\}} \prod_{j\in\Gamma}\int_{y_j}^{v_j} \varphi_{j,k_j}\,\mathrm{d}\mu_1 \prod_{l\in\Gamma'}\int_{a_l}^{y_l}\varphi_{l,k_l}\,\mathrm{d}\mu_1\\ &\qquad\text{(for some } v_j\in[y_j,b_j]\ \ (j=1,\ldots,m))\\ &= \prod_{j=1}^m \int_{a_j}^{v_j}\varphi_{j,k_j}\,\mathrm{d}\mu_1 \geqslant 0.\end{aligned}$$

Finally, since (i), (ii), (18), and (19) hold, it remains to prove that

$$\sup_{\boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]}\prod_{i=1}^m h_i(y_i)\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \sum_{\Gamma\subseteq\{1,\ldots,m\}} W(\Gamma,\boldsymbol{k},\boldsymbol{y}) \leqslant 4^m\Big\|\varphi\bigotimes_{i=1}^m h_i\Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}.$$

Let $\boldsymbol{y}\in(\boldsymbol{a},\boldsymbol{b}]$ be arbitrary. Clearly, it is enough to consider the following cases.

Case α: $\Gamma=\{1,\ldots,m\}$. From (iii), (ii), and (iv) we get

$$\Big|\prod_{i=1}^m h_i(y_i)\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} W(\Gamma,\boldsymbol{k},\boldsymbol{y})\Big| \leqslant 2^m\Big\|\varphi\bigotimes_{i=1}^m h_i\Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}. \tag{20}$$

Case β: $\Gamma=\emptyset$. We use (iii), (ii), and Theorem 2.10 to obtain (20):

$$\begin{aligned}\Big|\prod_{i=1}^m h_i(y_i)\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} W(\Gamma,\boldsymbol{k},\boldsymbol{y})\Big| &= \Big|(\mathrm{HK})\int_{[\boldsymbol{\xi},\boldsymbol{y}]}\varphi(\boldsymbol{t})\prod_{j=1}^m h_j(t_j)\,\mathrm{d}\boldsymbol{t}\Big|\\ &\qquad\text{(for some } \xi_i\in[a_i,y_i]\ \ (i=1,\ldots,m))\\ &\leqslant 2^m\Big\|\varphi\bigotimes_{i=1}^m h_i\Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}.\end{aligned}$$

Case γ: $\Gamma\subset\{1,\ldots,m\}$ is nonempty. Using (iii), (6), and (ii), we get (20), too. The proof is complete. □

4. Applications to multiple Fourier series

A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is said to be in $L^1(\mathbb{T}^m)$ if f is 2π-periodic in each variable, and $f \in L^1([-\pi,\pi]^m)$. The following theorem is a generalization of [28, Theorem 4] and [11, Theorem 3] from single to multiple Fourier series.

Theorem 4.1. *Let $g \in L^1(\mathbb{T}^m)$ and assume that $g(\boldsymbol{t}) \sim \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \sin k_i t_i$. If $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |b_{\boldsymbol{k}}| \prod_{i=1}^m k_i^{\alpha_i - 1}$ converges for some $\boldsymbol{\alpha} \in [0,1]^m$, then there exists $g_{\boldsymbol{\alpha}} \in \mathrm{HK}([0,\pi]^m)$ such that*

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty} \sup_{\boldsymbol{x}\in[0,\pi]^m} \left| \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} b_{\boldsymbol{k}} \prod_{i=1}^m \int_0^{x_i} \frac{\sin k_i t_i}{t_i^{\alpha_i}}\,\mathrm{d}t_i - (\mathrm{HK})\int_{[\boldsymbol{0},\boldsymbol{x}]} g_\alpha(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \right| = 0.$$

P r o o f. Since $p \in [0,1]$ implies

$$\sup_{k\in\mathbb{N}} \sup_{x\in[0,\pi]} \left\{ \frac{1}{k^{p-1}} \left| \int_0^x \frac{\sin kt}{t^p}\,\mathrm{d}t \right| \right\} + \sup_{k\in\mathbb{N}} \sup_{\delta\in(0,\pi]} \delta^p \int_\delta^\pi \left| \frac{\sin kt}{t^p} \right| \mathrm{d}t < \infty,$$

a simple application of Theorem 3.1 yields the theorem. The proof is complete. □

We now need the following result, which is a special case of Theorem 5.5.

Theorem 4.2. *Let $f \in L^1(\mathbb{T}^m)$, let*

$$f(\boldsymbol{t}) \sim \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k},\Gamma} \left\{ \prod_{i\in\Gamma} \cos k_i t_i \right\} \left\{ \prod_{i\in\Gamma'} \sin k_i t_i \right\}$$

and let

$$s_{\boldsymbol{n}} f(\boldsymbol{t}) := \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k},\Gamma} \left\{ \prod_{i\in\Gamma} \cos k_i t_i \right\} \left\{ \prod_{i\in\Gamma'} \sin k_i t_i \right\} \quad (\boldsymbol{n} \in \mathbb{N}^m,\ \boldsymbol{t} \in \mathbb{R}^m).$$

Then

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty} \|s_{\boldsymbol{n}} f - f\|_{\mathrm{HK}([-\pi,\pi]^m)} = 0.$$

We are ready to state and prove a higher-dimensional analogue of [3, Theorem 4.4] concerning sine series.

Theorem 4.3. *Let $\boldsymbol{\beta} \in (0,1)^m$ and let $g \in L^1(\mathbb{T}^m)$ with*

$$g(\boldsymbol{t}) \sim \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \sin k_i t_i.$$

If $b_{\boldsymbol{k}} \geqslant 0$ *for every* $\boldsymbol{k} \in \mathbb{N}^m$, *then* $\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m k_i^{\beta_i-1}$ *converges if and only if for each* $\boldsymbol{\alpha} \in [0,\pi)^m$ *there exists* $g_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathrm{HK}\Big(\prod_{i=1}^m [\alpha_i,\pi]\Big)$ *such that*

$$g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{t}) = g(\boldsymbol{t}) \prod_{i=1}^m (t_i-\alpha_i)^{-\beta_i}$$

for every $\boldsymbol{t} \in \prod_{i=1}^m (\alpha_i,\pi]$.

P r o o f. ($\Rightarrow$) Let $\alpha \in [0,\pi)$ be given. Since $p \in (0,1)$ implies

$$\begin{aligned}\sup_{k\in\mathbb{N}} \sup_{x\in[\alpha,\pi]} \left\{ \frac{1}{k^{p-1}} \left| \int_\alpha^x \frac{\sin kt}{(t-\alpha)^p} \,\mathrm{d}t \right| \right\} &+ \sup_{k\in\mathbb{N}} \sup_{\delta\in(0,\pi-\alpha]} \delta^p \int_{\alpha+\delta}^{\pi} \left| \frac{\sin kt}{(t-\alpha)^p} \right| \mathrm{d}t \\ &\leqslant \sup_{\theta>0} \left| \int_0^\theta \frac{\sin(x+\alpha)}{x^p} \,\mathrm{d}x \right| + \pi < \infty,\end{aligned}$$

a simple application of Theorem 3.1 yields the desired conclusion.

($\Leftarrow$) For this part of the proof we assume that $\boldsymbol{\alpha} = \boldsymbol{0}$ and $\boldsymbol{\beta} \in (0,1]^m$. Since

$$\inf_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \geqslant 0, \quad \inf_{n\in\mathbb{N}} \frac{1}{n^{p-1}} \int_0^\pi \frac{\sin nu}{u^p} \,\mathrm{d}u > 0 \quad (p \in (0,1])$$

and

$$\inf_{x\in[0,\pi]} \inf_{n\in\mathbb{N}} \int_0^x \sin nt \,\mathrm{d}t \geqslant 0,$$

the conclusion follows from Theorems 4.2 and 3.2. The proof is complete. □

From the proofs of Theorems 4.1 and 4.3, we get the following result.

Theorem 4.4. *Suppose that* $g \in L^1(\mathbb{T}^m)$ *with* $g(\boldsymbol{t}) \sim \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \sin k_i t_i$. *If* $b_{\boldsymbol{k}} \geqslant 0$ *for every* $\boldsymbol{k} \in \mathbb{N}^m$, *then* $\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}$ *converges if and only if there exists* $g_{\boldsymbol{0}} \in \mathrm{HK}([0,\pi]^m)$ *such that*

$$g_{\boldsymbol{0}}(\boldsymbol{t}) = g(\boldsymbol{t}) \Big/ \prod_{i=1}^m t_i$$

for every $\boldsymbol{t} \in (0,\pi]^m$.

Our next theorem resembles a result due to Hardy and Littlewood (cf. [8, Lemma 19] or [23]).

Theorem 4.5. *Let* $f \in L^1(\mathbb{T}^m)$ *and let* $a_{\boldsymbol{n}} = \int_{[-\pi,\pi]^m} f(\boldsymbol{t}) \prod_{i=1}^{m} \cos(n_i t_i/\pi)\,\mathrm{d}\mu_m(\boldsymbol{t})$ $(\boldsymbol{n} \in \mathbb{N}^m)$*. If* $a_{\boldsymbol{k}} \geqslant 0$ *for every* $\boldsymbol{k} \in \mathbb{N}^m$, *then* $\sum_{\boldsymbol{k}\in\mathbb{N}^m} a_{\boldsymbol{k}} \Big/ \prod_{i=1}^{m} k_i$ *converges if and only if there exists* $f_{\mathbf{0}} \in \mathrm{HK}([0,\pi]^m)$ *such that*

$$f_{\mathbf{0}}(\boldsymbol{x}) = \frac{1}{\prod\limits_{i=1}^{m} x_i} \int_{\prod\limits_{i=1}^{m}[-x_i,x_i]} f(\boldsymbol{t})\,\mathrm{d}\mu_m(\boldsymbol{t})$$

for every $\boldsymbol{x} \in (0,\pi]^m$.

P r o o f. According to Theorem 4.2,

$$\int_{\prod\limits_{i=1}^{m}[-x_i,x_i]} f(\boldsymbol{t})\,\mathrm{d}\mu_m(\boldsymbol{t}) \sim \sum_{\boldsymbol{k}\in\mathbb{N}^m} 2a_{\boldsymbol{k}} \prod_{i=1}^{m} \frac{\sin k_i x_i}{k_i}.$$

An appeal to Theorem 4.4 completes the proof. □

The following examples show that Theorem 3.1 is beyond the realm of Lebesgue integration.

Example 4.6. Let $g_1(x) = x^{-1} \sum_{k=1}^{\infty} k^{-3/2} \sin(2^k x)$ $(x \in (0,\pi])$ and let $g_1(0) = 0$. Then $g_1 \in \mathrm{HK}[0,\pi] \setminus L^1[0,\pi]$.

P r o o f. This follows from Theorem 3.1 and [3, p. 19]. □

Example 4.7. Let $m = 2$ and let

$$g_2(x,y) = \begin{cases} \sum\limits_{j=1}^{\infty} \sum\limits_{k=1}^{\infty} \dfrac{1}{(j+k)^3} \dfrac{\sin(2^j x)\sin(2^k y)}{\tan\dfrac{x}{2}\tan\dfrac{y}{2}} & \text{if } (x,y) \in (0,\pi)^2, \\ 0 & \text{if } (x,y) \in [0,\pi]^2 \setminus (0,\pi)^2. \end{cases}$$

Then $g_2 \in \mathrm{HK}([0,\pi]^2) \setminus L^1([0,\pi]^2)$.

P r o o f. By Theorems 4.1 and 2.9, $g_2 \in \mathrm{HK}([0,\pi]^2)$. It remains to apply [27, Theorems 1 and 2] to show that $g_2 \notin L^1([0,\pi]^2)$. But this is obvious, since the double series $\sum_{(j,k)\in\mathbb{N}^2} 1/(j+k)^6$ converges,

$$\sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} \sqrt{\sum_{p=j}^{\infty} \frac{1}{(p+k)^6}} + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} \sqrt{\sum_{q=k}^{\infty} \frac{1}{(j+q)^6}} < \infty$$

and

$$\sum_{(j,k)\in\mathbb{N}^2}\sqrt{\sum_{p=j}^{\infty}\sum_{q=k}^{\infty}\frac{1}{(p+q)^6}} \geqslant \sum_{(j,k)\in\mathbb{N}^2}\frac{1}{5(j+k)^2}=\infty.$$

□

5. Proof of Theorem 4.2

In 1912, W. H. Young [36] proved that Theorem 4.2 holds if $m = 2$. However, there is a gap in his proof; the claim on [36, p. 156, lines 18–19] need not be correct. More precisely, if $f \in L^1(\mathbb{T}^2)$, the assertion

$$\int_{[-\pi,\pi]^2}\left\{\int_{[0,x]\times[0,y]} f(s,t)\,\mathrm{d}\mu_2(s,t)\right\}\cos kx\,\mathrm{d}(x,y)=0$$

need not be true for every $k \in \mathbb{N}$. In this section we correct the proof and strengthen the result in other ways; see Theorem 5.5 for details.

Definition 5.1 ([26]). Let $\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ be a multiple sequence of real numbers. We consider the (formal) multiple series

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} u_{\boldsymbol{k}} := \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty} u_{\boldsymbol{k}}. \tag{21}$$

(i) The multiple series (21) *converges in Pringsheim's sense* to a real number s if for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}_0$ such that

$$\left|\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} u_{\boldsymbol{k}} - s\right| < \varepsilon$$

for every $\boldsymbol{n} \in \mathbb{N}_0^m$ satisfying $\min\{n_1,\ldots,n_m\} \geqslant N(\varepsilon)$.

(ii) The multiple series (21) *converges regularly* if for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}_0$ such that

$$\left|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} u_{\boldsymbol{k}}\right| < \varepsilon$$

for every $\boldsymbol{p},\boldsymbol{q} \in \mathbb{N}_0^m$ satisfying $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\|\boldsymbol{p}\| \geqslant N(\varepsilon)$.

The next theorem gives a simple necessary and sufficient condition for a multiple series to be regularly convergent.

Theorem 5.2 ([26, Theorem 1]). *Let $\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ be a multiple sequence of real numbers. The multiple series $\sum\limits_{\boldsymbol{k}\in\mathbb{N}_0^m} u_{\boldsymbol{k}}$ is regularly convergent if and only if*

(i) $\sum\limits_{\boldsymbol{k}\in\mathbb{N}_0^m} u_{\boldsymbol{k}}$ *converges in Pringsheim's sense, and*

(ii) *for each choice of the index $j \in \{1,\dots,m\}$ and for all fixed integral values of c_j, the $(m-1)$-multiple series*

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_j=c_j}} u_{\boldsymbol{k}}$$

are regularly convergent.

The following theorem shows that Fubini's theorem holds for regularly convergent multiple series.

Corollary 5.3 ([7, Corollary 2.10]). *Let $\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ be a multiple sequence of real numbers. If the multiple series $\sum\limits_{\boldsymbol{k}\in\mathbb{N}_0^m} u_{\boldsymbol{k}}$ is regularly convergent, then*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} u_{\boldsymbol{k}} = \sum_{k_{\sigma(1)}=0}^{\infty}\Big\{\dots\Big\{\sum_{k_{\sigma(m)}=0}^{\infty} u_{\boldsymbol{k}}\Big\}\dots\Big\}$$

for every permutation σ of the set $\{1,\dots,m\}$.

We need the following lemma to prove Theorem 5.5.

Lemma 5.4. *If $f \in L^1([-\pi,\pi]^m)$, then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\max_{\boldsymbol{x}\in[-\pi,\pi]^m}\Big|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}}\int_{[-\pi,\pi]^m} f(\boldsymbol{t})\prod_{i=1}^{m}\frac{\sin k_i(t_i-x_i)}{k_i}\,\mathrm{d}\mu_m(\boldsymbol{t})\Big| < \varepsilon$$

for every $\boldsymbol{p},\boldsymbol{q}\in\mathbb{N}^m$ satisfying $\boldsymbol{q}\geqslant\boldsymbol{p}$ and $\max\limits_{i=1,\dots,m} p_i \geqslant N$.

Proof. Let $\boldsymbol{x}\in[-\pi,\pi]^m$ be given and let $\varepsilon>0$. Since $f\in L^1([-\pi,\pi]^m)$ there exists $\eta\in(0,\frac{1}{8}\pi)$ such that

$$\int_{\prod\limits_{k=1}^{m} U(x_k)} |f| < \frac{\varepsilon}{2(4\pi)^m},$$

where $U(\theta)$ denotes the set $[-\pi,-\pi+\eta)\cup([\theta-\eta,\theta+\eta]\cap[-\pi,\pi])\cup(\pi-\eta,\pi]$.

Clearly, we can fix a positive integer N (independent of $\boldsymbol{x}$) such that

$$\sup_{\theta\in[\eta,2\pi-\eta]}\left|\sum_{k=p}^{q}\frac{\sin k\theta}{k}\right| < \frac{\varepsilon}{2\bigl(1+(4\pi)^{m-1}\|f\|_{L^1([-\pi,\pi]^m)}\bigr)}$$

for every integers $q \geqslant p \geqslant N$. Then, for every $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^m$ satisfying $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\max\limits_{i=1,\dots,m} p_i \geqslant N$, we have

$$\begin{aligned}
&\left|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}}\int_{[-\pi,\pi]^m} f(\boldsymbol{t})\prod_{i=1}^{m}\frac{\sin k_i(t_i-x_i)}{k_i}\,\mathrm{d}\mu_m(\boldsymbol{t})\right| \\
&\qquad\leqslant \|f\|_{L^1\left([-\pi,\pi]^m\setminus\prod_{i=1}^{m}U(x_i)\right)}(4\pi)^{m-1}\min_{i=1,\dots,m}\sup_{\theta\in[\eta,2\pi-\eta]}\left|\sum_{k_i=p_i}^{q_i}\frac{\sin k_i(\theta)}{k_i}\right| \\
&\qquad\quad + \|f\|_{L^1\left(\prod_{i=1}^{m}U(x_i)\right)}(4\pi)^m < \varepsilon \qquad \left(\text{since } \sup_{n\in\mathbb{N}}\sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{n}\frac{\sin k\theta}{k}\right| \leqslant 2\pi\right).
\end{aligned}$$

As $\boldsymbol{x} \in [-\pi,\pi]^m$ is arbitrary, the lemma is proved. □

The following theorem is a refinement of W. H. Young's theorem concerning double Fourier series (cf. [36, p. 155–156]).

Theorem 5.5. *Let $f \in L^1(\mathbb{T}^m)$ and assume that*

$$f(\boldsymbol{t}) \sim \sum_{\Gamma\subseteq\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k},\Gamma}\Bigl\{\prod_{i\in\Gamma}\cos k_i t_i\Bigr\}\Bigl\{\prod_{j\in\Gamma'}\sin k_j t_j\Bigr\}. \tag{22}$$

Then the following assertions hold.

(i) *For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\max_{\Gamma\subseteq\{1,\dots,m\}}\max_{\boldsymbol{x}\in[-\pi,\pi]^m}\left|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} c_{\boldsymbol{k},\Gamma}\Bigl\{\prod_{i\in\Gamma}\int_{-\pi}^{x_i}\cos k_i t_i\,\mathrm{d}t_i\Bigr\}\Bigl\{\prod_{j\in\Gamma'}\int_{-\pi}^{x_j}\sin k_j t_j\,\mathrm{d}t_j\Bigr\}\right| < \varepsilon$$

for every $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^m$ satisfying $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\max\limits_{i=1,\dots,m} p_i \geqslant N$.

(ii) *Let*

$$s_{\boldsymbol{n}}f(\boldsymbol{t}) := \sum_{\Gamma\subseteq\{1,\dots,m\}}\sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k},\Gamma}\Bigl\{\prod_{i\in\Gamma}\cos k_i t_i\Bigr\}\Bigl\{\prod_{i\in\Gamma'}\sin k_i t_i\Bigr\} \quad (\boldsymbol{n}\in\mathbb{N}^m,\ \boldsymbol{t}\in\mathbb{R}^m).$$

Then

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty}\|s_{\boldsymbol{n}}f - f\|_{\mathrm{HK}([-\pi,\pi]^m)} = 0.$$

(iii) (Parseval's formula) *If* ν *is a finite signed Borel measure on* $[-\pi,\pi)^m$, *then*

$$\int_{[-\pi,\pi]^m} f(\boldsymbol{t})\nu\Big(\prod_{i=1}^m[-\pi,t_i)\Big)\,\mathrm{d}\mu_m(\boldsymbol{t}) \tag{23}$$
$$= \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \int_{[-\pi,\pi]^m} \prod_{i\in\Gamma}\cos k_i t_i \prod_{j\in\Gamma'}\sin k_j t_j \nu\Big(\prod_{i=1}^m[-\pi,t_i)\Big)\,\mathrm{d}\mu_m(\boldsymbol{t});$$

the multiple series on the right being regularly convergent.

Proof. (i) Let $\Gamma\subseteq\{1,\dots,m\}$, let $\boldsymbol{x}\in[-\pi,\pi]^m$ and define

$$\psi_r(\alpha)=\begin{cases}\sin\alpha & \text{if } r\in\Gamma \text{ and } \alpha\in\mathbb{R},\\ -\cos\alpha & \text{if } r\in\Gamma' \text{ and } \alpha\in\mathbb{R}.\end{cases}$$

Then, for any $\boldsymbol{p},\boldsymbol{q}\in\mathbb{N}^m$ satisfying $\boldsymbol{q}\geqslant\boldsymbol{p}$, we have

$$\Big|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} c_{\boldsymbol{k},\Gamma}\Big\{\prod_{i\in\Gamma}\int_{-\pi}^{x_i}\cos k_i t_i\,\mathrm{d}t_i\Big\}\Big\{\prod_{j\in\Gamma'}\int_{-\pi}^{x_j}\sin k_j t_j\,\mathrm{d}t_j\Big\}\Big|$$
$$=\Big|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} c_{\boldsymbol{k},\Gamma}\Big\{\prod_{i\in\Gamma}\frac{\sin k_i x_i}{k_i}\Big\}\Big\{\prod_{j\in\Gamma'}\frac{-\cos k_j x_j+\cos k_j\pi}{k_j}\Big\}\Big|$$
$$=\Big|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}}\int_{[-\pi,\pi]^m} f(\boldsymbol{t})\prod_{i=1}^m\frac{\psi_i'(k_i t_i)(\psi_i(k_i x_i)-\psi_i(k_i\pi))}{\pi k_i}\,\mathrm{d}\mu_m(\boldsymbol{t})\Big|$$
$$\leqslant 4^m\max_{\Gamma\subseteq\{1,\dots,m\}}\max_{\boldsymbol{\theta}\in[-\pi,\pi]^m}\Big|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}}\int_{[-\pi,\pi]^m} f(\boldsymbol{t})\Big\{\prod_{i=1}^m\frac{\sin k_i(t_i-\theta_i)}{\pi k_i}\Big\}\,\mathrm{d}\mu_m(\boldsymbol{t})\Big|.$$

An appeal to Lemma 5.4 yields the desired conclusion.

(ii) Set $F(\boldsymbol{x}):=\int_{\prod_{i=1}^m[-\pi,x_i]} f(\boldsymbol{t})\,\mathrm{d}\mu_m(\boldsymbol{t})$ $(\boldsymbol{x}\in\mathbb{R}^m)$. Since (22) implies

$$\int_{[-\pi,\pi]^{m-1}}\Big|\int_{-\pi}^{\pi} f(\boldsymbol{t})\,\mathrm{d}\mu_1(t_{\sigma(1)})\Big|\,\mathrm{d}\mu_{m-1}(t_{\sigma(2)},\dots,t_{\sigma(m)})=0 \tag{24}$$

for every permutation σ of $\{1,\dots,m\}$, we infer from the absolute continuity of F that $F\in C(\mathbb{T}^m)$. Let

$$\sum_{\Gamma\subseteq\{1,\dots,m\}}\sum_{W\subseteq\Gamma'}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_i\geqslant1\,\forall\, i\in\Gamma\cup W\\ r_j=0\,\forall\, j\in\Gamma'\cap W'}}\frac{A_{\boldsymbol{k}}}{2^{m-\mathrm{card}(W\cup\Gamma)}}\prod_{i\in\Gamma}\sin k_i x_i\prod_{j\in W}\cos k_j x_j$$

be the multiple Fourier series of F. In view of (i), it suffices to prove that if $\Gamma \subseteq \{1,\dots,m\}$ and $W \subseteq \Gamma'$, then

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ k_i\geqslant 1\,\forall\, i\in\Gamma\cup W \\ k_j=0\,\forall j\in\Gamma'\cap W'}} \frac{A_{\boldsymbol{k}}}{2^{m-\operatorname{card}(W\cup\Gamma)}} \prod_{i\in\Gamma} \sin k_i x_i \prod_{j\in W} \cos k_j x_j = \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k},\Gamma} \Big\{\prod_{i\in\Gamma} \frac{\sin k_i x_i}{k_i}\Big\}\Big\{\prod_{j\in W}\Big(-\frac{\cos k_j x_j}{k_j}\Big) \prod_{l\in\Gamma'\cap W'} \frac{\cos k_l \pi}{k_l}\Big\}. \tag{25}$$

Let $\Gamma \subseteq \{1,\dots,m\}$ be given and let $W \subseteq \Gamma'$. We consider two cases.

Case 1: $W \neq \Gamma'$. Following the proof of (i), we let

$$\psi_r(\alpha) = \begin{cases} \sin\alpha & \text{if } r\in\Gamma, \\ -\cos\alpha & \text{if } r\in\Gamma'. \end{cases}$$

For each $\boldsymbol{k} \in \mathbb{N}_0^m$ satisfying $\{i\in\{1,\dots,m\}\colon\ k_i \neq 0\} = \Gamma\cup W$ we have

$$\begin{aligned}
&A_{\boldsymbol{k}} \prod_{j\in W}(-1) \\
&= \frac{1}{\pi^m}\int_{[-\pi,\pi]^m} F(\boldsymbol{t}) \prod_{i\in\Gamma}\sin k_i t_i \prod_{j\in W}(-\cos k_j t_j)\,\mathrm{d}\mu_m(\boldsymbol{t}) \\
&= \frac{1}{\pi^m}\int_{[-\pi,\pi]^m} f(\boldsymbol{t}) \prod_{i\in\Gamma\cup W} \frac{-\psi_i'(k_i\pi)+\psi_i'(k_i t_i)}{k_i} \prod_{l\in\Gamma'\cap W'} (\pi - t_l)\,\mathrm{d}\mu_m(\boldsymbol{t}) \\
&= \frac{1}{\pi^m}\int_{[-\pi,\pi]^m} f(\boldsymbol{t}) \prod_{i\in\Gamma\cup W} \frac{\psi_i'(k_i t_i)}{k_i} \prod_{l\in\Gamma'\cap W'} (-t_l)\,\mathrm{d}\mu_m(\boldsymbol{t}) \qquad \text{(by (22))} \\
&= \frac{1}{\pi^m}\int_{[-\pi,\pi]^m} f(\boldsymbol{t}) \prod_{i\in\Gamma\cup W} \frac{\psi_i'(k_i t_i)}{k_i} \prod_{l\in\Gamma'\cap W'} \Big\{\sum_{r_l=1}^{\infty} \frac{2(-1)^{r_l}\sin r_l t_l}{r_l}\Big\}\,\mathrm{d}\mu_m(\boldsymbol{t}) \\
&= \sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m \\ r_i=k_i\,\forall\, i\in\Gamma\cup W \\ r_j\geqslant 1\,\forall\, j\in\Gamma'\cap W'}} \frac{c_{\boldsymbol{r},\Gamma}}{\prod_{i=1}^m r_i} \prod_{l\in\Gamma'\cap W'} 2(-1)^{r_l}.
\end{aligned}$$

Case 2: $W = \Gamma'$. In this case we can follow the proof of case 1 to show that if $\boldsymbol{k}\in\mathbb{N}^m$, then

$$A_{\boldsymbol{r}} \prod_{j\in\Gamma'}(-1) = c_{\boldsymbol{k},\Gamma} \Big/ \prod_{i=1}^m k_i.$$

Now, we deduce from the above cases, (i), Theorem 5.2, and Corollary 5.3 that (25) holds:

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_i\geqslant 1\,\forall\, i\in\Gamma\cup W\\ k_j=0\,\forall\, j\in\Gamma'\cap W'}} \frac{A_{\boldsymbol{k}}}{2^{m-\mathrm{card}(W\cup\Gamma)}}\prod_{i\in\Gamma}\sin k_ix_i\prod_{j\in W}\cos k_jx_j\\
&\qquad=\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ k_i\geqslant 1\,\forall\, i\in\Gamma\cup W\\ k_j=0\,\forall\, j\in\Gamma'\cap W'}}\Bigg(\sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m\\ r_i=k_i\,\forall\, i\in\Gamma\cup W\\ r_j\geqslant 1\,\forall\, j\in\Gamma'\cap W'}}\frac{c_{\boldsymbol{r},\Gamma}}{\prod_{i=1}^m r_i}\Bigg)\prod_{i\in\Gamma\cup W}\psi_i(k_ix_i)\prod_{l\in\Gamma'\cap W'}(-1)^{r_l}\\
&\qquad=\sum_{\boldsymbol{k}\in\mathbb{N}^m}c_{\boldsymbol{k},\Gamma}\Big\{\prod_{i\in\Gamma}\frac{\sin k_ix_i}{k_i}\Big\}\Big\{\prod_{j\in W}\Big(-\frac{\cos k_jx_j}{k_j}\Big)\prod_{l\in\Gamma'\cap W'}\frac{\cos k_l\pi}{k_l}\Big\}.
\end{aligned}$$

Finally, since Theorem 2.9 and (i) imply that

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m}c_{\boldsymbol{k}}\int_{[-\pi,\pi]^m}\Big\{\prod_{i\in\Gamma}\cos k_it_i\Big\}\Big\{\prod_{i\in\Gamma'}\sin k_it_i\Big\}\nu\Big(\prod_{i=1}^m[-\pi,t_i)\Big)\,\mathrm{d}\mu_m(\boldsymbol{t})$$

is regularly convergent, (iii) follows from Theorems 5.3, 2.9, and (ii). The proof is complete. □

We observe that the proof of Theorem 4.2 depends on regularly convergent multiple series. On the other hand, the proof of Theorem 4.5 relies on Theorems 3.1 and 3.2 involving absolutely convergent multiple series. In view of [8, Lemma 19], [16, Theorem 4.5], Theorems 4.2, 4.5, 5.2, and the fact that Fubini's theorem is valid for the Henstock-Kurzweil integral, it is reasonable to believe that the following conjecture is true for every positive integer $m\geqslant 2$.

Conjecture 5.6. *Let $f\in L^1(\mathbb{T}^m)$. Then the multiple series*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m}\frac{1}{\prod_{i=1}^m k_i}\int_{[-\pi,\pi]^m}f(\boldsymbol{t})\prod_{i=1}^m\cos k_it_i\,\mathrm{d}\mu_m(\boldsymbol{t})$$

converges regularly if and only if the multiple Perron integral

$$(\mathrm{P})\int_{[0,\pi]^m}\frac{1}{\prod_{i=1}^m x_i}\Big\{\int_{\prod_{i=1}^m[-x_i,x_i]}f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\Big\}\,\mathrm{d}\boldsymbol{x}$$

exists.

6. Another convergence theorem for Henstock-Kurzweil integrals

The aim of this section is to establish a new convergence theorem (Theorem 6.3) involving the Henstock-Kurzweil integral and regularly convergent multiple series. We begin with the following important summation by parts theorem (cf. [19, Theorem 2.2]).

Theorem 6.1. *Let* $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ *be a multiple sequence of real numbers such that* $\lim_{\|\boldsymbol{n}\|\to\infty} c_{\boldsymbol{n}} = 0$ *and* $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} |\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})|$ *converges. If* $\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ *is a multiple sequence of real numbers and if* $\boldsymbol{n}\in\mathbb{N}_0^m$, *then*

$$\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k}}u_{\boldsymbol{k}} = \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ 0\leqslant k_l\leqslant n_l-1\,\forall\, l\in\Gamma\\ k_l\geqslant n_l\forall\, l\in\Gamma'}} \left\{\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}}) \sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^m\\ 0\leqslant j_l\leqslant k_l\,\forall\, l\in\Gamma\\ 0\leqslant j_l\leqslant n_l\,\forall\, l\in\Gamma'}} u_{\boldsymbol{j}}\right\}. \tag{26}$$

Theorem 6.1 leads us to the following generalized Dirichlet test; see [19, Theorem 2.3] for details.

Theorem 6.2. *Let* $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ *be a multiple sequence of real numbers such that* $\lim_{\|\boldsymbol{n}\|\to\infty} c_{\boldsymbol{n}} = 0$. *If* $\{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^m}$ *is a multiple sequence in a Banach space* $(\mathbb{B},\|\cdot\|)$ *and*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \{|\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})|\}\left\{\max_{\boldsymbol{0}\leqslant\boldsymbol{r}\leqslant\boldsymbol{k}}\left\|\sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{r}} x_{\boldsymbol{j}}\right\|\right\}$$

converges, then the following assertions hold:

(i) $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}}) \sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} x_{\boldsymbol{j}}$ *converges absolutely.*

(ii) *For each* $\varepsilon>0$ *there exists* $N(\varepsilon)\in\mathbb{N}_0$ *such that*

$$\left\|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} c_{\boldsymbol{k}}x_{\boldsymbol{k}}\right\| < \varepsilon$$

for every $\boldsymbol{p},\boldsymbol{q}\in\mathbb{N}_0^m$ *satisfying* $\boldsymbol{q}\geqslant\boldsymbol{p}$ *and* $\|\boldsymbol{p}\|\geqslant N(\varepsilon)$.

(iii) *We have*

$$\lim_{\min\{n_1,\dots,n_m\}\to\infty}\left\|\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k}}x_{\boldsymbol{k}} - \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}}) \sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} x_{\boldsymbol{j}}\right\| = 0.$$

In the next theorem, the set Ξ cannot be empty and the limiting function f need not be Lebesgue integrable on $[\boldsymbol{a}, \boldsymbol{b}]$; see Example 8.2 and [37, (1.9) Theorem, Chapter V] for details.

Theorem 6.3. *Let* $\{\varphi_{i,n}\}_{n=0}^{\infty} \subset L^1[a_i, b_i]$ $(i = 1, \dots, m)$ *and suppose that*

$$\Xi := \left\{ r \in \{1, \dots, m\} \colon \sup_{n \in \mathbb{N}_0} \Big\| \sum_{k=0}^{n} \varphi_{r,k} \Big\|_{\mathrm{HK}[a_r, b_r]} < \infty \right\}$$

is nonempty. If $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}_0^m}$ *is a multiple sequence of real numbers and if*

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} |\Delta_{\Gamma \cup \Xi}(c_{\boldsymbol{k}})| \prod_{l \in \Gamma' \cap \Xi'} \|\varphi_{l,k_l}\|_{\mathrm{HK}[a_l, b_l]} \times \prod_{j \in \Gamma} \max_{0 \leqslant q_j \leqslant k_j} \Big\| \sum_{r_j=0}^{q_j} \varphi_{j,r_j} \Big\|_{L^1[a_j + (b_j - a_j/(n+1), b_j]} \tag{27}$$

converges for every $\Gamma \subseteq \{1, \dots, m\}$ *and* $n \in \mathbb{N}$, *then there exists* $f \in \mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]$ *such that*

$$\lim_{\min\{n_1, \dots, n_m\} \to \infty} \Big\| \sum_{\boldsymbol{0} \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i,k_i} - f \Big\|_{\mathrm{HK}[\boldsymbol{a}, \boldsymbol{b}]} = 0. \tag{28}$$

Proof. We may assume that

$$\Xi = \left\{ r \in \{1, \dots, m\} \colon \sup_{n \in \mathbb{N}_0} \Big\| \sum_{k=0}^{n} \varphi_{r,k} \Big\|_{\mathrm{HK}[a_r, b_r]} \leqslant 1 \right\}. \tag{29}$$

Using (27) with $\Gamma = \{1, \dots, m\}$ we see that

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} |\Delta_{\{1, \dots, m\}}(c_{\boldsymbol{k}})| \prod_{j=1}^{m} \max_{0 \leqslant q_j \leqslant k_j} \Big\| \sum_{r_j=0}^{q_j} \varphi_{j,r_j} \Big\|_{L^1[a_j + (b_j - a_j)/(n+1), b_j]} \tag{30}$$

converges for every $n \in \mathbb{N}$, where an empty product is taken to be one. Hence, by Theorem 6.2, there exists a function $f \colon [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ with the following properties:

(31) $I \subset (\boldsymbol{a}, \boldsymbol{b}]$ and $I \in \mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$ imply $f \in L^1(I)$, the multiple series $\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} c_{\boldsymbol{k}} \int_I \prod_{i=1}^{m} \varphi_{i,k_i}(t_i) \, \mathrm{d}\mu_m(\boldsymbol{t})$ converges regularly and

$$\int_I f(\boldsymbol{t}) \, \mathrm{d}\mu_m(\boldsymbol{t}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} c_{\boldsymbol{k}} \int_I \prod_{i=1}^{m} \varphi_{i,k_i}(t_i) \, \mathrm{d}\mu_m(\boldsymbol{t}).$$

We shall next prove that

$$(32)\qquad \Big\| \sum_{\boldsymbol{p}\leqslant \boldsymbol{k}\leqslant \boldsymbol{q}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i,k_i} \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} \to 0 \quad \text{as } \max\{\|\boldsymbol{p}\|, \|\boldsymbol{q}\|\} \to \infty.$$

P r o o f of (32). Let $\kappa_0 > 0$ be given. Using (27) with $\Gamma = \emptyset$, we select $N \in \mathbb{N}$ such that

$$(33)\qquad \max_{i=1,\ldots,m} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ k_i \geqslant N}} |\Delta_\Xi(c_{\boldsymbol{k}})| \prod_{l\in\Xi'} \|\varphi_{l,k_l}\|_{\mathrm{HK}[a_l,b_l]} < \frac{\kappa_0}{8^m}\Big(1 + \max_{r\in\Xi} \sup_{n\in\mathbb{N}_0} \Big\| \sum_{k=0}^{n} \varphi_{r,k} \Big\|_{\mathrm{HK}[a_r,b_r]}\Big)^{-1}.$$

Hence, for any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}_0^m$ satisfying $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\|\boldsymbol{p}\| \geqslant N$, we have (32):

$$\begin{aligned}
&\Big\| \sum_{\boldsymbol{p}\leqslant \boldsymbol{k}\leqslant \boldsymbol{q}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i,k_i} \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} \\
&\leqslant \sum_{\substack{\boldsymbol{r}\in\mathbb{N}_0^m \\ r_j=0\,\forall\, j\in\Xi \\ p_l\leqslant r_l\leqslant q_l\,\forall\, l\in\Xi'}} \Big\| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ p_j\leqslant k_j\leqslant q_j\,\forall\, j\in\Xi \\ k_l=r_l\,\forall\, l\in\Xi'}} c_{\boldsymbol{k}} \bigotimes_{j\in\Xi} \varphi_{j,k_j} \bigotimes_{l\in\Xi'} \varphi_{l,k_l} \Big\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} \\
&\qquad \text{(by triangle inequality)} \\
&\leqslant 8^m \max_{r\in\Xi} \sup_{n\in\mathbb{N}_0} \Big\| \sum_{k=0}^{n} \varphi_{r,k} \Big\|_{\mathrm{HK}[a_r,b_r]} \Big(\sum_{\boldsymbol{k}\geqslant\boldsymbol{p}} |\Delta_\Xi(c_{\boldsymbol{k}})| \prod_{l\in\Xi'} \|\varphi_{l,k_l}\|_{\mathrm{HK}[a_l,b_l]} \Big) \\
&\qquad \text{(by Theorem 6.2 with } m = \operatorname{card}(\Xi), \\
&\qquad \text{Fubini's theorem and triangle inequality)} \\
&< \kappa_0 \qquad \text{(by (33)).}
\end{aligned}$$

The proof of (32) is complete. □

We shall next show that $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and G is the indefinite Henstock-Kurzweil integral of f, where the interval function $G\colon \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}]) \to \mathbb{R}$ is defined by

$$G([\boldsymbol{u},\boldsymbol{v}]) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{u_i}^{v_i} \varphi_{i,k_i}\,\mathrm{d}\mu_1;$$

the multiple series on the right being regularly convergent (cf. (32)). In view of (31), Theorem 2.7, and (32), it remains to prove that

$$(34)\qquad V_{\mathrm{HK}} G([\boldsymbol{a},\boldsymbol{b}] \setminus (\boldsymbol{a},\boldsymbol{b}]) = 0,$$

where $(\boldsymbol{a},\boldsymbol{b}] := \prod_{k=1}^{m} (a_k, b_k]$.

P r o o f of (34). Since $V_{\mathrm{HK}}G$ is an outer measure (cf. Theorem 2.5), it is enough to prove that

$$V_{\mathrm{HK}}G(Z_{\Gamma,n}) = 0 \tag{35}$$

whenever $n \in \mathbb{N}$ and $\Gamma \subset \{1,\dots,m\}$, where

$$[\boldsymbol{a},\boldsymbol{b}] \setminus (\boldsymbol{a},\boldsymbol{b}] = \bigcup_{n\in\mathbb{N}} \bigcup_{\Gamma\subset\{1,\dots,m\}} Z_{\Gamma,n},$$

$$Z_{\Gamma,n} := \prod_{k=1}^{m} Z_{\Gamma,n,k} \quad \text{and} \quad Z_{\Gamma,n,k} := \begin{cases} \{a_k\} & \text{if } k \in \Gamma', \\ \left[a_k + \dfrac{b_k - a_k}{n+1}, b_k\right] & \text{if } k \in \Gamma. \end{cases}$$

Let $\varepsilon > 0$ be given. From (27) we pick a positive integer K such that

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ \boldsymbol{k}\notin[0,K]^m}} |\Delta_{\Gamma\cup\Xi}(c_{\boldsymbol{k}})| \prod_{l\in\Gamma'\cap\Xi'} \|\varphi_{l,k_l}\|_{\mathrm{HK}[a_l,b_l]} \prod_{j\in\Gamma} \max_{0\leqslant q_j\leqslant k_j} \Big\| \sum_{r_j=0}^{q_j} \varphi_{j,r_j} \Big\|_{L^1(Z_{\Gamma,n,j})} < \frac{\varepsilon}{2}\Big(1 + \max_{r\in\Xi} \sup_{n\in\mathbb{N}_0} \Big\| \sum_{k=0}^{n} \varphi_{r,k} \Big\|_{\mathrm{HK}[a_r,b_r]}\Big)^{-1}. \tag{36}$$

Employing the uniform continuity of the indefinite Henstock-Kurzweil integral (cf. Theorem 2.3 (iv)), we select a $\eta(\Gamma,n) > 0$ such that

$$\max_{i\in\Gamma'} \max_{0\leqslant k_i\leqslant K} \sup_{\substack{[u,v]\subseteq[a_i,b_i] \\ 0<v-u<\eta(\Gamma,n)}} \|\varphi_{i,k_i}\|_{\mathrm{HK}[u,v]} < \frac{\varepsilon}{2}\Big(1 + \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} |\Delta_{\Gamma\cup\Xi}(c_{\boldsymbol{k}})| \prod_{l\in\Gamma} \Big\| \sum_{j_l=0}^{k_l} \varphi_{l,j_l} \Big\|_{L^1(Z_{\Gamma,n,l})}\Big)^{-1}. \tag{37}$$

Define a gauge $\delta_{\Gamma,n}$ on $Z_{\Gamma,n}$ by setting $\delta_{\Gamma,n}(\boldsymbol{x}) := \eta(\Gamma,n)$ and select any $\delta_{\Gamma,n}$-fine partial partition $P_{\Gamma,n} = \{(J_1,\boldsymbol{t}_1),\dots,(J_q,\boldsymbol{t}_q)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ with $\{\boldsymbol{t}_1,\dots,\boldsymbol{t}_q\} \subset Z_{\Gamma,n}$. Since all the integrals are real-valued, it suffices to prove that

$$\Big| \sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{I_i} \varphi_{i,k_i} \,\mathrm{d}\mu_1 \Big| < \varepsilon. \tag{38}$$

P r o o f of (38). Write $T = \Gamma \cup \Xi$ and $\Phi_{i,k_i} = \sum_{j_i=0}^{k_i} \varphi_{i,j_i}$ $(i = 1, \ldots, m)$. Direct computations give

$$
\begin{aligned}
&\Bigl|\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} \prod_{i=1}^m \int_{I_i} \varphi_{i,k_i}\,\mathrm{d}\mu_1\Bigr| \\
&\quad= \Bigl|\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} \Delta_T(c_{\boldsymbol{k}}) \Bigl(\prod_{i\in T'} \int_{I_i} \varphi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigl(\prod_{i\in T} \int_{I_i} \Phi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigr| \\
&\qquad\qquad\qquad\qquad\text{(by a card}(T)\text{-dimensional analogue of Theorem 6.2)} \\
&\quad\leqslant \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{K}} |\Delta_T(c_{\boldsymbol{k}})| \Bigl|\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} \Bigl(\prod_{i\in T'} \int_{I_i} \varphi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigl(\prod_{i\in T} \int_{I_i} \Phi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigr| \\
&\qquad+ \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m\\ \boldsymbol{k}\notin[0,K]^m}} |\Delta_T(c_{\boldsymbol{k}})| \Bigl|\sum_{(I,\boldsymbol{t})\in P_{\Gamma,n}} \Bigl(\prod_{i\in T'} \int_{I_i} \varphi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigl(\prod_{i\in T} \int_{I_i} \Phi_{i,k_i}\,\mathrm{d}\mu_1\Bigr)\Bigr| \\
&\quad= R_1 + R_2,
\end{aligned}
$$

say. To complete the proof of (38), we have to establish the following claims.

Claim 1. *If* $\Gamma = \emptyset$, *then* $\max\{R_1, R_2\} < \frac{1}{2}\varepsilon$.

P r o o f of Claim 1. Since $\operatorname{card}(P_{\emptyset,n}) = 1$, (37) and (29) imply that $R_1 < \frac{1}{2}\varepsilon$. Likewise, we infer from (36) and (29) that $R_2 < \frac{1}{2}\varepsilon$. □

Claim 2. *If* $\Gamma \subset \{1, \ldots, m\}$ *is nonempty, then* $\max\{R_1, R_2\} < \frac{1}{2}\varepsilon$.

P r o o f of Claim 2. Our choice of $P_{\Gamma,n}$ implies that $\mu_{\operatorname{card}(\Gamma)}\Bigl(\prod_{i\in\Gamma} I_i \cap \prod_{i\in\Gamma} U_i\Bigr) = 0$ whenever $\Bigl(\prod_{i=1}^m I_i, \boldsymbol{x}\Bigr)$ and $\Bigl(\prod_{i=1}^m U_i, \boldsymbol{y}\Bigr)$ are two distinct elements of $P_{\Gamma,n}$. Combining this with (37) and (29), we get $R_1 < \frac{1}{2}\varepsilon$:

$$
\begin{aligned}
R_1 &\leqslant \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{K}} |\Delta_T(c_{\boldsymbol{k}})| \prod_{i\in T'} \|\varphi_{i,k_i}\|_{\mathrm{HK}[a_i,b_i]} \\
&\quad\times \prod_{j\in\Xi\setminus\Gamma} \|\Phi_{j,k_j}\|_{\mathrm{HK}[a_i,b_i]} \prod_{l\in\Gamma} \|\Phi_{l,k_l}\|_{L^1(Z_{\Gamma,n,l})} < \frac{1}{2}\varepsilon.
\end{aligned}
$$

A similar reasoning shows that $R_2 < \frac{1}{2}\varepsilon$. The proof is complete. □

Corollary 6.4. *Suppose that $S \subseteq \{1, \dots, m\}$ is nonempty and let $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}_0^m}$ be a multiple sequence of real numbers such that $\lim\limits_{\|\boldsymbol{n}\| \to \infty} c_{\boldsymbol{n}} = 0$. If the multiple series*

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^m \\ k_i > 0 \, \forall \, i \in \Gamma' \cap S'}} \left| \frac{\Delta_{\Gamma \cup S}(c_{\boldsymbol{k}})}{\prod\limits_{i \in \Gamma' \cap S'} k_i} \right| \quad \text{converges for every } \Gamma \subseteq \{1, \dots, m\},$$

then the following assertions hold:

(i) *The multiple series*

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} c_{\boldsymbol{k}} \Big\{ \prod_{i \in S} \cos k_i x_i \Big\} \Big\{ \prod_{i \in S'} \sin k_i x_i \Big\}$$

converges regularly for all $\boldsymbol{x} \in (0, \pi]^m$.

(ii) *Let*

$$f_S(\boldsymbol{x}) := \begin{cases} \sum\limits_{\boldsymbol{k} \in \mathbb{N}_0^m} c_{\boldsymbol{k}} \Big\{ \prod\limits_{i \in S} \cos k_i x_i \Big\} \Big\{ \prod\limits_{i \in S'} \sin k_i x_i \Big\} & \text{if } \boldsymbol{x} \in (0, \pi]^m, \\ 0 & \text{otherwise,} \end{cases}$$

then $f_S \in \mathrm{HK}([0, \pi]^m)$ and

$$\lim_{\min\{n_1, \dots, n_m\} \to \infty} \Big\| \sum_{\boldsymbol{0} \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} c_{\boldsymbol{k}} \bigotimes_{i=1}^{m} \varphi_{i, k_i} - f_S \Big\|_{\mathrm{HK}([0, \pi]^m)} = 0.$$

Proof. Assertion (i) follows from Theorem 6.2. Since

$$\sup_{x \in \mathbb{R}} \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leqslant 2\pi,$$

assertion (ii) follows from Theorem 6.3. □

Remark 6.5. When $m = 2$ and $S = \{1, 2\}$, Corollary 6.4 refines [28, Theorem 1]. On the other hand, if $m = 2$ and $S = \{1\}$, Corollary 6.4 and Theorem 3.2 can be used to refine [30, Theorem 2]. The details will appear elsewhere.

7. A multidimensional analogue of Boas' theorem

The aim of this section is to prove a useful multidimensional analogue of

Theorem 7.1 (cf. [2, Theorem 4]). *Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\lim_{k\to\infty} b_k = 0$ and $\sum_{k=1}^{\infty} |b_k - b_{k+1}|$ converges. Then $\sum_{k=1}^{\infty} b_k/k$ converges if and only if $\lim_{x\to 0^+} \sum_{k=1}^{\infty} (b_k \cos kx)/k$ exists.*

We need the following lemmas.

Lemma 7.2. *Let $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that*

$$\lim_{|||\boldsymbol{n}|||\to\infty} c_{\boldsymbol{n}} = 0 \tag{39}$$

and

$$\max_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \left| \frac{\Delta_{\Gamma}(c_{\boldsymbol{k}})}{\prod_{i\in\Gamma'} k_i} \right| < \infty. \tag{40}$$

If $j \in \{1,\dots,m\}$ and $\alpha_j \in \mathbb{N}$, then the $(m-1)$-multiple series

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_j=\alpha_j}} c_{\boldsymbol{k}} \Bigg(\prod_{\substack{i=1\\ i\neq j}}^{m} k_i\Bigg)^{-1}$$

is absolutely convergent.

P r o o f. Since

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_j=\alpha_j}} |c_{\boldsymbol{k}}| \Bigg(\prod_{\substack{i=1\\ i\neq j}}^{m} k_i\Bigg)^{-1} \leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_j\geqslant\alpha_j}} |\Delta_{\{j\}}(c_{\boldsymbol{k}})| \Bigg(\prod_{\substack{i=1\\ i\neq j}}^{m} k_i\Bigg)^{-1}$$

the lemma follows. □

From the proof of Lemma 7.2 we get

Lemma 7.3. *Let $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that* (39) *and* (40) *hold. If $j \in \{1,\dots,m\}$, then*

$$\lim_{n_j\to\infty} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_j=n_j}} |c_{\boldsymbol{k}}| \Bigg(\prod_{\substack{i=1\\ i\neq j}}^{m} k_i\Bigg)^{-1} = 0.$$

For the rest of this paper, we write $(\boldsymbol{a}, \boldsymbol{b}) := \prod_{i=1}^{m} (a_i, b_i)$. In view of Theorem 6.2, the following theorem is a generalization of [18, Theorem 4.3].

Theorem 7.4. *Let $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that* (39) *and* (40) *hold. Suppose that $\{\varphi_{i,n}\}_{n=1}^{\infty} \subset C[a_i, b_i]$ $(i = 1, \ldots, m)$ and*

$$\max_{i=1,\ldots,m} \left\{ \sup_{n\in\mathbb{N}} \|\varphi_{i,n}\|_{C[a_i,b_i]} + \sup_{a_i<x_i<b_i} \sup_{n\in\mathbb{N}} \left| \frac{\varphi_{i,n}(x_i) - \varphi_{i,n}(a_i)}{n(x_i - a_i)} \right| + \sup_{x_i\in[a_i,b_i]} \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} (x_i - a_i)\varphi_{i,k}(x_i) \right| \right\} < \infty.$$

Then the following assertions hold.

(i) *If $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}] \setminus \{\boldsymbol{a}\}$, then $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(c_{\boldsymbol{k}} / \prod_{i=1}^{m} k_i\Big) \prod_{i=1}^{m} \varphi_{i,k_i}(x_i)$ converges regularly.*

(ii) *The function $\boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(c_{\boldsymbol{k}} / \prod_{i=1}^{m} k_i\Big) \prod_{i=1}^{m} \varphi_{i,k_i}(x_i)$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}] \setminus \{\boldsymbol{a}\}$.*

(iii) *If*

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a} \\ \boldsymbol{x}\in(\boldsymbol{a},\boldsymbol{b})}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i} \prod_{i=1}^{m} \varphi_{i,k_i}(x_i)$$

exists, then $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(c_{\boldsymbol{k}} / \prod_{i=1}^{m} k_i\Big) \prod_{i=1}^{m} \varphi_{i,k_i}(a_i)$ converges regularly.

(iv) *If $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(c_{\boldsymbol{k}} / \prod_{i=1}^{m} k_i\Big) \prod_{i=1}^{m} \varphi_{i,k_i}(a_i)$ is regularly convergent, then the function $\boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(c_{\boldsymbol{k}} / \prod_{i=1}^{m} k_i\Big) \prod_{i=1}^{m} \varphi_{i,k_i}(x_i)$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$.*

Proof. Without loss of generality we may suppose that

$$\max_{i=1,\ldots,m} \left\{ \sup_{n\in\mathbb{N}} \|\varphi_{i,n}\|_{C[a_i,b_i]} + \sup_{a_i<x_i<b_i} \sup_{n\in\mathbb{N}} \left| \frac{\varphi_{i,n}(x_i) - \varphi_{i,n}(a_i)}{n(x_i - a_i)} \right| + \sup_{x_i\in[a_i,b_i]} \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} (x_i - a_i)\varphi_{i,k}(x_i) \right| \right\} \leqslant \frac{1}{2}. \tag{41}$$

(i) Let $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}] \setminus \{\boldsymbol{a}\}$ and set

$$\Gamma_{\boldsymbol{x}} := \{i \in \{1, \ldots, m\} \colon x_i \neq a_i\}.$$

If $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^m$ and $\boldsymbol{q} \geqslant \boldsymbol{p}$, then it follows by summation by parts and (41) that

$$\text{(42)} \qquad \left| \sum_{\boldsymbol{p}+\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant \boldsymbol{q}} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \prod_{i=1}^m \varphi_{i,k_i}(x_i) \right|$$
$$\leqslant \sum_{\substack{\boldsymbol{r} \in \mathbb{N}^m \\ r_j = 1 \, \forall \, j \in \Gamma_{\boldsymbol{x}} \\ p_l + 1 \leqslant r_l \leqslant q_l \, \forall \, l \in \Gamma'_{\boldsymbol{x}}}} \left| \sum_{\substack{\boldsymbol{k} \in \mathbb{N}^m \\ p_j + 1 \leqslant k_j \leqslant q_j \, \forall \, j \in \Gamma_{\boldsymbol{x}} \\ k_l = r_l \, \forall \, l \in \Gamma'_{\boldsymbol{x}}}} c_{\boldsymbol{k}} \prod_{j \in \Gamma_{\boldsymbol{x}}} \frac{\varphi_{j,k_j}(x_j)}{k_j} \prod_{l \in \Gamma'_{\boldsymbol{x}}} \frac{1}{2k_l} \right|$$
$$\leqslant \frac{2^m}{\prod\limits_{i \in \Gamma_{\boldsymbol{x}}} (x_i - a_i)} \sum_{\substack{\boldsymbol{r} \in \mathbb{N}^m \\ r_j = 1 \, \forall \, j \in \Gamma_{\boldsymbol{x}} \\ p_l + 1 \leqslant r_l \leqslant q_l \, \forall \, l \in \Gamma'_{\boldsymbol{x}}}} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}^m \\ p_j + 1 \leqslant k_j \, \forall \, j \in \Gamma_{\boldsymbol{x}} \\ k_l = r_l \, \forall \, l \in \Gamma'_{\boldsymbol{x}}}} \left| \frac{\Delta_{\Gamma_{\boldsymbol{x}}}(c_{\boldsymbol{k}})}{\prod\limits_{l \in \Gamma'_{\boldsymbol{x}}} 2k_l} \right|.$$

It is now easy to check that (i) is a consequence of (42), (40), and Definition 5.1 (ii).

(ii) We infer from (42) that the regularly convergent series $\sum\limits_{\boldsymbol{k} \in \mathbb{N}^m} \left(c_{\boldsymbol{k}} / \prod\limits_{i=1}^m k_i \right) \prod\limits_{i=1}^m \varphi_{i,k_i}$ is uniformly convergent on each compact interval $\prod\limits_{i=1}^m [u_i, v_i] \subset [\boldsymbol{a}, \boldsymbol{b}] \setminus \{\boldsymbol{a}\}$. Hence (ii) follows.

(iii) Let $\Phi(\boldsymbol{x}) := \prod\limits_{i=1}^m \varphi_{i,k_i}(x_i)$. In view of (41), Lemma 7.2, and Theorem 5.2, it suffices to prove that

$$\text{(43)} \qquad \lim_{\substack{\boldsymbol{\delta} \to \boldsymbol{0} \\ \boldsymbol{\delta} \in \prod_{i=1}^m (0, b_i - a_i)}} \left| \sum_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant (\lfloor 1/\delta_1 \rfloor, \ldots, \lfloor 1/\delta_m \rfloor)} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \Phi(\boldsymbol{a}) - \sum_{\boldsymbol{k} \in \mathbb{N}^m} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \Phi(\boldsymbol{a} + \boldsymbol{\delta}) \right| = 0.$$

To prove (43), we select any $\boldsymbol{\delta} \in \prod\limits_{i=1}^m (0, \min\{1, b_i - a_i\})$ and a direct computation yields

$$\left| \sum_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant (\lfloor 1/\delta_1 \rfloor, \ldots, \lfloor 1/\delta_m \rfloor)} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \Phi(\boldsymbol{a}) - \sum_{\boldsymbol{k} \in \mathbb{N}^m} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \Phi(\boldsymbol{a} + \boldsymbol{\delta}) \right|$$
$$\leqslant \left| \sum_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant (\lfloor 1/\delta_1 \rfloor, \ldots, \lfloor 1/\delta_m \rfloor)} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} (\Phi(\boldsymbol{a}) - \Phi(\boldsymbol{a} + \boldsymbol{\delta})) \right|$$
$$+ \left| \sum_{\boldsymbol{k} \in \mathbb{N}^m \setminus \prod_{i=1}^m [1, \lfloor 1/\delta_i \rfloor]} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \Phi(\boldsymbol{a} + \boldsymbol{\delta}) \right| = S_{\boldsymbol{\delta}} + T_{\boldsymbol{\delta}},$$

say.

Using [18, Lemma 4.2] and (41), we get

$$S_{\boldsymbol{\delta}} \leqslant \sum_{j=1}^{m} \Big(\frac{1}{\lfloor \delta_j \rfloor}\Big)^{-1} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i \geqslant 1\,\forall\, i\in\{1,\ldots,m\}\setminus\{j\} \\ 1\leqslant k_j \leqslant \lfloor 1/\delta_j \rfloor}} |c_{\boldsymbol{k}}| \prod_{\substack{i=1 \\ i\neq j}}^{m} \frac{1}{k_i}$$

and hence Lemma 7.3 yields

$$S_{\boldsymbol{\delta}} \to 0 \quad \text{as} \quad \|\boldsymbol{\delta}\| \to 0. \tag{44}$$

It remains to prove that $T_{\boldsymbol{\delta}} \to 0$ as $\|\boldsymbol{\delta}\| \to 0$. We write

$$U_\Gamma(\boldsymbol{\delta}) = \Bigg| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i > \lfloor 1/\delta_i \rfloor\,\forall\, i\in\Gamma \\ 1\leqslant k_j \leqslant \lfloor 1/\delta_j \rfloor\,\forall\, j\in\Gamma'}} c_{\boldsymbol{k}} \prod_{i=1}^{m} \frac{\varphi_{i,k_i}(a_i+\delta_i)}{k_i} \Bigg|$$

and observe that

$$T_{\boldsymbol{\delta}} \leqslant \sum_{\emptyset\neq\Gamma\subseteq\{1,\ldots,m\}} U_\Gamma(\boldsymbol{\delta}).$$

Therefore it is enough to prove that

$$\max_{\emptyset\neq\Gamma\subseteq\{1,\ldots,m\}} U_\Gamma(\boldsymbol{\delta}) \to 0 \quad \text{as} \quad \|\boldsymbol{\delta}\| \to 0. \tag{45}$$

Let $\Gamma \subseteq \{1,\ldots,m\}$ be nonempty and write $q = \max_{i\in\Gamma} i$. Then the triangle inequality and (41) yield

$$U_\Gamma(\boldsymbol{\delta}) \leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_q=1 \\ k_i\geqslant 1\,\forall\, i\in\{1,\ldots,m\}\setminus\{q\}}} \prod_{\substack{i=1 \\ i\neq q}}^{m} \frac{1}{2k_i} \Bigg| \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_q > \lfloor 1/\delta_q \rfloor \\ j_i=k_i\,\forall\, i\in\{1,\ldots,m\}\setminus\{q\}}} c_{\boldsymbol{j}} \frac{\varphi_{q,j_q}(a_q+\delta_q)}{j_q} \Bigg|. \tag{46}$$

In order to obtain an appropriate upper bound for the right-hand side of (46), we let $k_i \in \mathbb{N}$ $(i \in \{1,\dots,m\} \setminus \{q\})$ and get

$$
\begin{aligned}
(47)\quad & \left| \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_q > \lfloor 1/\delta_q \rfloor \\ j_i = k_i \,\forall\, i \in \{1,\dots,m\}\setminus\{q\}}} c_{\boldsymbol{j}} \frac{\varphi_{q,j_q}(a_q+\delta_q)}{j_q} \right| \\
&\leqslant \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_q > \lfloor 1/\delta_q \rfloor \\ j_i = k_i \,\forall\, i \in \{1,\dots,m\}\setminus\{q\}}} |\Delta_{\{q\}}(c_{\boldsymbol{j}})| \left| \sum_{r_q=\lfloor 1/\delta_q \rfloor+1}^{j_q} \frac{\varphi_{q,r_q}(a_q+\delta_q)}{r_q} \right| \\
&\qquad \text{(by single summation by parts and triangle inequality)} \\
&\leqslant \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_q > \lfloor 1/\delta_q \rfloor \\ j_i = k_i \,\forall\, i \in \{1,\dots,m\}\setminus\{q\}}} |\Delta_{\{q\}}(c_{\boldsymbol{j}})| \frac{4}{\delta_q(1+\lfloor 1/\delta_q \rfloor)} \\
&\qquad \text{(by single summation by parts and (41))} \\
&\leqslant 8 \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_q > \lfloor 1/\delta_q \rfloor \\ j_i = k_i \,\forall\, i \in \{1,\dots,m\}\setminus\{q\}}} |\Delta_{\{q\}}(c_{\boldsymbol{j}})|.
\end{aligned}
$$

Combining (46), (47), and (40) yields (45). This completes the proof of (iii).

(iv) In view of (43) and Fubini's theorem for regularly convergent multiple series (cf. Corollary 5.3), it suffices to prove that if $S \subset \{1,\dots,m\}$ is nonempty and if $\eta_i \in (0, b_i - a_i)$ for every $i \in S$, then

$$
(48)\quad \left| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ 1\leqslant k_i \leqslant \lfloor 1/\eta_i \rfloor \,\forall\, i\in S \\ k_j \geqslant 1 \,\forall\, j \in S'}} c_{\boldsymbol{k}} \prod_{i=1}^{m} \frac{\varphi_{i,k_i}(a_i)}{k_i} - \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i\in S} \frac{\varphi_{i,k_i}(a_i+\eta_i)}{k_i} \prod_{j\in S'} \frac{\varphi_{j,k_j}(a_j)}{k_j} \right| = o(\max_{i\in S} \eta_i).
$$

A modification of the proof of (43) yields (44), since (39), (40), and (41) imply

$$
\max_{\emptyset\neq\Gamma\subseteq S} \sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m \\ r_i \geqslant 1 \,\forall\, i\in S \\ r_j = 1 \,\forall\, j\in S'}} \prod_{l\in\Gamma'} \frac{1}{r_l} \left| \Delta_\Gamma \left(\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i = r_i \,\forall\, i\in S \\ k_j \geqslant 1 \,\forall\, j\in S'}} c_{\boldsymbol{k}} \prod_{j\in S'} \frac{\varphi_{j,k_j}(a_j)}{k_j} \right) \right| \leqslant \max_{\emptyset\neq\Gamma\subseteq S} \sum_{\boldsymbol{r}\in\mathbb{N}^m} \left| \frac{\Delta_\Gamma(c_{\boldsymbol{r}})}{\prod_{j\in\Gamma'} r_j} \right| < \infty
$$

and

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_i=n_i\,\forall\, i\in S\\ k_j\geqslant 1\,\forall\, j\in S'}} \left| c_{\boldsymbol{k}} \prod_{j\in S'} \frac{\varphi_{j,k_j}(a_j)}{k_j} \right| \leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ k_i\geqslant n_i\,\forall\, i\in S}} \left| \frac{\Delta_S(c_{\boldsymbol{k}})}{\prod_{j\in S'} k_j} \right| \to 0 \text{ as } \max_{i\in S} n_i \to \infty.$$

The proof is complete. □

Using Theorem 7.4 with $[\boldsymbol{a},\boldsymbol{b}] = [0,\pi]^m$, $c_{\boldsymbol{k}} = b_{\boldsymbol{k}}$ and $\psi_{i,k}(x) = \cos kx$ ($k \in \mathbb{N}$, $i = 1,\dots,m$), we deduce the following m-dimensional analogue of Theorem 7.1.

Theorem 7.5. *Let $\{b_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that*

$$\lim_{|||\mathbf{n}|||\to\infty} b_{\mathbf{n}} = 0 \tag{49}$$

and

$$\max_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \left| \Delta_\Gamma(b_{\boldsymbol{k}}) \Big/ \prod_{i\in\Gamma'} k_i \right| < \infty. \tag{50}$$

Then the following assertions hold.

(i) *If $\boldsymbol{x} \in [0,\pi]^m \setminus \{\mathbf{0}\}$, then $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}}/\prod_{i=1}^m k_i\Big) \prod_{i=1}^m \cos k_i x_i$ converges regularly.*

(ii) *The function $\boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}}/\prod_{i=1}^m k_i\Big) \prod_{i=1}^m \cos k_i x_i$ is continuous on $[0,\pi]^m \setminus \{\mathbf{0}\}$.*

(iii) *If $\lim_{\substack{\boldsymbol{x}\to\mathbf{0}\\ \boldsymbol{x}\in(0,\pi)^m}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}}/\prod_{i=1}^m k_i\Big) \prod_{i=1}^m \cos k_i x_i$ exists, then $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}}/\prod_{i=1}^m k_i\Big)$ is regularly convergent.*

(iv) *If $\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}/\prod_{i=1}^m k_i$ is regularly convergent, then the function*

$$\boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}} \Big/ \prod_{i=1}^m k_i\Big) \prod_{i=1}^m \cos k_i x_i$$

is continuous on $[0,\pi]^m$.

The next example shows that Theorem 7.4 is a proper generalization of [18, Theorem 4.3].

Example 7.6. Let $c_n = \sum_{k=n}^\infty (-1)^{k-1} 1/(k(\ln(k+1))^2)$ for $n = 1,2,\dots$. Clearly,

$$\lim_{n\to\infty} c_n = 0, \quad \sum_{k=1}^\infty \Big(|c_k - c_{k+1}| + \frac{|c_n|}{n}\Big) < \infty$$

and

$$\lim_{\substack{(\varepsilon,\delta)\to(0,0)\\(\varepsilon,\delta)\in(0,\pi)^2}} \sum_{(j,k)\in\mathbb{N}^2} \frac{c_j c_k}{jk} \cos j\varepsilon \cos k\delta \tag{51}$$

exists.

On the other hand, since $\sum\limits_{k=1}^{\infty} |c_k - c_{k+1}| \ln(k+1) = \infty$, [18, Theorem 4.3] cannot be used to deduce (51).

8. A Henstock-Kurzweil integrability theorem for multiple sine series

The next result is a generalization of [18, Theorem 5.2]; in particular, we give a negative answer to an open problem of Móricz (cf. [29, Remark 1 (ii)]).

Theorem 8.1. *Let $\{b_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that*

$$\lim_{\|\mathbf{n}\|\to\infty} b_{\mathbf{n}} = 0 \tag{52}$$

and

$$\max_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \left| \frac{\Delta_\Gamma(b_{\boldsymbol{k}})}{\prod\limits_{i\in\Gamma'} k_i} \right| < \infty. \tag{53}$$

Then the following assertions hold.

(i) $\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod\limits_{i=1}^{m} \sin k_i t_i$ *converges regularly for every* $\boldsymbol{t} \in [0,\pi]^m$.

(ii) *If* $\boldsymbol{x} \in [0,\pi)^m \setminus \{\mathbf{0}\}$, *then the function* $\boldsymbol{t} \mapsto \sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod\limits_{i=1}^{m} \sin k_i t_i$ *is Henstock-Kurzweil integrable on* $\prod\limits_{i=1}^{m} [x_i,\pi]$ *and*

$$(\mathrm{HK}) \int_{\prod\limits_{i=1}^{m}[x_i,\pi]} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i \,\mathrm{d}\boldsymbol{t} = \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{x_i}^{\pi} \sin k_i t_i \,\mathrm{d}t_i. \tag{54}$$

(iii) *The function* $\boldsymbol{t} \mapsto \sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod\limits_{i=1}^{m} \sin k_i t_i$ *is Henstock-Kurzweil integrable on* $[0,\pi]^m$ *if and only if* $\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} \Big(b_{\boldsymbol{k}} / \prod\limits_{i=1}^{m} k_i\Big)$ *is regularly convergent. In either case,*

$$(\mathrm{HK}) \int_{[0,\pi]^m} \left\{ \sum_{\mathbf{1}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i - \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i \right\} \mathrm{d}\boldsymbol{t} \to 0 \tag{55}$$

as $\min\{n_1,\dots,n_m\} \to \infty$.

P r o o f. (i) If $\boldsymbol{t} \in [0,\pi]^m$ and $\prod_{i=1}^{m} \sin t_i = 0$, then the result is obvious. On the other hand, if $\boldsymbol{t} \in [0,\pi]^m$ and $\prod_{i=1}^{m} \sin t_i \neq 0$, then the result follows from Theorem 6.2.

(ii) Using Theorem 6.3 with $[\boldsymbol{a},\boldsymbol{b}] = \prod_{i=1}^{m} [x_i,\pi]$ ($\boldsymbol{x} \in [0,\pi)^m \setminus \{\boldsymbol{0}\}$) and $\varphi_{i,k}(t) = \sin kt$ ($k \in \mathbb{N},\ i = 1,\dots,m$), we get the result.

(iii) We infer from (54) and Theorem 2.8 that the function $\boldsymbol{t} \mapsto \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i$ belongs to $\mathrm{HK}([0,\pi]^m)$ if and only if

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{x_i}^{\pi} \sin k_i t_i \,\mathrm{d}t_i \tag{56}$$

exists, which is easily seen to be equivalent to the assertion

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} \Bigg\{ \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i} \Big(\prod_{i\in\Gamma} \cos k_i\pi\Big)\Big(\prod_{i\in\Gamma'} \cos k_i x_i\Big)\Bigg\} \tag{57}$$

exists. Hence assertion (57) holds if and only if

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i} \prod_{i=1}^{m} \cos k_i x_i \tag{58}$$

exists, since Theorem 7.5 (ii), (52), and (53) imply that

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} \Bigg\{ \sum_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i} \Big(\prod_{i\in\Gamma} \cos k_i\pi\Big)\Big(\prod_{i\in\Gamma'} \cos k_i x_i\Big)\Bigg\}$$

exists. It is now clear that the first assertion of statement (iii) follows from parts (iii) and (iv) of Theorem 7.5.

Finally, Theorem 2.8, (54), and Theorem 7.5 yield (55):

$$\begin{aligned}
&(\mathrm{HK}) \int_{[0,\pi]^m} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i \,\mathrm{d}\boldsymbol{t} \\
&\quad = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} (\mathrm{HK}) \int_{\prod_{i=1}^{m}[x_i,\pi]} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \sin k_i t_i \,\mathrm{d}\boldsymbol{t} \\
&\quad = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in[0,\pi]^m}} \Bigg\{ \sum_{\Gamma\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i} \Big(\prod_{i\in\Gamma} \cos k_i\pi\Big)\Big(\prod_{i\in\Gamma'} \cos k_i x_i\Big)\Bigg\} \\
&\quad = \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^{m} \int_{x_i}^{\pi} \sin k_i t_i \,\mathrm{d}t_i.
\end{aligned}$$

□

The following example shows that Theorem 6.3 is, in some sense, sharp.

Example 8.2 (cf. [1]). Let $m=2$ and let $b_{j,k}=1/(\ln(j+k+2))^2$ $((j,k)\in\mathbb{N}^2)$. Then

$$\sum_{(j,k)\in\mathbb{N}^2}\Big\{|\Delta_{\{1,2\}}(b_{j,k})|+\frac{1}{k}|b_{j,k}-b_{j+1,k}|+\frac{1}{j}|b_{j,k}-b_{j,k+1}|\Big\}<\infty.$$

According to parts (i) and (ii) of Theorem 8.1, the double trigonometric series $\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ converges regularly for every $(x,y)\in[-\pi,\pi]^2$, and the function $(x,y)\mapsto\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ is Henstock-Kurzweil integrable on every interval $[a_1,b_1]\times[a_2,b_2]\subset[0,\pi]^2\setminus\{(0,0)\}$. On the other hand, since $\sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} b_{j,k}/jk=\infty$, we infer from part (iii) of Theorem 8.1 that the function $(x,y)\mapsto\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ cannot be Henstock-Kurzweil integrable on $[0,\pi]^2$. However, this does not contradict Theorem 6.3 because $\sup\limits_{n\in\mathbb{N}}\sup\limits_{x\in[0,\pi]}\Big|\sum\limits_{k=1}^{n}\int_0^x\sin kt\,\mathrm{d}t\Big|=\infty$.

The following corollary refines [18, Theorem 5.2].

Corollary 8.3. *Let $\{b_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^m}$ be a multiple sequence of real numbers such that $\lim\limits_{\|\boldsymbol{n}\|\to\infty} b_{\boldsymbol{n}}=0$ and $\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m}|\Delta_{\{1,\dots,m\}}(b_{\boldsymbol{k}})|(\ln(\|\boldsymbol{k}\|+1))^{m-1}$ converges. Then the multiple series $\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m}\Big(b_{\boldsymbol{k}}/\prod\limits_{i=1}^{m}k_i\Big)$ converges regularly if and only if the improper Lebesgue integral*

$$\lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0}\\ \boldsymbol{\delta}\in[0,\pi]^m}}\int_{\prod_{i=1}^{m}[\delta_i,\pi]}\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod_{i=1}^{m}\sin k_ix_i\,\mathrm{d}\mu_m(\boldsymbol{x}) \tag{59}$$

exists.

P r o o f. For each $\boldsymbol{x}\in[0,\pi)^m\setminus\{\boldsymbol{0}\}$, we apply Theorem 6.2 with $c_{\boldsymbol{n}}=b_{\boldsymbol{n}}$, $(\mathbb{B},\|\cdot\|)=\Big(L^1\Big(\prod\limits_{i=1}^{m}[x_i,\pi]\Big),\|\cdot\|_{L^1\left(\prod_{i=1}^{m}[x_i,\pi]\right)}\Big)$, $x_{\boldsymbol{k}}=\prod\limits_{i=1}^{m}\varphi_{i,k_i}$ and $\varphi_{i,k}(t)=\sin kt$ $(k\in\mathbb{N},\ i=1,\dots,m)$, to conclude that the function $\boldsymbol{t}\mapsto\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod\limits_{i=1}^{m}\sin k_it_i$ belongs to $L^1\Big(\prod\limits_{i=1}^{m}[x_i,\pi]\Big)$ and

$$\int_{\prod_{i=1}^{m}[x_i,\pi]}\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod_{i=1}^{m}\sin k_it_i\,\mathrm{d}\mu_m(\boldsymbol{t})=\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod_{i=1}^{m}\int_{x_i}^{\pi}\sin k_it_i\,\mathrm{d}t_i. \tag{60}$$

Since our assumptions and Theorem 6.2 imply that (52) and (53) hold, the corollary follows from Theorems 8.1 and 2.8. □

Let χ_X denote the characteristic function of a set $X \subseteq \mathbb{R}$. The following example shows that Theorem 8.1 is a proper generalization of Corollary 8.3.

Example 8.4 (cf. [24, Example 2.6]). Let $m = 1$, let $b_1 = 0$ and let

$$b_k = \sum_{j=k}^{\infty} (-1)^{\ln j / \ln 2} \frac{\chi_{\{2^r \colon r \in \mathbb{N}\}}(j)}{(\ln j)^{3/2}} \quad (k \in \mathbb{N} \setminus \{1\}).$$

Using Theorem 8.1 with $m = 2$, we see that the function

$$(x, y) \mapsto \sum_{(j,k) \in \mathbb{N}^2} b_j b_k \sin jx \sin ky$$

is Henstock-Kurzweil integrable on $[0, \pi]^2$. In view of Corollary 8.3, it remains to check that $\sum_{k=1}^{\infty} b_k \sin kx$ is not a Fourier-Lebesgue series. But this assertion follows from [24, Example 2.6].

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