Pratulananda Das; Debraj Chandra

Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 54 (2013), No. 1, 83--96

Persistent URL: http://dml.cz/dmlcz/143154

# Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence

Pratulananda Das, Debraj Chandra

*Abstract.* In this paper we extend the notion of quasinormal convergence via ideals and consider the notion of $\mathcal{I}$-quasinormal convergence. We then introduce the notion of $\mathcal{I}QN$ ($\mathcal{I}wQN$) space as a topological space in which every sequence of continuous real valued functions pointwise converging to 0, is also $\mathcal{I}$-quasinormally convergent to 0 (has a subsequence which is $\mathcal{I}$-quasinormally convergent to 0) and make certain observations on those spaces.

*Keywords:* ideal, filter, $\mathcal{I}$-quasinormal convergence, Chain Condition, $AP$-ideal, $\mathcal{I}QN$ space, $\mathcal{I}wQN$ space

*Classification:* Primary 54G99; Secondary 54C30, 40G15

## 1. Introduction

We start by recalling the definition of asymptotic density as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K_n$ denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of the set $K_n$. The asymptotic density of the subset $K$ is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$$

provided the limit exists.

Using this idea of asymptotic density, the notion of convergence of a real sequence had been extended to statistical convergence by Fast [19] (see also [31]) as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space $(X, \rho)$ is said to be statistically convergent to $\ell$ if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : d(x_k, \ell) \geq \varepsilon\}$ has asymptotic density zero. A lot of investigations have been done on this very interesting convergence and its topological consequences after the initial works by Fridy [20] and Šalat [30].

On the other hand, in [24] an interesting generalization of the notion of statistical convergence was proposed. Namely it is easy to check that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible (or free) ideal of $\mathbb{N}$.

The research of the second author was done when the author was a junior research fellow of the Council of Scientific and Industrial Research, HRDG, India. The first author is also thankful to CSIR for granting the project No. 25(0186)/10/EMR-II during the tenure of which this work was done.

A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set $Y$ is said to be an ideal in $Y$ if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$. Here we consider an ideal of $\mathbb{N}$ and without any loss of generality we also assume that $\bigcup_{A \in \mathcal{I}} A = \mathbb{N}$ which implies that $\{k\} \in \mathcal{I}$ for each $k \in \mathbb{N}$. Such ideals were sometimes called admissible ideals in the literature [24], [26], [14] (which are also called free ideals). If $\mathcal{I}$ is a proper ideal in $Y$ (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in $Y$. It is called the filter associated with the ideal $\mathcal{I}$. Thus one may consider an arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$ and define $\mathcal{I}$-convergence of a sequence by replacing the sets of density zero by the members of the ideal. Following the general line of [24] (see also [26]), ideals were used to study nets in topological and uniform spaces ([27], [12], [13]), to study certain variants of open covers and selection principles [16], [10], to study convergence of sequences of functions and its applications to measure theory ([2], [25], [28]).

The notion of quasinormal convergence was introduced by Bukovská in [3], [4] though it should be mentioned that Császár and Laczkovich [8] defined the same notion with the name ‘equal convergence’ in 1975 and again studied it in [9]. Bukovský, Reclaw and Repický introduced the notions of $QN$ and $wQN$ spaces in [5] as topological spaces not distinguishing pointwise and quasinormal convergence of real functions and established many fundamental and interesting properties of these spaces in [5], [6] and recently more work was done on these spaces relating them with certain covering properties by Bukovský and Hales [7]. A brief history of studies of spaces not distinguishing between two types of convergences and many important references can be found in the two beautifully written papers [5] and [6].

As a natural consequence we try to unify both these lines of investigations and first extend the notion of quasinormal convergence via ideals to $\mathcal{I}$-quasinormal convergence. Then we introduce the main notions of $\mathcal{I}QN$ ($\mathcal{I}wQN$) spaces as topological spaces in which every sequence of continuous real valued functions pointwise converging to 0, is $\mathcal{I}$-quasinormally convergent to 0 (has a subsequence which is $\mathcal{I}$-quasinormally convergent to 0). We make certain observations of these spaces basically following the line of investigation of [5].

## 2. Basic definitions and properties

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $\mathcal{I}$ will stand for a non-trivial proper admissible ideal of $\mathbb{N}$.

Recall that the usual definition of convergence of a sequence was extended in two ways by using an ideal in [24] as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \dots < m_k < \dots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$.

An ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an $AP$-ideal (or said to satisfy the property (AP) [24]) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of $\mathcal{I}$ there is a sequence

$\{B_1, B_2, \dots\}$ of sets such that $A_i \Delta B_i \ \ (i = 1, 2, \dots)$ is finite and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$. These types of ideals have also been called $P$-ideals (see [2], [25], [28], [14]). The ideal $\mathcal{I}_{fin}$ of all finite subsets of $\mathbb{N}$ as well as the ideal $\mathcal{I}_d$ are simple examples of $AP$-ideals. Other examples of $AP$-ideals can be seen from [25], [28]. Also a very useful fact is that the notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of real sequences coincide if and only if the ideal $\mathcal{I}$ is an $AP$-ideal (see [24], [26] and for more applications [11]).

The following property of an ideal will play a very important role in many results of this paper.

We say that a subset $\mathcal{B}$ of an ideal $\mathcal{I}$ is a "basis" if every element of $\mathcal{I}$ is a subset of some element of $\mathcal{B}$. We say that $\mathcal{I}$ satisfies the "Chain Condition" if there exists a sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathcal{I}$ with $C_1 \subset C_2 \subset C_3 \subset \dots$ such that for any $A \in \mathcal{I}$ there exists $k \in \mathbb{N}$ such that $A \subset C_k$. Therefore an ideal satisfies the Chain Condition if and only if it possesses a countable basis. Note that the ideal $\mathcal{I}_{fin}$ clearly satisfies the Chain Condition. Another non-trivial example of an ideal with Chain Condition is the following. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} A_j$ be a decomposition of $\mathbb{N}$ such that each $A_j$ is infinite and $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $\mathcal{I}_0$ denote the class of all $A \subset \mathbb{N}$ which intersect at most a finite number of $A_j$'s. Then $\mathcal{I}_0$ is a non-trivial ideal satisfying the Chain Condition. But this ideal is not an $AP$-ideal as can be seen from [24], [26] where it was established that any metric space (or topological space) with at least one limit point has a sequence which is $\mathcal{I}_0$-convergent but not $\mathcal{I}_0^*$-convergent.

Following [24] the usual ideas of pointwise and uniform convergence of a sequence of functions were extended via ideals first in [2] and then studied in ([2], [25], [28]) which we now recall. Let $X$ be a nonempty set and let $f_n,\ f$ be real valued functions defined on $X$. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions is said to be $\mathcal{I}$-pointwise convergent to $f$ if for each $x \in X$ and for each $\varepsilon > 0$ there exists an $A = A(x, \varepsilon) \in \mathcal{I}$ such that $n \in \mathbb{N} \setminus A$ implies $|f_n(x) - f(x)| < \varepsilon$ and in this case we write $f_n \xrightarrow{\mathcal{I}} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to $f$ if for any $\varepsilon > 0$ there exists $A = A(\varepsilon) \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus A$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$.

The important notion of quasinormal convergence (which was earlier introduced as equal convergence in [8]) was introduced in [3, 4] as follows. A function $f$ is said to be the quasinormal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there is a sequence of positive reals $\varepsilon_n \to 0$ such that for every $x \in X$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq n_0$.

We are now in a position to introduce our main definitions.

**Definition 2.1.** Let $X$ be a nonempty set and $f_n,\ f$ be real valued functions defined on $X$. We say that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-quasinormally convergent to $f$ on $X$ (written as $f_n \xrightarrow{\mathcal{I}QN} f$ on $X$) if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of nonnegative reals $\mathcal{I}$-converging to 0 such that for each $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$.

This convergence can also be called $\mathcal{I}$-equal convergence following the terminology of [8] which has been very recently used to study certain properties concerning $\mathcal{I}$-equal limits of real functions in [15].

**Definition 2.2.** A topological space $X$ is called an $\mathcal{I}QN$ space if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous real valued functions pointwise converging to zero on $X$, we have $f_n \xrightarrow{\mathcal{I}QN} 0$.

**Definition 2.3.** A topological space $X$ is called an $\mathcal{I}wQN$ space if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous real valued functions pointwise converging to zero on $X$, there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $f_{n_k} \xrightarrow{\mathcal{I}QN} 0$ on $X$.

**Definition 2.4.** A set $X \subset [0,1]$ is called an $\mathcal{I}QN$ set if $X$ with the subspace topology induced from the usual topology is an $\mathcal{I}QN$ space.

**Definition 2.5.** A set $X \subset [0,1]$ is called an $\mathcal{I}wQN$ set if $X$ with the subspace topology is an $\mathcal{I}wQN$ space.

We start with two results providing some necessary and sufficient (Theorem 2.1) and sufficient (Theorem 2.2) conditions for $\mathcal{I}$-quasinormal convergence which will play important roles throughout the paper.

**Theorem 2.1.** *Let $\mathcal{I}$ be an ideal satisfying the Chain Condition. Let $f$, $f_n$, $n = 1,2,3,\dots$ be real valued functions defined on a set $X$. The following conditions are equivalent.*

(i) $f_n \xrightarrow{\mathcal{I}QN} f$ *on* $X$.

(ii) *There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on $X_k$ for every $k = 1,2,3,\dots$ .*

(iii) *There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset X_3 \dots$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on $X_k$ for every $k = 1,2,3,\dots$ .*

*If $X$ is a topological space and $f_n$, $n = 1,2,3,\dots$ are continuous, then* (i), (ii), (iii) *are equivalent to:*

(iv) *There are closed sets $X_k \subset X$, $k = 1,2,3,\dots$, $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset X_3 \dots$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on $X_k$ for every $k = 1,2,3,\dots$ .*

Proof: (i)$\Rightarrow$(iii) Assume (i), i.e. $f_n \xrightarrow{\mathcal{I}QN} f$. Then there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and for every $x \in X$ there is a set $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in \mathbb{N} \setminus A_x$. Since $\mathcal{I}$ satisfies the Chain Condition, there exists a sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$ with $C_1 \subset C_2 \subset C_3 \subset \dots$ such that for every $A \in \mathcal{I}$ there exists some $C_k \in \mathcal{I}$ with $A \subset C_k$. Now define $X_k = \{x \in X : |f_n(x) - f(x)| < \varepsilon_n \text{ for all } n \in \mathbb{N} \setminus C_k\}$, $k \in \mathbb{N}$. Then clearly $X_1 \subset X_2 \subset X_3 \subset \dots$ . Further observe that for any $x \in X$, if $A_x \in \mathcal{I}$ is the set witnessing $\mathcal{I}$-quasinormal convergence as defined above, then $A_x \subset C_k$ for some $k \in \mathbb{N}$. Consequently $x \in X_k$. Hence $X = \bigcup_{k\in\mathbb{N}} X_k$. It is now easy to observe that $f_n \xrightarrow{\mathcal{I}-u} f$ on $X_k$. Indeed, take $\varepsilon > 0$. Let $B = \{n \in \mathbb{N} : \varepsilon_n \geq \varepsilon\}$. Then $B \in \mathcal{I}$, since $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$. If $x \in X_k$, then $|f_n(x) - f(x)| < \varepsilon$ for $n \in (\mathbb{N} \setminus C_k) \cap (\mathbb{N} \setminus B) = \mathbb{N} \setminus (C_k \cup B)$ and $C_k \cup B \in \mathcal{I}$. This proves (iii).

(ii)$\Rightarrow$(i) Now assume (ii), i.e. suppose that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $|f_n(x) - f(x)| \leq \varepsilon_{in}$ for all $x \in X_i$ when $n \notin M(i) \in \mathcal{I}$, where $\{\varepsilon_{in}\}_{n\in\mathbb{N}}$ is a sequence of positive reals depending on $i$ such that $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_{in} = 0$ for a fixed $i$. We can select sets $M_k \in \mathcal{I}$ such that $M_1 \subset M_2 \subset \cdots \subset M_k \subset \ldots$ and $\varepsilon_{kn} < \frac{1}{k}$ whenever $n \notin M_k$, for $k = 1, 2, 3, \ldots$ . Define

$$\begin{aligned} \varepsilon_n &= 1 \quad \text{if } \ n \in M_2 \\ &= \frac{1}{k} \quad \text{if } \ n \in M_{k+1} \setminus M_k \\ &= 0 \quad \text{if } \ n \notin \bigcup_{k\in\mathbb{N}} M_k. \end{aligned}$$

Then $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and furthermore $|f_n(x) - f(x)| \leq \varepsilon_{in} < \varepsilon_n$ for $x \in X_i$ and if $n \notin M(i) \cup M_i \in \mathcal{I}$ which shows that $f_n \xrightarrow{\mathcal{I}QN} f$. So (i) follows. Since (iii)$\Rightarrow$(ii), so it now follows that (i), (ii) and (iii) are equivalent.

Now let $X$ be a topological space and $f_n$, $n = 1, 2, 3, \ldots$ be continuous. Evidently (iv) implies (iii). Assume (i). Let us define $X_k = \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon_n + \varepsilon_m$ for all $m, n \in \mathbb{N} \setminus C_k\}$, $k \in \mathbb{N}$. Suppose as before $\mathcal{I}$ satisfies the Chain Condition with the sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$. Clearly $X_k$ is closed for $k = 1, 2, 3, \ldots$ as $f_n$'s are continuous functions and $X_1 \subset X_2 \subset X_3 \subset \ldots$ . If $x \in X$ then from the proof of (i)$\Rightarrow$(iii), it readily follows that $x \in X_k$ for some $k \in \mathbb{N}$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on each $X_k$. So (iv) is proved. Hence (i), (ii) and (iii) are equivalent to (iv). □

**Remark 2.1.** The first part of the above theorem can be further generalized in the following manner: Let $X$ be a topological space and $f_n$, $n = 1, 2, 3, \ldots$ be real valued continuous functions defined on $X$ such that $f_n \xrightarrow{\mathcal{I}QN} f$ on $X$ to some real valued function $f$ defined on $X$. If the ideal $\mathcal{I}$ has a basis of cardinality $\kappa$, then there exists a family of sets $\mathcal{K}$ such that $|\mathcal{K}| = \kappa$, $X = \bigcup \mathcal{K}$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on every $K \in \mathcal{K}$.

**Note 2.1.** Note that we require the additional hypothesis on the ideal to prove the necessity part but we do not require any additional assumption for the sufficiency part.

**Corollary 2.1.** *Let $X = \bigcup_{k\in\mathbb{N}} X_k$. If $f_n \xrightarrow{\mathcal{I}QN} f$ on each $X_k$, $k = 1, 2, 3, \ldots$, then $f_n \xrightarrow{\mathcal{I}QN} f$ on $X$.*

**Example 2.1.** This example shows that there exist functions $f$ and $f_n$, $n = 1, 2, 3, \ldots$ such that $f_n \xrightarrow{\mathcal{I}} f$ but $f_n \overset{\mathcal{I}QN}{\nrightarrow} f$. Let $\mathcal{I}$ be an admissible ideal satisfying the Chain Condition and $\mathcal{I} \neq \mathcal{I}_{fin}$. Let $C$ be an infinite member of $\mathcal{I}$. Let $\mathbb{Q} = \{r_k : k \in \mathbb{N} \cup \{0\}\}$ be a one to one enumeration of rational numbers. Let

$$\begin{aligned} f(x) = &\ 0 && \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \\ = &\ 2^{-k} && \text{if } \ x = r_k, \ k = 0, 1, 2, \dots . \end{aligned}$$

Clearly $f$ is not continuous on any interval. For every $n \in \mathbb{N} \setminus C$ choose a positive real $\delta_n \leq 2^{-n}$ such that $\delta_n \leq \frac{1}{2}|r_i - r_j|$, $i = 0, 1, 2, \dots, n$, $j = 0, 1, 2, \dots, n$, $i \neq j$. Let

$$\begin{aligned} f_n(x) = &\ 0 && \text{if } \ x \in \mathbb{R} \setminus \bigcup_{i=0}^{n} (r_i - \delta_i, r_i + \delta_i) \\ = &\ 2^{-i} && \text{for } \ x = r_i, \ i = 0, 1, 2, \dots, n \\ = &\ 2^{-i}(1 - \frac{|x - r_i|}{\delta_i}) && \text{for } \ x \in (r_i - \delta_i, r_i + \delta_i), \ i = 1, 2, 3, \dots, n. \end{aligned}$$

for $n \in \mathbb{N} \setminus C$ and $f_n = n$ for each $n \in C$.

Clearly $f_n \xrightarrow{\mathcal{I}} f$ (though $f_n$ does not converge to $f$ pointwise) on $\mathbb{R}$. But $f_n \overset{\mathcal{I}QN}{\nrightarrow} f$ on $\mathbb{R}$. Otherwise if $f_n \xrightarrow{\mathcal{I}QN} f$ on $\mathbb{R}$ then by Theorem 2.1, $\mathbb{R} = \bigcup_{k=0}^{\infty} E_k$ where $E_k$'s are closed and $f_n \xrightarrow{\mathcal{I}-u} f$ on every $E_k$ for $k = 0, 1, 2, \dots$ . By the Baire category theorem, there is $k$ such that $Int\ E_k \neq \emptyset$, i.e. there are $a, b$, $a < b$ such that $[a, b] \subseteq E_k$. Since each $f_n$ is continuous and $f_n \xrightarrow{\mathcal{I}-u} f$ on $[a, b]$, it follows that $f$ being the $\mathcal{I}$-uniform limit of continuous functions on $[a, b]$ is continuous on $[a, b]$ (see [2]), which is a contradiction.

**Example 2.2.** This example shows that there exist $f$, $f_n$, $n = 0, 1, 2, \dots$ such that $f_n \xrightarrow{\mathcal{I}QN} f$ but $f_n \overset{\mathcal{I}-u}{\nrightarrow} f$. Let $\mathcal{I}$ be any admissible ideal and $\mathcal{I} \neq \mathcal{I}_{fin}$. Let $C$ be any infinite member of $\mathcal{I}$. Take $f_n(x) = x^n$ if $n \notin C$ and $f_n(x) = n$ for all $x \in [0, 1]$ if $n \in C$. Let $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. Clearly $f_n \xrightarrow{\mathcal{I}QN} f$ on $[0, 1]$. As $f$ is not continuous, $f_n \overset{\mathcal{I}-u}{\nrightarrow} f$ on $[0, 1]$. Note that $\{f_n\}_{n \in \mathbb{N}}$ does not converge to $f$ quasinormally.

A quasiordering $\leq^*$ is defined on $\mathbb{N}^{\mathbb{N}}$ by eventual dominance:

$$f \leq^* g \ \text{ if } \ f(n) \leq g(n) \ \text{ for all but finitely many } n.$$

We say that a subset $Y$ of $\mathbb{N}^{\mathbb{N}}$ is bounded if there exists $g$ in $\mathbb{N}^{\mathbb{N}}$ such that for each $f \in Y$, $f \leq^* g$. Otherwise we say that $Y$ is unbounded. Moreover, $\mathfrak{b}$ is defined as

$$\mathfrak{b} = \min\{|B| : B \ \text{ is an unbounded subset of } \ \mathbb{N}^{\mathbb{N}}\}.$$

It is known that $\aleph_0 < \mathfrak{b} \leq \mathfrak{c}$ but not necessarily $\mathfrak{b} = \aleph_1$ ([31], see also [4], [5]).

**Theorem 2.2.** *Let $\mathcal{I}$ be an AP-ideal. Let $X = \bigcup_{s\in S} X_s$, $|S| < \mathfrak{b}$. If the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{I}$-quasinormally to $f$ on every $X_s$, $s \in S$, then it does so on $X$.*

Proof: From hypothesis, for each $s \in S$, there is a sequence $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to zero and witnessing $\mathcal{I}$-quasinormal convergence on $X_s$. Since $\mathcal{I}$ is an $AP$-ideal, $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to zero. So we can actually take $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ to be a decreasing sequence of positive reals witnessing the $\mathcal{I}$-quasinormal convergence on $X_s$. Now let us define

$$h_s(k) = \min\{n \in \mathbb{N} : {\varepsilon_n}^s \le \frac{1}{k+1}\,,\ n > h_s(k-1)\}.$$

Since the family $\{h_s : s \in S\}$ is of power less than $\mathfrak{b}$, there exists a function $g : \mathbb{N} \to \mathbb{N}$ with the above described condition. Moreover, we can assume that $g$ is strictly increasing. Define

$$\begin{aligned} \varepsilon_n = {} & 1 && \text{if } n < g(1), \\ = {} & \frac{1}{k+1} && \text{if } g(k) \le n < g(k+1). \end{aligned}$$

If $x \in X$, then $x \in X_s$ for some $s \in S$. Since $f_n \xrightarrow{\mathcal{I}QN} f$ on $X_s$ we have $A = \{n \in \mathbb{N} : |f_n(x) - f(x)| \ge {\varepsilon_n}^s\} \in \mathcal{I}$. Consequently $\mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$ and $n \in \mathbb{N} \setminus A$ implies $|f_n(x) - f(x)| < {\varepsilon_n}^s$. Also there is a natural number $k$ such that $h_s(n) \le g(n)$ for $n \ge k$. Since we have already observed that $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to zero, so there exists a set $B_s \in \mathcal{F}(\mathcal{I})$ such that $\{{\varepsilon_n}^s\}_{n\in B_s}$ converges to zero. Hence if $n \in (\mathbb{N}\setminus A)\cap B_s$ and $n \ge g(k)$ then $g(l) \le n < g(l+1)$ for some $l \ge k$. Since $g(l) \ge h_s(l)$, we have $|f_n(x) - f(x)| < {\varepsilon_n}^s \le \frac{1}{l+1} \le \varepsilon_n$ and this proves the theorem. □

**Lemma 2.1.** *Continuous image of an $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*

The proof is omitted.

**Lemma 2.2.** *Continuous image of an $\mathcal{I}wQN$ space is an $\mathcal{I}wQN$ space.*

The proof is omitted.

**Lemma 2.3.** *Every countable space (more generally a space of cardinality less than $\mathfrak{b}$) is an $\mathcal{I}QN$ space (provided $\mathcal{I}$ is an AP-ideal).*

Proof: Let $X$ be countable and let $X = \{a_k : k \in \mathbb{N}\}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on $X$ pointwise converging to zero. Write $X = \bigcup_{k=1}^{\infty} X_k$, $X_k = \{a_k\}$ for $k = 1, 2, 3, \ldots$ . Each $X_k$ is closed and $f_n \xrightarrow{\mathcal{I}-u} 0$ on each $X_k$ as $X_k$ is a singleton. Hence by Theorem 2.1, $f_n \xrightarrow{\mathcal{I}QN} 0$ on $X$ and so $X$ is an $\mathcal{I}QN$ space.

If $X$ is of cardinality less than $\mathfrak{b}$, say $X = \{a_s : s \in S\}$, where $|S| < \mathfrak{b}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on $X$ pointwise converging to zero. Write $X = \bigcup_{s\in S} X_s$, where $X_s = \{a_s\}$ for $s \in S$. Now

$f_n \xrightarrow{\mathcal{I}-u} 0$ on each $X_s$ and hence by Theorem 2.1 and Theorem 2.2, $f_n \xrightarrow{\mathcal{I}QN} 0$ on $X$. Hence $X$ is an $\mathcal{I}QN$ space. □

Let $X$ be a perfectly normal topological space. We define

**Definition 2.6.** Let non($\mathcal{I}QN$ space) be the minimal cardinality of a perfectly normal space which is not an $\mathcal{I}QN$ space.

**Definition 2.7.** Let non($\mathcal{I}QN$ set) be the minimal cardinality of a subspace of $[0,1]$ which is not an $\mathcal{I}QN$ set.

**Definition 2.8.** Let add($\mathcal{I}QN$ space) be the minimal cardinal number $\alpha$ such that there is a perfectly normal non-$\mathcal{I}QN$ space (i.e. a perfectly normal space which is not an $\mathcal{I}QN$ space) which can be expressed as the union $X = \bigcup_{\xi<\alpha} X_\xi$, where $X_\xi$'s are $\mathcal{I}QN$ spaces.

**Definition 2.9.** Let add($\mathcal{I}QN$ set) be the minimal cardinal number $\alpha$ such that there is a perfectly normal non-$\mathcal{I}QN$ set which can be expressed as the union $X = \bigcup_{\xi<\alpha} X_\xi$, where $X_\xi$'s are $\mathcal{I}QN$ sets.

**Theorem 2.3.** *We have that*

(i) *add($\mathcal{I}QN$ set) $\geq$ add($\mathcal{I}QN$ space) $\geq$ $\mathfrak{b}$, where the second inequality holds provided $\mathcal{I}$ is an AP-ideal;*
(ii) *add($\mathcal{I}QN$ set) $\leq$ non($\mathcal{I}QN$ set).*

Proof: (i) If $X$ is an $\mathcal{I}QN$ set then it is obviously an $\mathcal{I}QN$ space. Hence add($\mathcal{I}QN$ set)$\geq$ add($\mathcal{I}QN$ space). By Theorem 2.2, if $X = \bigcup_{s\in S} X_s$, $|S| < \mathfrak{b}$ and $X_s$ is an $\mathcal{I}QN$ space for each $s \in S$, then $X$ becomes an $\mathcal{I}QN$ space. So from definition of add($\mathcal{I}QN$ space) it follows that add($\mathcal{I}QN$ space) $\geq \mathfrak{b}$.

(ii) It follows directly from Definition 2.7 and Definition 2.9. □

## 3. Some further observations on $\mathcal{I}QN$ and $\mathcal{I}wQN$ spaces

**Theorem 3.1.** *Let $\mathcal{I}$ be an AP-ideal.*

(a) *A closed subset of a perfectly normal $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*
(b) *A closed subset of a perfectly normal $\mathcal{I}wQN$ space is an $\mathcal{I}wQN$ space.*
(c) *An $F_\sigma$ subset of a perfectly normal $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*

Proof: (a) Let $X$ be a perfectly normal $\mathcal{I}QN$ space and $A \subseteq X$ is closed. Let $f_n : A \to \mathbb{R}$ be a sequence of continuous functions and $f_n \to 0$ on $A$. Since $A$ is a closed subset of a perfectly normal space, there exist open sets $B_1 \supset B_2 \supset \ldots$ such that $\bigcap_{n=1}^\infty B_n = A$. For each $n \in \mathbb{N}$, let $h_n : X \to \mathbb{R}$ be continuous such that $h_n |_A = f_n$ and $h_n(x) = 0$ for all $x \in X \setminus B_n$. Then $h_n \to 0$ on $X$ and since $X$ is an $\mathcal{I}QN$ space so $h_n \xrightarrow{\mathcal{I}QN} 0$ on $X$. Thus there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \geq 0$ and $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that for each $x \in X$, the set $\{n : |h_n(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Thus for each $x \in A$, $\{n : |f_n(x)| \geq \varepsilon_n\} = \{n : |h_n(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Hence $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A$. Thus $A$ is an $\mathcal{I}QN$ space.

(b) The proof is similar to that of (a) and so is omitted.

(c) By Theorem 2.3(i), add($\mathcal{I}QN$ space) $\geq \mathfrak{b}$ and $\mathfrak{b} > \aleph_0$, so it is sufficient to prove the assertions for closed subsets and by $(a)$ the result holds. $\square$

**Remark 3.1.** In [5] it was proved that $\mathfrak{b} \geq$ add($wQN$set)$\geq$add($wQN$space)$\geq \mathfrak{h}$ (see [5, Theorem 3.3]) which was subsequently used to prove that an $F_\sigma$ subset of a perfectly normal $wQN$ space is a $wQN$ space (see [5, Theorem 4.1]). We could neither prove nor disprove a result similar to [5, Theorem 3.3] for $\mathcal{I}wQN$ spaces and so we leave it as an open problem. It is easy to observe that if a similar result can be established then Theorem 3.1(c) is also true for $\mathcal{I}wQN$ spaces.

**Theorem 3.2.** *Let $(X, \rho)$ be a separable metric space and let $A$ be a subset of $X$ without isolated points. If $A$ is an $\mathcal{I}wQN$ space then $A$ is meager in $X$, provided $\mathcal{I}$ satisfies the Chain Condition.*

Proof: Let $B = \{r_n : n \in \mathbb{N}\}$ be a countable dense subset of $\bar{A}$. For every $n \in \mathbb{N}$ choose a sequence $\{x_{n,m}\}_{m\in\mathbb{N}}$ from $A$ such that $x_{n,m} \to r_n$, $x_{n,m} \neq r_n$ for each $m \in \mathbb{N}$. Let $f_{n,m} : X \to [0, \frac{1}{2^{n-1}}]$ be a continuous function such that $f_{n,m}(x_{n,m}) = \frac{1}{2^{n-1}}$ and $f_{n,m}(x) = 0$ for all those $x \in X$ for which $\rho(x, x_{n,m}) \geq \frac{1}{2}\rho(r_n, x_{n,m})$. Let us define $h_m(x) = \sum_{n=1}^{\infty} f_{n,m}(x)$, $x \in X$, $m = 1, 2, 3, \dots$ . Then every $h_m$ is a continuous function from $X$ into $[0, 2]$ and $h_m \to 0$ on $X$.

Suppose on the contrary that $A$ is not meager in $X$ though $A$ is an $\mathcal{I}wQN$ space i.e. there exists a subsequence $\{h_{m_k}\}_{k\in\mathbb{N}}$ of the sequence $\{h_m\}_{m\in\mathbb{N}}$ converging $\mathcal{I}$-quasinormally to zero on $A$. By Theorem 2.1, there exist closed sets $A_l \subset X$, $l = 1, 2, 3, \dots$, $A \subset \bigcup_{l=1}^{\infty} A_l$ such that

$$h_{m_k} \xrightarrow{\mathcal{I}-u} 0 \ \text{ on } \ A \cap A_l, \ l = 1, 2, 3, \dots \ . \tag{1}$$

Moreover, we can assume that $A_l \subset \bar{A}$ (otherwise we can just replace $A_l$ by $A_l \cap \bar{A}$). Since $A$ is not meager, there exists a $p \in \mathbb{N}$ such that $Int(A_p) \neq \emptyset$. Since $B$ is dense in $\bar{A}$, there is some $r_n \in Int(A_p)$. Consequently

$$x_{n,m} \in Int(A_p) \ \text{ for all } \ m \geq m_1 \ \text{ for some } \ m_1 \in \mathbb{N}. \tag{2}$$

Thus whenever $m \notin C$ where $C = \{1, 2, \dots, m_1\} \in \mathcal{I}$, we have that $\sup\{h_m(x) : x \in A \cap A_p\} \geq h_m(x_{n,m}) \geq f_{n,m}(x_{n,m}) = \frac{1}{2^{n-1}}$. The first inequality follows from (2) and the second inequality follows from the definition of $h_m$. Now

$$\{m_k : \sup_{x\in A\cap A_p} h_{m_k}(x) \geq \frac{1}{2^{n-1}}\} = \mathbb{N} \setminus C \notin \mathcal{I}$$

as $C \in \mathcal{I}$. So $h_{m_k} \overset{\mathcal{I}-u}{\nrightarrow} 0$ on $A \cap A_p$ which is a contradiction to (1). This implies that $A$ is not an $\mathcal{I}wQN$ space. This completes the proof of the theorem. $\square$

**Corollary 3.1.** *If $A$ is an $\mathcal{I}wQN$ subspace of a separable metric space, then $A$ is perfectly meager. Especially, any $\mathcal{I}wQN$ set is perfectly meager, provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: If $P$ is a perfect set, then $P \cap A = A_0 \cup A_1$, where $A_0$ is countable and $A_1$ is dense in itself and closed in $A$. Since $A_0$ is countable, it is meager. Again since $A_1$ is a closed subset of the $\mathcal{I}wQN$ space $A$, hence by Theorem 3.1(b), $A_1$ is also an $\mathcal{I}wQN$ space. Observe that $A_1$ being dense in itself has no isolated points and hence by Theorem 3.2, $A_1$ is meager. Thus $P \cap A$ is the union of two meager sets and so it is meager. As $P \cap A$ is meager for any perfect set $P$, hence $A$ is perfectly meager.

Similarly we can prove the assertion for any $\mathcal{I}wQN$ set because $[0,1]$ with the subspace topology is a separable metric space. $\square$

**Corollary 3.2.** *If $A$ is an $\mathcal{I}wQN$ set, then for the Lebesgue measure $\nu$ on $[0,1]$, the inner measure $\nu_*(A)$ of $A$ is zero provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: Suppose that $\nu_*(A) > 0$. Then from regularity we can find a compact set $K$ such that $K \subset A$ and $0 < \nu_*(K) \leq \nu_*(A)$. The compact set $K$ contains a perfect subset $K_1$. The perfect set $K_1$ is not perfectly meager and hence it is not an $\mathcal{I}wQN$ set. This is a contradiction to the fact that $K_1$ is an $\mathcal{I}wQN$ set by Theorem 3.1(b). Hence $\nu_*(A) = 0$. $\square$

**Remark 3.2.** The above result is also true for every Radon measure on $[0,1]$ (by a Radon measure we mean a finite diffused regular Borel measure on $[0,1]$, see [22]).

**Corollary 3.3.** *If $X$ is an $\mathcal{I}wQN$ set then $X$ is zero dimensional provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

The proof is similar to the proof of Corollary 3.2 and so we omit it.

**Corollary 3.4.** *If $X$ is completely regular $\mathcal{I}wQN$ space, then $X$ has a basis consisting of clopen sets. Moreover, if $X$ is also perfectly normal then every open subset of $X$ can be expressed as countable union of clopen sets provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: Let $A$ be an open subset of $X$ and let $x \in A$. As $X$ is completely regular, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$, $f(y) = 1$ for $y \in X \setminus A$. Clearly $f(X) \subset [0,1]$ and as $f$ is continuous so by Lemma 2.2, $f(X)$ is an $\mathcal{I}wQN$ set. Then by Corollary 3.3, $f(X)$ is zero dimensional. Since $f(X)$ is Hausdorff, there exists a basic open set $U$ of $f(X)$ such that $0 \in U$ but $1 \notin U$. Also as $f(X)$ is zero dimensional, $U$ can be chosen as clopen in $f(X)$. Now $f^{-1}(U)$ is a clopen subset of $A$ (because $f(y) = 1$ for all $y \in X \setminus A$ and $1 \notin U$) and $x \in f^{-1}(U)$ (as $f(x) = 0$ and $0 \in U$). Thus for any $x \in X$ and for any open set $A \subset X$ containing $x$, there is a clopen subset of $A$ containing $x$. Hence $X$ has a basis consisting of clopen sets.

If $X$ is perfectly normal then $X$ is normal and every closed set in $X$ is a $G_\delta$ set in $X$. We know that in a normal space $Z$, we can always find a continuous function $g : Z \to [0,1]$ such that $g(x) = 0$ for $x \in A$ and $g(x) > 0$ for $x \notin A$ if and only if $A$ is a closed and $G_\delta$ set in $Z$. Let $A$ be open in $X$. Then $X \setminus A$ is closed and so is a $G_\delta$ set in $X$. Now by the above stated property of the normal space, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in X \setminus A$ and $f(x) < 1$ if $x \in A$. Let $c \in A$. Then $f(c) \in f(A)$ and so $f(c) < 1$ and consequently there exists a clopen set $U_c \subset f(X)$ such that $f(c) \in U_c$ and $1 \notin U_c$. Indeed, $f(X) \subset [0,1]$ is an $\mathcal{I}wQN$ space by Lemma 2.2, and so $f(X)$ is an $\mathcal{I}wQN$ set which implies that $f(X)$ is zero dimensional by Corollary 3.3. Now $f^{-1}(U_c)$ is a clopen subset of $A$ containing $c$ (because $1 \notin U_c$ and $f(x) = 1$ for $x \in X \setminus A$). Then we have $A = \bigcup_{c\in A} f^{-1}(U_c)$. Note that $X \setminus A = \bigcap_{n\in\mathbb{N}} G_n$ where $G_n$ is open for $n = 1, 2, 3, \ldots$ (since $X \setminus A$ being closed is also $G_\delta$). Clearly $A = \bigcup_{n\in\mathbb{N}}(X \setminus G_n)$ where $X \setminus G_n$ is open for $n = 1, 2, 3, \ldots$ . Hence we can choose countably many $f^{-1}(U_{c_n})$, $n = 1, 2, 3, \ldots$ such that $A = \bigcup_{n\in\mathbb{N}} f^{-1}(U_{c_n})$ and so $A$ is the union of a countably many clopen sets. □

Let $\alpha \leq \mathfrak{c}$ be a regular cardinal. We now consider the following definition.

**Definition 3.1** ([5])**.** A set $X \subset [0,1]$ is called an $\alpha$-Sierpiński set if $|X| \geq \alpha$ and for every zero Lebesgue measure set $A$, $|A \cap X| < \alpha$.

It is known that Martin axiom implies the existence of a $\mathfrak{c}$-Sierpiński set [5].

Though an Egoroff-like theorem was established in [27] for ideals, a notion of convergence weaker than $\mathcal{I}$-uniform convergence was used there. This result was called weak Egoroff's theorem and it was observed ([27, Theorem 3.1]) that for every analytic $AP$-ideal $\mathcal{I}$, weak Egoroff's theorem holds. Following the terminology of [27] we say that Egoroff's theorem holds for the ideal $\mathcal{I}$ if for any finite measure space $(X, \mathcal{S}, \nu)$ and for any real valued continuous functions $f$, $f_n$, $n = 1, 2, 3, \ldots$ defined almost everywhere on $X$ such that $f_n \xrightarrow{\mathcal{I}} f$ almost everywhere on $X$, for every $\varepsilon > 0$ there is a measurable set $H_\varepsilon$ such that $\nu(X \setminus H_\varepsilon) < \varepsilon$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on $H_\varepsilon$.

**Remark 3.3.** In [27] it was further established that Egoroff's theorem holds true for a non-pathological ideal $\mathcal{I}$ if and only if it is isomorphic to $\mathcal{I}_{fin}$ or $\varphi \times \mathcal{I}_{fin}$ ([27, Theorem 3.4]). It is still an open problem whether there exists a pathological analytic $AP$-ideal for which Egoroff's theorem holds ([27, Problem 1]). We establish the following result for an ideal for which Egoroff's theorem holds. We do not know whether the result can be proved for ideals for which weak Egoroff's theorem hold and leave it as an open problem.

**Theorem 3.3.** *If $X$ is $\mathfrak{b}$-Sierpiński set, then every subset is an $\mathcal{I}QN$ set, for an $AP$-ideal $\mathcal{I}$ for which Egoroff's theorem holds.*

Proof: As in [5, Theorem 4.7] let $A \subset X$ and $f_n : A \to \mathbb{R}$ be a continuous function for $n = 1, 2, 3, \ldots$ and $f_n \to 0$ on $A$. We can assume that all $f_n$ are defined and continuous on a $G_\delta$ set $G \supset A$. Let $C \subset G$ be the Borel set of those $x \in G$ for which $f_n(x) \to 0$. Evidently $A \subset C$.

Now from our assumption of Egoroff's theorem for $\mathcal{I}$ on the finite measure space $(C, \nu)$, where $\nu$ stands here for the Lebesgue measure on $C$, for every $n \in \mathbb{N}$ we can choose a measurable set $H_n \subset C$ such that $f_n \xrightarrow{\mathcal{I}-u} 0$ on $H_n$ and $\nu(C \setminus H_n) < \frac{1}{n}$. Define $H = \bigcup_{n\in\mathbb{N}} H_n$. Then $f_n \xrightarrow{\mathcal{I}QN} 0$ on $H$ by Corollary 2.1 and $\nu(C \setminus H) = \nu(\bigcap_{n\in\mathbb{N}} C \setminus H_n) \leq \frac{1}{n}$ for each $n \in \mathbb{N}$ and so $\nu(C\setminus H) = 0$. Since $|A\cap(C\setminus H)| < \mathfrak{b}$, we have $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A\cap(C\setminus H)$ by Theorem 2.2. Thus $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A\cap(C\setminus H)$ and also $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A \cap H$. Consequently $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A$ which implies that $A$ is an $\mathcal{I}QN$ space. Clearly $A \subset X \subset [0,1]$, i.e. $A$ is an $\mathcal{I}QN$ set. □

We have already proved that continuous image of an $\mathcal{I}QN$ space ($\mathcal{I}wQN$ space) is also an $\mathcal{I}QN$ space ($\mathcal{I}wQN$ space) in Lemma 2.1 and Lemma 2.2. Below we prove a related result.

**Theorem 3.4.** *Let $f : X \to Y$ be a mapping from an $\mathcal{I}QN$ space $X$ into a metric space $Y$. If $f$ is an $\mathcal{I}$-quasinormal limit of a sequence of continuous mappings, then $f(X) \subseteq Y$ is an $\mathcal{I}QN$ space, provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

PROOF: Let $f_n : X \to Y$ be continuous functions for $n = 1, 2, 3, \ldots$ and $f_n \xrightarrow{\mathcal{I}QN} f$ on $X$. Then by Theorem 2.1, there exist closed sets $X_k$, $k = 1, 2, 3, \ldots$, $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on $X_k$, $k = 1, 2, 3, \ldots$ . Now $f$ being the $\mathcal{I}$-uniform limit of a sequence of continuous functions on $X_k$ is continuous on each $X_k$, $k = 1, 2, 3, \ldots$ (see [2]). Since each $X_k$ is closed in $X$ which is an $\mathcal{I}QN$ space so $X_k$ is also an $\mathcal{I}QN$ space by Theorem 3.1 and also $f(X_k) \subset Y$ is an $\mathcal{I}QN$ space by Lemma 2.1. As $f(X) = \bigcup_{k\in\mathbb{N}} f(X_k)$, $f(X)$ is an $\mathcal{I}QN$ space by Theorem 2.3(i). □

**Concluding remarks.** This is only an introduction into what seems to be an interesting line of investigation when one replaces the finiteness in a definition by members of an ideal as was previously done in ([2], [10]–[16], [24]–[28]) and a lot of investigation has to be done to understand the behaviors of the new notions. In particular we would like to raise the following questions which seem very natural.

**Problem 3.1.** We proved almost all the results under some assumption on the ideal (either taking it as an $AP$-ideal or requiring it to satisfy the Chain Condition). Are they essential? Can the results be proved for any admissible ideal (or at least under weaker assumption)?

**Problem 3.2.** At least under certain suitable assumption, many properties and behavior of $QN$ spaces and $\mathcal{I}QN$ spaces ($wQN$ spaces and $\mathcal{I}wQN$ spaces) appear to be the same. Then is every $\mathcal{I}QN$ space actually a $QN$ space? And is a $\mathcal{I}wQN$ space a $wQN$ space? We could neither prove nor disprove it.

**Acknowledgment.** We are thankful to the learned referee for pointing out some mistakes and several valuable suggestions which improved the presentation of the paper.

## References


[1] Balcar B., Pelant J., Simon P., *The space of ultrafilters on* $\mathbb{N}$ *covered by nowhere dense sets*, Fund. Math. **110** (1980), 11–24.

[2] Balcerzak M., Dems K., Komisarski A., *Statistical convergence and ideal convergence for sequence of functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 715–729.

[3] Bukovská Z., *Thin sets in trigonometrical series and quasinormal convergence*, Math. Slovaca **40** (1990), 53–62.

[4] Bukovská Z., *Quasinormal convergence*, Math. Slovaca **41** (1991), no. 2, 137–146.

[5] Bukovský L., Reclaw I., Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology Appl. **41** (1991), no. 1–2, 25–40.

[6] Bukovský L., Reclaw I., Repický M., *Spaces not distinguishing convergence of real valued functions*, Topology Appl. **112** (2001), no. 1, 13–40.

[7] Bukovský L., Haleš J., *QN spaces, wQN spaces and covering properties*, Topology Appl. **154** (2007), no. 4, 848–858.

[8] Császár Á., Laczkovich M., *Discrete and equal convergence*, Studia Sci. Math. Hungar. **10** (1975), 463–472.

[9] Császár Á., Laczkovich M., *Some remarks on discrete Baire classes*, Acta. Math. Acad. Sci. Hungar. **33** (1979), 51–70.

[10] Chandra D., Das P., *Some further investigations of open covers and selection principles using ideals*, Topology Proc. **39** (2012), 281–291.

[11] Das P., Ghosal S.K., *Some further results on* $\mathcal{I}$*-Cauchy sequences and condition* $(AP)$, Comput. Math. Appl. **59** (2010), no. 8, 2597–2600.

[12] Das P., Ghosal S.K., *On* $\mathcal{I}$*-Cauchy nets and completeness*, Topology Appl. **157** (2010), no. 7, 1152–1156.

[13] Das P., Ghosal S.K., *When* $\mathcal{I}$*-Cauchy nets in complete uniform spaces are* $\mathcal{I}$*-convergent*, Topology Appl. **158** (2011), no. 13, 1529–1533.

[14] Das P., *Some further results on ideal convergence in topological spaces*, Topology Appl. **159** (2012), 2621–2625.

[15] Das P., Dutta S., *On some types of convergence of sequences of functions in ideal context*, Filomat **27** (2013), no. 1, 147–154.

[16] Das P., *Certain types of covers and selection principles using ideals*, Houston J. Math. **39** (2013), no. 2, 447–460.

[17] Denjoy A., *Leçons sur le calcul des coefficients d'une série trigonométrique,* $2^e$ *partie*, Paris, 1941.

[18] Di Maio G., Kočinac Lj.D.R., *Statistical convergence in topology*, Topology Appl. **156** (2008), 28–45.

[19] Fast H., *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[20] Fridy J.A., *On statistical convergence*, Analysis **5** (1985), 301–313.

[21] Gerlits J., Nagy Z., *Some properties of* $C(X)$*, I*, Topology Appl. **14** (1984), 145–155.

[22] Jacobs K., *Measure and Integral*, Academic Press, New York-London, 1978.

[23] Kostyrko P., Šalát T., Wilczyński W., $\mathcal{I}$*-convergence*, Real Anal. Exchange **26** (2000/2001), no. 2, 669–685.

[24] Komisarski A., *Pointwise* $\mathcal{I}$*-convergence and* $\mathcal{I}$*-convergence in measure of sequences of functions*, J. Math. Anal. Appl. **340** (2008), 770–779.

[25] Lahiri B.K., Das P., $\mathcal{I}$ *and* $\mathcal{I}^*$*-convergence in topological spaces*, Math. Bohemica **130** (2005), 153–160.

[26] Lahiri B.K., Das P., $\mathcal{I}$ *and* $\mathcal{I}^*$*-convergence of nets*, Real Anal. Exchange **33** (2008), no. 2, 431–442.

[27] Mrożek N., *Ideal version of Egoroff's theorem for analytic* $P$*-ideals*, J. Math. Anal. Appl. **349** (2009), 452–458.

[28] Papanastassiou N., *On a new type of convergence of sequences of functions*, Atti Sem. Mat. Fis. Univ. Modena **50** (2002), no. 2, 493–506.

[29] Šalát T., *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.
[30] Schoenberg I.J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.
[31] Van Douven E.K., *The integers and topology*, Handbook of Set-theoritic Topology, K. Kunen and J.E. Vaughan (eds.), North-Holland, Amsterdam, 1984.

Department of Mathematics, Jadavpur University, Jadavpur, Kol-32, West Bengal, India

*E-mail:* pratulananda@yahoo.co.in
debrajchandra1986@gmail.com

(*Received* May 10, 2012, *revised* November 21, 2012)