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Associative and Lie deformations of Poisson algebras

Elisabeth Remm

Abstract. Considering a Poisson algebra as a nonassociative algebra satisfying the Markl-Remm identity, we study deformations of Poisson algebras as deformations of this nonassociative algebra. We give a natural interpretation of deformations which preserve the underlying associative structure and of deformations which preserve the underlying Lie algebra and we compare the associated cohomologies with the Poisson cohomology parametrizing the general deformations of Poisson algebras.

1 Introduction

The Poisson bracket is a multiplication which naturally appears when studying deformations of associative commutative algebras. For instance the algebra $\mathcal{C}^\infty(\mathbb{R}^2)$ with its ordinary multiplication $\mu = \mu_0$ admits a formal deformation $\sum_0^\infty t^n \mu_n$ such that the skew-symmetric bracket $\{a, b\} = \mu_1(a, b) - \mu_1(b, a)$ is the classical Poisson bracket (recalled in Section 2). This deformation is connected to the star product and then to the theory of deformation quantization (see Section 1 of [10]). This naturally leads to study deformations of Poisson algebras. But a Poisson algebra is usually defined by two multiplications, an associative commutative one $a * b$ and a Lie bracket $\{a, b\}$ (also called Poisson bracket) which are linked by the Leibniz rule $\{a * b, c\} = a * \{b, c\} + \{a, c\} * b$. The deformations of Poisson algebras which are classically considered consist of those deforming the Lie bracket while the associative product remains unchanged. The first studied Poisson algebras were defined on associative algebras of functions whose product is undeformable. This explains why this type of deformations, that we call Lie deformations of Poisson algebras, were first studied. They are parametrized by the Poisson-Lichnerowicz cohomology. Here we want to give a general approach of deformations of Poisson algebras, that is, we make deformations where both products are deformed. We then use the presentation of Poisson algebras in [13] with a single nonassociative multiplication which capture all informations. Then we find the Lie deformations

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as a particular case of deformations of this single multiplication but also the associative deformations obtained by deforming the associative product and letting the Lie bracket unchanged. We call Poisson-Hochschild the cohomology parametrizing the associative deformations (see Section 4.2). We then describe the Poisson cohomology parametrizing the general deformations of Poisson algebras and study the interactions between Poisson, Poisson-Lichnerovicz and Poisson-Hochschild cohomologies.

2 Generalities on Poisson algebras

2.1 Definition

Let $\mathbb{K}$ be a field of characteristic 0. A $\mathbb{K}$-Poisson algebra is a $\mathbb{K}$-vector space $\mathcal{P}$ equipped with two bilinear products denoted by $x*y$ and $\{x,y\}$, with the following properties:

1. The couple $(\mathcal{P},*)$ is an associative commutative $\mathbb{K}$-algebra.
2. The couple $(\mathcal{P},\{\cdot,\cdot\})$ is a $\mathbb{K}$-Lie algebra.
3. The products $*$ and $\{\cdot,\cdot\}$ satisfy the Leibniz rule:

$$\{x*y,z\}=x*\{y,z\}+\{x,z\}*y$$

for any $x,y,z\in\mathcal{P}$.

The product $\{\cdot,\cdot\}$ is usually called Poisson bracket and the Leibniz identity means that the Poisson bracket acts as a derivation of the associative product.

Classical examples: Poisson structures on the polynomial algebra. The polynomial algebra $A_n=\mathbb{C}[x_1,\ldots,x_n]$ is provided with several Poisson algebra structures. These examples are well studied, see, for example, [2], [8], [20] for results on classifications, or [16] for the study of the Poisson-Lichnerowicz cohomology.

2.2 Non standard example: Poisson algebras defined by a contact structure

The first Poisson structures appeared in classical mechanics. In 1809 Siméon Denis Poisson introduced a bracket in the algebra of smooth functions on $\mathbb{R}^{2r}$:

$$\{f,g\}=\sum_{i=1}^{r}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right).$$

This classical example has a natural generalization in symplectic geometry ([3]): Let (M^{2p},θ) be a symplectic manifold. For any Pfaffian form α on M^{2p}, we will denote by X_α the vector field defined by $\alpha=i(X_\alpha)\theta$, where $i(X)$ is the interior product by X: $(i(X)\theta)(Y)=\theta(X,Y)$. The Poisson bracket of two Pfaffian forms α,β on M^{2p} is the Pfaffian form $\{\alpha,\beta\}=i([X_\alpha,X_\beta])\theta$. If $\mathcal{D}(M^{2p})$ denotes the associative commutative algebra of smooth functions on M^{2p}, we provide it with a Poisson algebra structure letting $\{f,g\}=-\theta(X_{df},X_{dg})$. This Poisson bracket satisfies $d(\{f,g\})=\{df,dg\}$.

We can also define a Poisson bracket in contact geometry ([5]). Let (M^{2p+1},α) be a contact manifold, that is, α is a Pfaffian form on the $(2p+1)$-dimensional

differential manifold M^{2p+1} satisfying $(\alpha \wedge (d\alpha)^p)(x) \neq 0$ for any $x \in M^{2p+1}$. There exists one and only one vector field Z_α on M^{2p+1}, called the Reeb vector field of α, such that $\alpha(Z_\alpha) = 1$ and $i(Z_\alpha)d\alpha = 0$ at any point of M^{2p+1}. Let $\mathcal{D}_\alpha(M^{2p+1})$ be the set of first integrals of Z_α, that is,

$$\mathcal{D}_\alpha(M^{2p+1}) = \{f \in \mathcal{D}(M^{2p+1}),\ Z_\alpha(f) = 0\}\,.$$

Since we have $Z_\alpha(f) = i(Z_\alpha)df = 0$, then df is invariant by Z_α.

Lemma 1. *$\mathcal{D}_\alpha(M^{2p+1})$ is a commutative associative subalgebra of $\mathcal{D}(M^{2p+1})$.*

Proof. This is a consequence of the classical formulae

$$Z_\alpha(f+g) = Z_\alpha(f) + Z_\alpha(g) \text{ and } Z_\alpha(fg) = \big(Z_\alpha(f)\big)g + f\big(Z_\alpha(g)\big)\,.$$

□

Lemma 2. *For any non zero Pfaffian form β on M^{2p+1} satisfying $\beta(Z_\alpha) = 0$, there exists a vector field X_β with $\beta(Y) = d\alpha(X_\beta, Y)$ for any vector field Y. Two vector fields X_β and X'_β with this property satisfy $i(X_\beta - X'_\beta)d\alpha = 0$.*

This means that X_β is uniquely defined up to a vector field belonging to the distribution given by the characteristic space of $d\alpha$,

$$A(d\alpha)_x = \{X_x \in T_xM^{2p+1}, i(X_x)d\alpha(x) = 0\}\,.$$

In any Darboux open set, the contact form writes as $\alpha = x_1dx_2 + \cdots + x_{2p-1}dx_{2p} + dx_{2p+1}$. The Reeb vector field is $Z_\alpha = \partial/\partial x_{2p+1}$ and the form β satisfying $\beta(Z_\alpha) = 0$ writes as $\beta = \sum_{i=1}^{2p} \beta_i dx_i$. Then we have

$$X_\beta = \sum_{i=1}^{p} (\beta_{2i}\partial/\partial x_{2i-1} - \beta_{2i-1}\partial/\partial x_{2i})\,.$$

For any $f \in \mathcal{D}_\alpha(M^{2p+1})$, we writes X_f for X_{df}.

Theorem 1. *The algebra $\mathcal{D}_\alpha(M^{2p+1})$ is a Poisson algebra.*

Proof. (see [5]). Let f_1, f_2 be in $\mathcal{D}_\alpha(M^{2p+1})$. Since we have

$$d\alpha(X_{f_1}, X_{f_2}) = d\alpha(X_{f_1} + U_1, X_{f_2} + U_2)$$

for any $U_1, U_2 \in A(d\alpha)$, the bracket

$$\{f_1, f_2\} = d\alpha(X_{f_1}, X_{f_2})$$

is well defined. It is a Poisson bracket. □

2.3 Poisson algebra viewed as nonassociative algebra

In [13], we prove that any Poisson structure on a $\mathbb{K}$-vector space is also given by a nonassociative product denoted by xy and satisfying the nonassociative identity

$$3A(x,y,z) = (xz)y + (yz)x - (yx)z - (zx)y\,, \tag{1}$$

where $A(x,y,z)$ is the associator $A(x,y,z) = (xy)z - x(yz)$. In fact, if $\mathcal{P}$ is a Poisson algebra with associative product $x * y$ and Poisson bracket $\{x,y\}$, then xy is given by $xy = \{x,y\} + x * y$. Conversely, the Poisson bracket and the associative product of $\mathcal{P}$ are the skew-symmetric part and the symmetric part of the product xy. Thus it is equivalent to present a Poisson algebra classically or by this nonassociative product.

If $\mathcal{P}$ is a Poisson algebra given by the nonassociative product (1), we denote by $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra on the same vector space $\mathcal{P}$ whose Lie bracket is $\{x,y\} = \frac{xy-yx}{2}$ and by $\mathcal{A}_{\mathcal{P}}$ the commutative associative algebra, on the same vector space, whose product is $x * y = \frac{xy+yx}{2}$.

In [7], we have studied algebraic properties of the nonassociative algebra $\mathcal{P}$. In particular we have proved that this algebra is flexible, power-associative, and admits a Pierce decomposition.

Remark 1. A class of Poisson algebras is already defined with a single noncommutative multiplication but starting with a Jordan algebra. In [19], a noncommutative Jordan algebra is viewed as a Jordan commutative algebra J with an additional skew-symmetric operator $[\cdot,\cdot] : J \times J \to J$ such that

$$[x^2, y] = 2[x,y] \cdot x\,.$$

This definition is equivalent to consider only one multiplication satisfying

$$(xy)x - x(yx) = (x^2y)x - x^2(yx).$$

A particular class of such algebras for which $A^{(+)}$ is associative corresponds to Poisson algebras.

2.4 Classification of complex Poisson algebras of dimension 2 and 3

If e is an idempotent of the associative algebra, then the Leibniz rule implies that it is in the center of the Lie algebra corresponding to the Poisson bracket. In fact if e satisfies $e * e = e$, thus $\{e * e, x\} = 2e * \{e,x\} = \{e,x\}$. But if y is a non zero vector with $e * y = \lambda y$, then

$$(e * e) * y = e * y = \lambda y = e * (e * y) = \lambda^2 y.$$

This gives $\lambda^2 = \lambda$, that is, $\lambda = 0$ or 1. Since we have $e * \{e,x\} = 2^{-1}\{e,x\}$, the vector $\{e,x\}$ is zero for any x and e is in the center of the Lie algebra corresponding to the Poisson bracket. This remark simplifies the determination of all possible Poisson brackets when the associative product is fixed. In the following, we give the associative and Lie products in a fixed basis $\{e_i\}$ and the null products or

the products which are deduced by commutativity or skew-symmetry are often not written.

Dimension 2

algebra	associative product	Lie product
$\mathcal{P}_1^2$	$e_1 * e_i = e_i\,,\ i = 1, 2$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_2^2$	$e_1 * e_i = e_i\,,\ i = 1, 2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_3^2$	$e_1 * e_1 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_4^2$	$e_1 * e_1 = e_1$	$\{e_i, e_j\} = 0$
$\mathcal{P}_5^2$	$e_i * e_j = 0$	$\{e_1, e_2\} = e_2$
$\mathcal{P}_6^2$	$e_i * e_j = 0$	$\{e_i, e_j\} = 0$

Dimension 3

$\mathcal{P}_1^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$ $e_2 * e_2 = e_2$ $e_3 * e_3 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_2^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$ $e_2 * e_2 = e_2$ $e_3 * e_3 = e_2 - e_1$	$\{e_i, e_j\} = 0$
$\mathcal{P}_3^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_4^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$ $e_3 * e_3 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_5^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$	$\{e_2, e_3\} = e_3$
$\mathcal{P}_6^3$	$e_1 * e_i = e_i\,,\ i = 1, 2, 3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_7^3$	$e_1 * e_i = e_i\,,\ i = 1, 2$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_8^3$	$e_1 * e_1 = e_1$	$\{e_2, e_3\} = e_3$
$\mathcal{P}_9^3$	$e_1 * e_1 = e_1$	$\{e_i, e_i\} = 0$
$\mathcal{P}_{10}^3$	$e_1 * e_i = e_i\,,\ i = 1, 2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{11}^3$	$e_1 * e_1 = e_1$ $e_2 * e_2 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{12}^3(b)$	$e_1 * e_1 = e_2$	$\{e_1, e_3\} = e_2 + be_3$
$\mathcal{P}_{13}^3$	$e_1 * e_1 = e_2$	$\{e_1, e_3\} = e_3$
$\mathcal{P}_{14}^3$	$e_1 * e_1 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{15}^3$	$e_1 * e_1 = e_2$ $e_1 * e_2 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{16}^3(a)$	$e_i * e_j = 0$	any Lie algebra

It is also possible to establish this classification in small dimension starting from the nonassociative product. We can use, for example, technics used in [4] where we classify all the complex 2-dimensional algebras (and in particular the Poisson algebras).

3 Deformations of Poisson algebras

In this section we recall briefly the classical notion of formal deformations of a $\mathbb{K}$-algebra. These deformations are parametrized by a cohomology, called deformation cohomology, which is often difficult to define globally and to compute explicitly. But using the operadic approach, we can sometimes obtain this cohomology using the associated operad: when the operad is Koszul, which is the case for the operad associated to Poisson algebras. When the operad is non Koszul the operadic and deformation cohomologies differ and the last one is even more complicated to describe see [14]. Using the Markl-Remm definition of a Poisson algebra, we describe the formal deformations. So in this section, we mean by Poisson algebra a $\mathbb{K}$-algebra defined by a nonassociative product satisfying Identity (1).

3.1 Formal deformations of a Poisson algebra

Let R be a complete local augmented ring such that the augmentation ε takes values in $\mathbb{K}$. If B is an R-Poisson algebra, we consider the $\mathbb{K}$-Poisson algebra $\overline{B} = \mathbb{K} \otimes_R B$ given by $\alpha(\beta \otimes b) = \alpha\beta \otimes b$, with $\alpha, \beta \in \mathbb{K}$ and $b \in B$. It is clear that $\overline{B}$ satisfies (1). An R-deformation of a $\mathbb{K}$-Poisson algebra A is an R-Poisson algebra B with a $\mathbb{K}$-algebra homomorphism

$$\varrho \colon \overline{B} \to A\,.$$

A formal deformation of A is an R-deformation with $R = \mathbb{K}[[t]]$, the local ring of formal series on $\mathbb{K}$. We assume also that B is an R-free module isomorphic to $R \otimes A$.

Let $\mathbb{K}[\Sigma_3]$ be the $\mathbb{K}$-group algebra of the symmetric group Σ_3. We denote by τ_{ij} the transposition exchanging i and j and by c the cycle $(1, 2, 3)$. Every $\sigma \in \Sigma_3$ defines a natural action on any $\mathbb{K}$-vector space W by:

$$\begin{array}{rccc} \Phi_\sigma \colon & W^{\otimes^3} & \longrightarrow & W^{\otimes^3} \\ & x_1 \otimes x_2 \otimes x_3 & \longrightarrow & x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}. \end{array}$$

We extend this action of Σ_3 to an action of the algebra $\mathbb{K}[\Sigma_3]$. If $v = \Sigma_i a_i \sigma_i \in \mathbb{K}[\Sigma_3]$, then

$$\Phi_v = \Sigma_i a_i \Phi_{\sigma_i}.$$

Consider v_P the vector of $\mathbb{K}[\Sigma_3]$

$$v_P = 3Id - \tau_{23} + \tau_{12} - c + c^2.$$

Let $\mathcal{P}$ be a Poisson algebra and μ_0 its (nonassociative) multiplication. Identity (1) writes as

$$(\mu_0 \circ_1 \mu_0) \circ \Phi_{v_P} - 3(\mu_0 \circ_2 \mu_0) = 0$$

where $\circ_1$ and $\circ_2$ are the $comp_i$ operations given by

$$(\mu \circ_1 \mu')(x,y,z) = \mu(\mu'(x,y),z)\,,$$
$$(\mu \circ_2 \mu')(x,y,z) = \mu(x,\mu'(y,z))$$

for any bilinear maps μ and μ'.

Theorem 2. *A formal deformation B of the $\mathbb{K}$-Poisson algebra A is given by a family of linear maps*

$$\{\mu_i : A \otimes A \to A,\ i \in \mathbb{N}\}$$

satisfying

(i) μ_0 *is the multiplication of* A,

(ii) (D_k): $\displaystyle\sum_{\substack{i+j=k,\\ i,j\geq 0}} (\mu_i \circ_1 \mu_j) \circ \Phi_{v_P} = 3 \sum_{\substack{i+j=k,\\ i,j\geq 0}} \mu_i \circ_2 \mu_j$ *for each* $k \geq 1$.

Proof. The multiplication in B is determined by its restriction to $A \otimes A$ ([1]). We expand $\mu(x,y)$ for x, y in A into the power series

$$\mu(x,y) = \mu_0(x,y) + t\mu_1(x,y) + t^2\mu_2(x,y) + \cdots + t^n\mu_n(x,y) + \cdots$$

then μ is a Poisson product if and only if the family $\{\mu_i\}$ satisfies condition (D_k) for each k. □

Remark 2. As R is a complete ring, this formal expansion is convergent. It is also the case if R is a valued local ring (see [6]).

Let $\mathbb{K} = \mathbb{C}$ or an algebraically closed field. If $\{e_1, \ldots, e_n\}$ is a fixed basis of $\mathbb{K}^n$, we denote by $\mathcal{P}_n$ the set of all Poisson algebra structures on $\mathbb{K}^n$, that is, the set of structure constants $\{\Gamma_{ij}^k\}$ given by $\mu(e_i, e_j) = \sum_{k=1}^n \Gamma_{ij}^k e_k$. Relation (1) is equivalent to

$$\sum_{l=1}^{n} \big(3\Gamma_{ij}^l\Gamma_{lk}^s - 3\Gamma_{il}^s\Gamma_{jk}^l - \Gamma_{ik}^l\Gamma_{lj}^s - \Gamma_{jk}^l\Gamma_{li}^s + \Gamma_{ji}^l\Gamma_{lk}^s + \Gamma_{ki}^l\Gamma_{lj}^s\big) = 0.$$

Thus $\mathcal{P}_n$ is an affine algebraic variety. If we replace $\mathcal{P}_n$ by a differential graded scheme, we call *Deformation Cohomology,* the cohomology of the tangent space of this scheme.

Remark 3. This cohomology of deformation is defined in same manner for any $\mathbb{K}$-algebra and more generally for any n-ary algebra. If we denote by $H_{\mathrm{def}}(A) = \bigoplus_{n\geq 0} H_{\mathrm{def}}^n(A)$ the deformation cohomology of the algebra A, then $H_{\mathrm{def}}^0(A) = \mathbb{K}$, $H_{\mathrm{def}}^1(A)$ is the space of outer derivations of A and the coboundary operator δ_{def}^1 corresponds to the operator of derivation, and the space of 2-cocycles is determined by the linearization of the identities defining A. Thus, in any case, the three first spaces of cohomology are easy to compute. But the determination of the spaces $H_{\mathrm{def}}^n(A)$ for $n \geq 3$ is usually not easy; we cannot deduce for example $H_{\mathrm{def}}^3(A)$ directly from the knowledge of $H_{\mathrm{def}}^2(A)$. However we have the following result:

Proposition 1. *Let $\mathcal{P}_A$ be the quadratic operad related to A. If $\mathcal{P}_A$ is a Koszul operad, then $H_{\text{def}}(A)$ coincides with the natural operadic cohomology.*

For example, if A is a Lie algebra or an associative algebra, the corresponding operads $\mathcal{L}ie$ and $\mathcal{A}ss$ are Koszul and $H_{\text{def}}(A)$ coincides with the operadic cohomology, that is, respectively, the Chevalley-Eilenberg cohomology and the Hochschild cohomology. Examples of determination of $H_{\text{def}}(A)$ in the non-Koszul cases can be found in [9], [17]. A theory of deformations on non-Koszul operads in presented in [14].

3.2 The operadic cohomology of a Poisson algebra

Let $\mathcal{P}oiss$ be the quadratic binary operad associated with Poisson algebras. Recall briefly its definition. Let $E = \mathbb{K}[\Sigma_2]$ be the $\mathbb{K}$-group algebra of the symmetric group on two elements. The basis of the free $\mathbb{K}$-module $\mathcal{F}(E)(n)$ consists of the "n-parenthesized products" of n variables $\{x_1, \ldots, x_n\}$. Let R be the $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$ generated by the vector

$$u = 3x_1(x_2x_3) - 3(x_1x_2)x_3 + (x_1x_3)x_2 + (x_2x_3)x_1 - (x_2x_1)x_3 - (x_3x_1)x_2\,.$$

Then $\mathcal{P}oiss$ is the binary quadratic operad with generators E and relations R. It is given by

$$\mathcal{P}oiss(n) = (\mathcal{F}(E)/\mathcal{R})(n) = \frac{\mathcal{F}(E)(n)}{\mathcal{R}(n)}$$

where $\mathcal{R}$ is the operadic ideal of $\mathcal{F}(E)$ generated by R satisfying $\mathcal{R}(1) = \mathcal{R}(2) = 0$, $\mathcal{R}(3) = R$. The dual operad $\mathcal{P}oiss^!$ is equal to $\mathcal{P}oiss$, that is, $\mathcal{P}oiss$ is self-dual. In [18] we defined, for a binary quadratic operad $\mathcal{E}$, an associated quadratic operad $\widetilde{\mathcal{E}}$ which gives a functor

$$\mathcal{E} \otimes \widetilde{\mathcal{E}} \to \mathcal{E}\,.$$

In the case $\mathcal{E} = \mathcal{P}oiss$, we have $\widetilde{\mathcal{E}} = \mathcal{P}oiss^! = \mathcal{P}oiss$. All these properties show that the operad $\mathcal{P}oiss$ is a Koszul operad (see also [12]). In this case the cohomology of deformation of $\mathcal{P}oiss$-algebras coincides with the natural operadic cohomology. An explicit presentation of the space of k-cochains is given in [15]:

$$\mathcal{C}^k(\mathcal{P},\mathcal{P}) = \mathcal{L}in(\mathcal{P}oiss(n)^! \otimes_{\Sigma_n} V^{\otimes n}, V) = End(\mathcal{P}^{\otimes k}, \mathcal{P})$$

where V is the underlying vector space (here $\mathbb{C}^n$). The cohomology associated with the complex $(\mathcal{C}^k(\mathcal{P},\mathcal{P}), \delta_P^k)_k$ where δ_P^k denotes the coboundary operator

$$\delta_P^k : \mathcal{C}^k(\mathcal{P},\mathcal{P}) \to \mathcal{C}^{k+1}(\mathcal{P},\mathcal{P})\,,$$

is denoted by $H_P^*(\mathcal{P},\mathcal{P})$. We will describe the coboundary operators δ_P^2 in Subsection 3.3 and δ_P^k in Section 5.

Consequence: The deformation cohomology of a Poisson algebra. If $\mathcal{P}$ is a Poisson algebra, then $H_{\text{def}}(\mathcal{P})$ is the operadic cohomology $H_P^*(\mathcal{P},\mathcal{P})$ or briefly $H_P^*(\mathcal{P})$.

3.3 Some relations on the coboundary operator δ_P^2

Let $\mathcal{P}$ be a Poisson algebra whose nonassociative product $\mu_0(X,Y)$ is denoted by $X\cdot Y$. Let $\mathfrak{g}_\mathcal{P}$ and $\mathcal{A}_\mathcal{P}$ be its corresponding Lie and associative algebras. We denote by $H_C^*(\mathfrak{g}_\mathcal{P},\mathfrak{g}_\mathcal{P})$ the Chevalley-Eilenberg cohomology of $\mathfrak{g}_\mathcal{P}$ and by $H_H^*(\mathcal{A}_\mathcal{P},\mathcal{A}_\mathcal{P})$ the Hochschild cohomology of $\mathcal{A}_\mathcal{P}$. A important part of this work devoted to describe the coboundary operator and its links with the Chevalley-Eilenberg and Hochschild coboundary operators. We focus in this section on the degree 2 because it is related to the parametrization of deformations. The condition (D_1) writes as

$$(\mu_0\circ_1\mu_1+\mu_1\circ_1\mu_0)\circ\Phi_{v_P}=3(\mu_0\circ_2\mu_1+\mu_1\circ_2\mu_0)\,,$$

that is,

$$\begin{aligned}&3\mu_1(\mu_0(x,y),z)-3\mu_1(x,\mu_0(y,z))-\mu_1(\mu_0(x,z),y)-\mu_1(\mu_0(y,z),x)\\&\quad+\mu_1(\mu_0(y,x),z)+\mu_1(\mu_0(z,x),y)+3\mu_0(\mu_1(x,y),z)-3\mu_0(x,\mu_1(y,z))\\&\quad-\mu_0(\mu_1(x,z),y)-\mu_0(\mu_1(y,z),x)+\mu_0(\mu_1(y,x),z)+\mu_0(\mu_1(z,x),y)=0\end{aligned}$$

for any $x,y,z\in\mathcal{P}$. If φ is a 2-cocycle of $H^2_{\mathrm{def}}(\mathcal{P})$, this implies

$$\delta_P^2\varphi=(\varphi\circ_1\mu+\mu\circ_1\varphi)\circ\Phi_{v_P}-3(\varphi\circ_2\mu+\mu\circ_2\varphi)\circ\Phi_{Id}.$$

Recall that $v_P=3Id-\tau_{23}+\tau_{12}-c+c^2$.

Let $\varphi\colon\mathcal{P}^{\otimes^2}\to\mathcal{P}$ be a bilinear map and μ be the nonassociative multiplication of the Poisson algebra $\mathcal{P}$. We denote by $\varphi_a=\frac{\varphi-\widetilde{\varphi}}{2}$ and $\varphi_s=\frac{\varphi+\widetilde{\varphi}}{2}$ the skew-symmetric and symmetric parts of φ with $\widetilde{\varphi}(X,Y)=\varphi(Y,X)$. We consider the following trilinear maps:

$$\mathcal{L}_C(\varphi)=\frac{1}{2}[\varphi\circ_1\mu\circ\Phi_{Id+c+c^2-\tau_{12}-\tau_{13}-\tau_{23}}+(\mu\circ_1\varphi-\mu\circ_2\varphi)\circ\Phi_{Id+c+c^2}]\,,$$

$$\mathcal{L}_H(\varphi)=\frac{1}{2}[\mu\circ_1\varphi\circ\Phi_{Id-c}-\mu\circ_2\varphi\circ\Phi_{Id-c^2}+\varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}}-\varphi\circ_2\mu\circ\Phi_{Id+\tau_{13}}].$$

If $\varphi=\varphi_a$, that is, if φ is skew-symmetric, then $\mathcal{L}_C(\varphi_a)=\delta^2_{C,\{\,,\}}\varphi_a$ where $\delta^2_{C,\{\cdot,\cdot\}}$ is the Chevalley-Eilenberg coboundary operator of the cohomology of the Lie algebra $\mathfrak{g}_\mathcal{P}$ associated with $\mathcal{P}$. Similarly if $\varphi=\varphi_s$, that is, if φ is symmetric, then $\mathcal{L}_H(\varphi_s)=\delta^2_{H,*}\varphi_s$ where $\delta^2_{H,*}$ is the Hochschild coboundary operator of the cohomology of the associative algebra $\mathcal{A}_\mathcal{P}$ associated with $\mathcal{P}$. Since no confusions are possible we will write δ^*_C and δ^*_H in place of $\delta^*_{C,\{\cdot,\cdot\}}$ $\delta^*_{H,*}$. Then for any bilinear map φ on $\mathcal{P}^{\otimes^2}$ with skew-symmetric part φ_a and symmetric part φ_s, we obtain

$$\begin{aligned}4\delta^2_C\varphi_a&=(\mu\circ_1\varphi+\varphi\circ_1\mu-\mu\circ_2\varphi+\varphi\circ_2\mu)\circ\Phi_V\\&\qquad\text{with}\quad V=Id-\tau_{12}-\tau_{13}-\tau_{23}+c+c^2\,,\\4\mathcal{L}_C(\varphi_s)&=(\mu\circ_1\varphi-\mu\circ_2\varphi)\circ\Phi_W+(\varphi\circ_1\mu+\varphi\circ_2\mu)\circ\Phi_V\\&\qquad\text{with}\quad W=Id+\tau_{12}+\tau_{13}+\tau_{23}+c+c^2\,,\\4\mathcal{L}_H(\varphi_a)&=\mu\circ_1\varphi\circ\Phi_{Id-\tau_{12}+\tau_{13}-c}+\mu\circ_2\varphi\circ\Phi_{-Id-\tau_{13}+\tau_{23}+c^2}\\&\quad+\varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}+\tau_{13}+c}+\varphi\circ_2\mu\circ\Phi_{-Id-\tau_{13}-\tau_{23}-c^2}\,,\\4\delta^2_H\varphi_s&=\mu\circ_1\varphi\circ\Phi_{Id+\tau_{12}-\tau_{13}-c}+\mu\circ_2\varphi\circ\Phi_{-Id+\tau_{13}-\tau_{23}+c^2}\\&\quad+\varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}-\tau_{13}-c}+\varphi\circ_2\mu\circ\Phi_{-Id+\tau_{13}-\tau_{23}+c^2}\,.\end{aligned}$$

At least we introduce the following operators, $\mathcal{L}_1$ which acts on the space of skew-symmetric bilinear maps and $\mathcal{L}_2$ which acts on the space of symmetric bilinear maps on $\mathcal{P}$:

$$\begin{aligned}
4\mathcal{L}_1(\varphi_a) &= \mu \circ_1 \varphi \circ \Phi_{\tau_{13}-\tau_{23}-c+c^2} + \mu \circ_2 \varphi \circ \Phi_{-Id-\tau_{12}+\tau_{23}+c} \\
&\quad + \varphi \circ_1 \mu \circ \Phi_{Id+\tau_{12}} + \varphi \circ_2 \mu \circ \Phi_{-\tau_{13}-c^2}, \\
4\mathcal{L}_2(\varphi_s) &= \mu \circ_1 \varphi \circ \Phi_{2Id+2\tau_{12}-\tau_{13}-\tau_{23}-c-c^2} + \mu \circ_2 \varphi \circ \Phi_{-Id+\tau_{12}-\tau_{13}+c} \\
&\quad + \varphi \circ_1 \mu \circ \Phi_{Id+\tau_{12}+2\tau_{13}-4c} + \varphi \circ_2 \mu \circ \Phi_{-4Id+\tau_{13}+2\tau_{23}+c^2}.
\end{aligned}$$

Lemma 3. *We have $\mathcal{L}_1(\varphi_a) = 0$ if and only if φ_a is a skew derivation of the associative product associated with μ, that is:*

$$\varphi_a(x * y, z) = x * \varphi_a(y, z) + \varphi_a(x, z) * y\,.$$

Proof.

$$\begin{aligned}
&\varphi_a(x * y, z) - x * \varphi_a(y, z) - \varphi_a(x, z) * y \\
&\quad = \tfrac{1}{2}\big(\varphi_a(xy + yx, z) - x\varphi_a(y, z) - \varphi_a(y, z)x - \varphi_a(x, z)y - y\varphi_a(x, z)\big) \\
&\quad = \tfrac{1}{2}\big(\varphi_a(xy, z) + \varphi_a(yx, z) + x\varphi_a(z, y) + \varphi_a(z, y)x + \varphi_a(z, x)y + y\varphi_a(z, x)\big) \\
&\quad = \tfrac{1}{4}\mathcal{L}_1(\varphi_a)(x, y, z).
\end{aligned}$$

□

Proposition 2. *For every bilinear map φ on $\mathcal{P}$, we have*

$$\delta_P^2\varphi = 2\big(\delta_C^2\varphi_a + \mathcal{L}_C(\varphi_s) + \delta_H^2\varphi_s + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)\big)\,. \tag{2}$$

Corollary 1. *Let φ be a bilinear map and φ_a and φ_s the skew-symmetric and the symmetric parts of φ. We have:*

$$12\delta_C^2\varphi_a = \delta_P^2\varphi \circ \Phi_{Id-\tau_{12}-\tau_{13}-\tau_{23}+c+c^2} \tag{3}$$

and

$$12\delta_H^2\varphi_s = \delta_P^2\varphi \circ \Phi_{Id-\tau_{13}+\tau_{23}-c^2}. \tag{4}$$

4 Particular deformations: Lie and associative deformations of a Poisson algebra

In this section we study two particular types of deformations. Usually, only Lie deformations of Poisson algebras are considered. This is a consequence of the classical problem of considering Poisson algebras on the associative commutative algebra of differential functions on a manifold. In this context, the associative algebra is preserved when we consider deformations of Poisson structures on this algebra, for example in problems of deformation quantization. Moreover, such an associative structure is rigid, so it is not appropriate to consider deformations of this multiplication. As consequence, the corresponding deformation cohomology is the Poisson-Lichnerowicz cohomology [11]. So the first particular type we consider, the Lie deformations, is when we deform the Poisson bracket and let the associative

product unchanged. We study a second special case which is non classical, the associative deformations. It consists in deformations of the associative product with a preserved Poisson bracket. Such deformations appear naturally when the Poisson bracket is a rigid Lie bracket. These deformations are parametrized by a cohomology defined by a subcomplex of the Poisson complex. We called it Poisson--Hochschild cohomology and describe it explicitely.

4.1 Lie deformations

Definition 1. We say that the formal deformation μ of the Poisson multiplication μ_0 is a Lie formal deformation if the corresponding commutative associative multiplication is conserved, that is, if

$$\mu_0(x,y)+\mu_0(y,x)=\mu(x,y)+\mu(y,x)$$

for any x, y.

As $\mu(x,y)=\mu_0(x,y)+\sum_{n\geq 1}t^n\mu_n(x,y)$, if μ is a Lie deformation of μ_0, then

$$\mu(x,y)+\mu(y,x)=\mu_0(x,y)+\mu_0(y,x)+\sum_{n\geq 1}t^n\big(\mu_n(x,y)+\mu_n(y,x)\big)\,.$$

So

$$\sum_{n\geq 1}t^n\big(\mu_n(x,y)+\mu_n(y,x)\big)=0$$

and

$$\mu_n(x,y)+\mu_n(y,x)=0$$

for any $n\geq 1$. Each bilinear maps μ_n is skew-symmetric. In particular μ_1 is skew-symmetric and $(\mu_1)_s=0$. As $\delta_P^2\mu_1=0$, Relation (2) writes as

$$\delta_C^2\mu_1+\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0\,.$$

But, from (3), $\delta_P^2\mu_1=0$ implies $\delta_C^2\mu_1=0$. Thus we have $\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0$. Since μ_1 is skew-symmetric:

$$\begin{aligned}\mathcal{L}_H(\mu_1)(x,y,z)&=\mu_1(x,y)*z-x*\mu_1(y,z)+\mu_1(x*y,z)-\mu_1(x,y*z)\\&=-\mu_1(x,y*z)+\mu_1(x,y)*z+y*\mu_1(x,z)+\mu_1(x*y,z)\\&\quad-x*\mu_1(y,z)-\mu_1(x,z)*y\\&=\mathcal{L}_1(\mu_1)(x,y,z)+\mathcal{L}_1(\mu_1)(y,z,x)\,.\end{aligned}$$

So

$$\mathcal{L}_H(\mu_1)=\mathcal{L}_1(\mu_1)\circ\Phi_{Id+c}\,.$$

We deduce that

$$\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=\mathcal{L}_1(\mu_1)\circ\Phi_{2Id+c}$$

and $\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0$ implies $\mathcal{L}_1(\mu_1)=0$.

Theorem 3. *If $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n \mu_n(x,y)$ is a Lie deformation of the Poisson product μ_0, then μ_1 is a skew-symmetric map satisfying*

$$\begin{cases} \delta_C^2 \mu_1 = 0, \\ \mathcal{L}_1(\mu_1) = 0. \end{cases}$$

Recall that Poisson-Lichnerowicz cohomology [11] is associated with the complex

$$(\mathcal{C}^*_{PL}(\mathcal{P},\mathcal{P}), \delta^*_C)$$

where the cochains are the skew-symmetric multilinear maps $\mathcal{P} \times \cdots \times \mathcal{P} \to \mathcal{P}$ satisfying the Leibniz rule in each of their arguments (such maps are called skew-symmetric multiderivations of the algebra $\mathcal{P}$). The coboundary operators coincide with the Chevalley-Eilenberg coboundary operator denoted by δ^*_C. Of course $\mathcal{C}^n_{PL}(\mathcal{P},\mathcal{P})$ is a vector subspace of $\mathcal{C}^n_P(\mathcal{P},\mathcal{P})$. The previous theorem shows that if φ is a 2-cochain of $\mathcal{C}^2_{PL}(\mathcal{P},\mathcal{P})$, thus its classes of cohomology in $H^2_{PL}(\mathcal{P},\mathcal{P})$ and $H^2_P(\mathcal{P},\mathcal{P})$ are equal.

4.2 Associative deformations of Poisson algebras

Definition 2. We say that the formal deformation μ of the Poisson multiplication μ_0 is an associative formal deformation if the corresponding Lie multiplication is conserved, that is, if

$$\mu_0(x,y) - \mu_0(y,x) = \mu(x,y) - \mu(y,x)$$

for any x, y.

As $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n \mu_n(x,y)$, if μ is an associative deformation of μ_0, then

$$\mu(x,y) - \mu(y,x) = \mu_0(x,y) - \mu_0(y,x) + \sum_{n\geq 1} t^n \big(\mu_n(x,y) - \mu_n(y,x)\big)\,.$$

Thus

$$\sum_{n\geq 1} t^n \big(\mu_n(x,y) - \mu_n(y,x)\big) = 0$$

and

$$\mu_n(x,y) - \mu_n(y,x) = 0$$

for any $n \geq 1$. Each bilinear maps μ_n is symmetric. In particular μ_1 is symmetric and $(\mu_1)_a = 0$. Since $\delta^2_P \mu_1 = 0$, Relation (2) writes as

$$\mathcal{L}_C(\mu_1) + \delta^2_H \mu_1 + \mathcal{L}_2(\mu_1) = 0\,.$$

But, from (4), $\delta^2_P \mu_1 = 0$ implies $\delta^2_H \mu_1 = 0$. Thus

$$\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1) = 0\,.$$

Lemma 4. *When φ is a symmetric map with $\delta_H^2\varphi = 0$,*

$$\begin{aligned}\mathcal{L}_C(\varphi)(x,y,z) &= \{\varphi(x,y),z\} + \{\varphi(y,z),x\} + \{\varphi(z,x),y\} \\ &\quad + \varphi(\{x,y\},z) + \varphi(\{y,z\},x) + \varphi(\{z,x\},y)\,, \\ \mathcal{L}_2(\varphi)(x,y,z) &= \{y,\varphi(x,z)\} - \{z,\varphi(x,y)\} + 3\varphi(x,\{z,y\})\,.\end{aligned}$$

This is a direct consequence of the definition of $\mathcal{L}_C(\varphi_s)$ and $\mathcal{L}_2(\varphi_s)$ when φ is a symmetric bilinear map, replacing $\mu_0(x,y) - \mu_0(y,x)$ by $2\{x,y\}$.

We deduce

$$\begin{aligned}\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big)(x,y,z) &= 2\{\mu_1(x,y),z\} + \{\mu_1(y,z),x\} + \mu_1(\{x,y\},z) \\ &\quad + \mu_1(\{z,x\},y) + 2\mu_1(\{z,y\},x) \\ &= 2\{\mu_1(x,y),z\} - 2\mu_1(\{y,z\},x) - 2\mu_1(\{x,z\},y) \\ &\quad + \{\mu_1(y,z),x\} - \mu_1(\{y,x\},z) - \mu_1(\{z,x\},y) \\ &= 2\Delta\mu_1(x,y,z) + \Delta\mu_1(y,z,x)\end{aligned}$$

with

$$\Delta\mu_1(x,y,z) = \{\mu_1(x,y),z\} - \mu_1(\{y,z\},x) - \mu_1(\{x,z\},y)\,.$$

We deduce that

$$\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big) = \Delta\mu_1 \circ \Phi_{2Id+c}\,.$$

But Φ_{2Id+c} is an invertible map on $\mathcal{P}^{\otimes^3}$. Then $\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big) = 0$ if and only if

$$\Delta\mu_1(x,y,z) = \{\mu_1(x,y),z\} - \mu_1\big(\{y,z\},x\big) - \mu_1\big(\{x,z\},y\big) = 0\,.$$

Definition 3. Let $\mathcal{P}$ be a Poisson algebra and let $\{x,y\}$ be its Poisson bracket. A bilinear map φ on $\mathcal{P}$ is called a Lie biderivation if

$$\{\varphi(x_1,x_2),x_3\} - \varphi\big(x_1,\{x_2,x_3\}\big) - \varphi\big(\{x_1,x_3\},x_2\big) = 0$$

for any $x_1, x_2, x_3 \in \mathcal{P}$.

We deduce that μ_1, which is a symmetric map, is a Lie biderivation.

Theorem 4. *If $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n\mu_n(x,y)$ is an associative deformation of the Poisson product μ_0, then μ_1 is a symmetric map such that*

1. $\delta_H^2\mu_1 = 0$.
2. *μ_1 is a Lie biderivation.*

In case of Lie deformation of the Poisson product μ_0, we have seen that the relations concerning μ_1 can be interpreted in terms of Poisson-Lichnerowicz cohomology. We propose a similar approach for the Lie deformations of μ_0.

Recall that $x * y$ the associative commutative product associated with the Poisson product μ_0, that is $x * y = \dfrac{\mu_0(x,y) + \mu_0(y,x)}{2}$.

Lemma 5. *Let φ be a symmetric bilinear map on $\mathcal{P}$ which is a Lie biderivation. If $\delta_H^2\varphi$ is the Hochschild coboundary operator, we have*

$$\begin{aligned}\delta_H^2\varphi(x_1,x_2,x_3) &= x_1 * \varphi(x_2,x_3) - \varphi(x_1 * x_2, x_3)\\ &\quad + \varphi(x_1, x_2 * x_3) - \varphi(x_1,x_2) * x_3\end{aligned}$$

and

$$\begin{aligned}\{\delta_H^2\varphi(x_1,x_2,x_3),x_4\} &= \delta_H^2\varphi(\{x_1,x_4\},x_2,x_3) + \delta_H^2\varphi(x_1,\{x_2,x_4\},x_3)\\ &\quad + \delta_H^2\varphi(x_1,x_2,\{x_3,x_4\})\end{aligned}$$

for any $x_1,x_2,x_3,x_4 \in \mathcal{P}$.

Proof. As φ is a Lie biderivation, we have

$$\{\varphi(x_1,x_2),x_3\} - \varphi(x_1,\{x_2,x_3\}) - \varphi(\{x_1,x_3\},x_2) = 0\,.$$

Thus, using the definition of $\delta_H^2\varphi$, we obtain

$$\begin{aligned}\{\delta_H^2\varphi(x_1,x_2,x_3),x_4\} &= \{x_1 * \varphi(x_2,x_3),x_4\} - \{\varphi(x_1 * x_2,x_3),x_4\}\\ &\quad + \{\varphi(x_1,x_2 * x_3),x_4\} - \{\varphi(x_1,x_2) * x_3,x_4\}\\ &= x_1 * \{\varphi(x_2,x_3),x_4\} - x_3 * \{\varphi(x_1,x_2),x_4\}\\ &\quad + \varphi(x_2,x_3) * \{x_1,x_4\} - \varphi(x_1,x_2) * \{x_3,x_4\}\\ &\quad - \{\varphi(x_1 * x_2,x_3),x_4\} + \{\varphi(x_1,x_2 * x_3),x_4\}\end{aligned}$$

As φ is a Lie biderivation,

$$\begin{aligned}\{\delta_H^2\varphi(x_1,x_2,x_3),x_4\} &= x_1 * \varphi(\{x_2,x_4\},x_3) + x_1 * \varphi(x_2,\{x_3,x_4\})\\ &\quad - x_3 * \varphi(\{x_1,x_4\},x_2) - x_3 * \varphi(x_1,\{x_2,x_4\})\\ &\quad + \varphi(x_2,x_3) * \{x_1,x_4\} - \varphi(x_1,x_2) * \{x_3,x_4\}\\ &\quad - \varphi(x_1 * \{x_2,x_4\},x_3) - \varphi(x_2 * \{x_1,x_4\},x_3)\\ &\quad - \varphi(x_1 * x_2,\{x_3,x_4\}) + \varphi(\{x_1,x_4\},x_2 * x_3)\\ &\quad + \varphi(x_1,x_2 * \{x_3,x_4\}) + \varphi(x_1,x_3 * \{x_2,x_4\})\,.\end{aligned}$$

But

$$\begin{aligned}\delta_H^2\varphi(\{x_1,x_4\},x_2,x_3) &= \{x_1,x_4\} * \varphi(x_2,x_3) - \varphi(\{x_1,x_4\} * x_2,x_3)\\ &\quad + \varphi(\{x_1,x_4\},x_2 * x_3) - \varphi(\{x_1,x_4\},x_2) * x_3\\ \delta_H^2\varphi(x_1,\{x_2,x_4\},x_3) &= x_1 * \varphi(\{x_2,x_4\},x_3) - \varphi(x_1 * \{x_2,x_4\},x_3)\\ &\quad + \varphi(x_1,\{x_2,x_4\} * x_3) - \varphi(x_1,\{x_2,x_4\}) * x_3\\ \delta_H^2\varphi(x_1,x_2,\{x_3,x_4\}) &= x_1 * \varphi(x_2,\{x_3,x_4\}) - \varphi(x_1 * x_2,\{x_3,x_4\})\\ &\quad + \varphi(x_1,x_2 * \{x_3,x_4\}) - \varphi(x_1,x_2) * \{x_3,x_4\}\,.\end{aligned}$$

As the product $*$ is commutative, we deduce

$$\begin{aligned}\{\delta_H^2\varphi(x_1,x_2,x_3),x_4\} &= \delta_H^2\varphi(\{x_1,x_4\},x_2,x_3) + \delta_H^2\varphi(x_1,\{x_2,x_4\},x_3)\\ &\quad + \delta_H^2\varphi(x_1,x_2,\{x_3,x_4\})\,.\end{aligned}$$

□

Observe that the last identity is not a consequence of the symmetry of φ. It is satified for any bilinear Lie biderivation. Now, we can generalize these identities.

Definition 4. Let ϕ be a k-linear map on $\mathcal{P}$. We say that ϕ is a Lie k-derivation if

$$\{\phi(x_1,\dots,x_k),x_{k+1}\}=\sum_{i=1}^{k}\phi(x_1,\dots,\{x_i,x_{k+1}\},\dots,x_k)$$

for any $x_1,\dots,x_{k+1}\in\mathcal{P}$, where $\{x,y\}$ denotes the Lie bracket associated with the Poisson product.

For example, from the previous lemma, if φ is a Lie 2-derivation (or biderivation), then $\delta_H^2\varphi$ is a Lie 3-derivation.

For any $(k-1)$-linear map on $\mathcal{P}$, let $\delta_H^{k-1}\varphi$ the k-linear map given by

$$\begin{aligned}\delta_H^{k-1}\varphi(x_1,\dots,x_k)&=x_1*\varphi(x_2,\cdots,x_k)-\varphi(x_1*x_2,\cdots,x_k)+\varphi(x_1,x_2*x_3,\cdots,x_k)\\&\quad+\cdots+(-1)^{k-1}\varphi(x_1,x_2,\cdots,x_{k-1}*x_k)\\&\quad+(-1)^k\varphi(x_1,x_2,\cdots,x_{k-1})*x_k.\end{aligned}$$

This operator is the coboundary operator of the Hochschild complex related to the associative operad $\mathcal{A}ss$.

Theorem 5. *If φ is a Lie k-derivation of $\mathcal{P}$, then $\delta_H^k\varphi$ is a Lie $(k+1)$-derivation of $\mathcal{P}$.*

Proof. It is analogous to the proof detailed for $k=3$. It depends only of the symmetry of the associative product $x*y$. □

Recall that a k-linear map φ on a vector space is called commutative if it satisfies $\varphi\circ\phi_{V_k}=0$ where $V_k=\sum_{\sigma\in\Sigma_k}\varepsilon(\sigma)\sigma=0$.

Lemma 6. *For any k-linear commutative map φ on $\mathcal{P}$, the $(k+1)$-linear map $\delta_H^k\varphi$ is commutative.*

Proof. In fact, consider the first term of $\delta_H^k\varphi(x_1,\dots,x_{k+1})$, that is,

$$x_1*\varphi(x_2,\dots,x_{k+1})\,.$$

We have

$$\sum_{\sigma\in\Sigma^i_{k+1}}\varepsilon(\sigma)x_i*\varphi(x_{\sigma(1)},\dots,x_{\sigma(i-1)},x_{\sigma(i+1)},\dots,x_{\sigma(k+1)})=0$$

because φ is commutative, where $\Sigma^i_{k+1}=\{\sigma\in\Sigma_{k+1},\sigma(i)=i\}$. The same trick vanishes the last terms, that is,

$$\sum_{\sigma\in\Sigma k+1}\varphi\big(x_\sigma(1),x_\sigma(2),\cdots,x_{\sigma(k)}\big)*x_{\sigma(k+1)}\,.$$

The terms in between vanishes two by two when we compose with Φ_{V_k}. □

Let $C^k_{PH}(\mathcal{P},\mathcal{P})$ be the vector space constituted by k-linear maps on $\mathcal{P}$ which are commutative and which are Lie k-derivations. From the previous result, the image of the $C^k_{PH}(\mathcal{P},\mathcal{P})$ by the map δ^k_H is contained in $C^{k+1}_{PH}(\mathcal{P},\mathcal{P})$. As these maps coincide with the coboundary operators of the complex, we obtain a complex $(C^k_{PH}(\mathcal{P},\mathcal{P}),\delta^k_H)$ whose associated cohomology is called the Poisson-Hochschild cohomology.

Theorem 6. *Let $\mathcal{P}$ be a Poisson algebra whose (nonassociative) product is denoted μ_0. For any associative deformation $\mu = \sum_{n\geq 0} t^i\mu_i$ of μ_0, the linear term μ_1 is a 2-cocycle for the Poisson-Hochschild cohomology.*

4.3 Example: Poisson structures on rigid Lie algebras

Such Poisson structures have been studied in [8], [7]. We will study these structures in terms of Poisson-Hochschild cohomology. Consider, for example, the 3-dimensional complex Poisson algebra given, in a basis $\{e_1,e_2,e_3\}$, by

$$e_1e_2 = 2e_2\,,\quad e_1e_3 = -2e_3\,,\quad e_2e_3 = e_1\,.$$

If $\{\cdot,\cdot\}$ and $*$ denote respectively the Lie bracket and the commutative associative product attached with the Poisson product, we have

$$\{e_1,e_2\} = 2e_2\,,\quad \{e_1,e_3\} = -2e_3\,,\quad \{e_2,e_3\} = e_1$$

and

$$e_i * e_j = 0\,,$$

for any i,j. If φ is a Lie biderivation, it satisfies

$$\{\varphi(e_i,e_j),e_k\} = \varphi(\{e_i,e_k\},e_j) + \varphi(\{e_j,e_k\},e_i)\,.$$

This implies $\varphi = 0$ and the Poisson algebra is rigid.

5 Coboundary operators of the general Poisson cohomology

In this section, we describe relations between the coboundary operators δ^k_P of the Poisson cohomology (the operadic cohomology or the deformation cohomology) of a Poisson algebra $\mathcal{P}$ and the corresponding operators of the Poisson-Lichnerowicz and Poisson-Hochschild cohomology of $\mathcal{P}$.

5.1 The cases $k = 0$ and $k = 1$

- $k = 0$. We put

$$H^0_P(\mathcal{P},\mathcal{P}) = \{X\in\mathcal{P} \text{ such that } \forall Y\in\mathcal{P},\ X\cdot Y = 0\}\,.$$

- $k = 1$. For $f\in End(\mathcal{P},\mathcal{P})$, we put

$$\delta^1_P f(X,Y) = f(X)\cdot Y + X\cdot f(Y) - f(X\cdot Y)$$

for any $X,Y\in\mathcal{P}$. Then we have

$$H^1_P(\mathcal{P},\mathcal{P}) = H^1_C(\mathfrak{g}_\mathcal{P},\mathfrak{g}_\mathcal{P})\cap H^1_H(\mathcal{A}_\mathcal{P},\mathcal{A}_\mathcal{P})\,.$$

5.2 Description of δ_P^2

In Section 4, we have seen that

$$\begin{aligned}\delta_P^2\varphi(x,y,z) = {} & 3\varphi(x\cdot y,z) - 3\varphi(x,y\cdot z) - \varphi(x\cdot z,y) - \varphi(y\cdot z,x)\\ & + \varphi(y\cdot x,z) + \varphi(z\cdot x,y) + 3\varphi(x,y)\cdot z - 3x\cdot\varphi(y,z)\\ & - \varphi(x,z)\cdot y - \varphi(y,z)\cdot x + \varphi(y,x)\cdot z + \varphi(z,x)\cdot y\end{aligned}$$

and

$$\delta_P^2\varphi = 2\big(\delta_C^2\varphi_a + \mathcal{L}_C(\varphi_s) + \delta_H^2\varphi_s + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)\big)\,.$$

Let us compare this operator with the corresponding Poisson-Lichnerowicz and Poisson-Hochschild ones.

Example 1. Assume that the Poisson product is skew-symmetric. Then $\{x,y\} = x\cdot y$ and $x * y = 0$. If $\varphi \in \mathcal{C}_P^2(\mathcal{P},\mathcal{P})$ is also skew-symmetric, then

$$\begin{aligned}\delta_P^2\varphi(x,y,z) &= 2\varphi(x\cdot y,z) + 2\varphi(y\cdot z,x) - 2\varphi(x\cdot z,y)\\ &\quad + 2\varphi(x,y)\cdot z + 2\varphi(y,z)\cdot x - 2\varphi(x,z)\cdot y\\ &= \delta_{PL}^2\varphi(x,y,z)\,,\end{aligned}$$

that is, the coboundary operator of the Poisson-Lichnerowicz cohomology.

The results of the previous sections imply:

Theorem 7. *Let φ be in $C_P^2(\mathcal{P},\mathcal{P})$, φ_s and φ_a be its symmetric and skew-symmetric parts. Then the following propositions are equivalent:*

1. $\delta_P^2\varphi = 0$.
2. $\begin{cases} i)\ \delta_C^2\varphi_a = 0,\ \delta_H^2\varphi_s = 0,\\ ii)\ \mathcal{L}_C(\varphi_s) + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s) = 0.\end{cases}$

Applications. Suppose that φ is skew-symmetric. Then $\varphi = \varphi_a$ and $\varphi_s = 0$. Then $\delta_P^2\varphi = 0$ if and only if $\delta_C^2\varphi = 0$ and $\mathcal{L}_H(\varphi) + \mathcal{L}_1(\varphi) = 0$. Morever if we suppose than φ is a biderivation on each argument, that is, $\mathcal{L}_1(\varphi) = 0$, then $\delta_P^2\varphi = 0$ if and only if $\mathcal{L}_H(\varphi) = 0$. But we have seen in Section 3 that

$$\mathcal{L}_H(\varphi) = \mathcal{L}_1(\varphi)\circ\Phi_{Id+c}.$$

Thus $\mathcal{L}_H(\varphi) = 0$ as soon as $\mathcal{L}_1(\varphi) = 0$.

Proposition 3. *Let φ be a skew-symmetric map which is a biderivation, that is, φ is a Poisson-Lichnerowicz 2-cochain. Then $\varphi \in Z_{PL}^2(\mathcal{P},\mathcal{P})$ if and only if $\varphi \in Z_P^2(\mathcal{P},\mathcal{P})$.*

Similarly, if φ is symmetric, then $\delta_P^2\varphi = 0$ if and only if $\delta_H^2\varphi = 0$ and $\mathcal{L}_C(\varphi) + \mathcal{L}_2(\varphi) = 0$. If φ be a skew-symmetric map which is a Lie biderivation, that is, if φ is a Poisson-Hochschild 2-cochain, then $\varphi \in Z_{PH}^2(\mathcal{P},\mathcal{P})$ if and only if $\varphi \in Z_P^2(\mathcal{P},\mathcal{P})$.

5.3 The case $k \geq 3$

Let $\mathcal{P}$ be a Poisson algebra and $H^*_{def}(\mathcal{P})$ or $H^*_P(\mathcal{P},\mathcal{P})$ its operadic cohomology. We propose here to describe $H^n_P(\mathcal{P},\mathcal{P})$ for $n \geq 3$. Let φ be a n-cochain of $C^n_P(\mathcal{P},\mathcal{P})$, that is, a n-linear map on $\mathcal{P}$. Its skew-symmetric part is the skew-symmetric n-linear map

$$\varphi_a = \frac{1}{n!}\varphi \circ \Phi_{V_n}$$

with $V_n = \sum_{\sigma\in\Sigma_n} \varepsilon(\sigma)\sigma$; its symmetric part is the symmetric n-linear map

$$\varphi_s = \frac{1}{n!}\varphi \circ \Phi_{W_n}$$

with $W_n = \sum_{\sigma\in\Sigma_n} \sigma$. We denote by δ^n_P, δ^n_C and δ^n_H respectively the coboundary operators associated with the Poisson cohomology of $\mathcal{P}$, the Chevalley-Eilenberg cohomology of $\mathfrak{g}_{\mathcal{P}}$ and the Hochschild cohomology of $\mathcal{A}_{\mathcal{P}}$.

The formulae (3) and (4) can be generalized as follows

$$2(n+1)!\delta^n_C\varphi_a = \delta^n_p\varphi \circ \Phi_{V_n}\,, \tag{5}$$

$$2(n+1)!\delta^n_H\varphi_s = \delta^n_p\varphi \circ \Phi_{U_{H,n}}\,, \tag{6}$$

where $U_{H,n} = \sum_{\sigma\in\Sigma_{1,n}} \sigma + (-1)^n \sum_{\sigma\in\Sigma_{n,n}} \sigma$ with $\Sigma_{i,n} = \{\sigma \in \Sigma_n, \sigma(1) = i\}$.

Proposition 4. *Let φ be a n-cochain of the Poisson complex of the Poisson algebra $\mathcal{P}$. Then*

$$\delta^n_P\varphi = 0 \Rightarrow \begin{cases} \delta^n_C\varphi_a = 0\,, \\ \delta^n_H\varphi_s = 0\,. \end{cases}$$

Let us consider $\mathcal{L}_{1,n}$ acting on the skew-symmetric n-linear map by

$$\begin{aligned} 2(n-1)!\mathcal{L}_{1,n}\varphi_a = &\sum_{\sigma^{-1}\in\Sigma_{i,i+1,n}} \varepsilon(\sigma)\varphi \circ_{\sigma^{-1}(1)} \mu \circ \Phi_{(Id+\tau_{12})\circ\sigma} \\ &+ (-1)^{n-1} \sum_{\sigma^{-1}\in\Sigma_{n,n}} \varepsilon(\sigma)\mu \circ_1 \varphi \circ \Phi_{(Id+\tau_{12})\circ\sigma} \\ &- \sum_{\sigma^{-1}\in\Sigma_{1,n}} \varepsilon(\sigma)\mu \circ_2 \varphi \circ \Phi_{(Id+\tau_{12})\circ\sigma} \end{aligned}$$

where $\Sigma_{i,i+1,n} = \{\sigma \in \Sigma_n, \sigma(1) = i, \sigma(2) = i+1\}$.

Lemma 7. *φ_a is a skew-symmetric n-derivation, that is, a skew-symmetric n-linear map which is a derivation for the associative product $x * y$ on each argument, if and only if $\mathcal{L}_{1,n}\varphi_a = 0$.*

Now we define the operator $\mathcal{L}_{H,n}$which acts on the the skew-symmetric n-linear map by

$$\mathcal{L}_{H,n}\varphi_a = \mathcal{L}_{1,n}\varphi_a \circ \Phi_{Id+c_n+c_n^2+\cdots+c_n^{n-2}}$$

where $c_n \in \Sigma_n$ is the cycle $(1, 2, \ldots, n)$.

Proposition 5. *Let φ be a skew-symmetric linear map on $\mathcal{P}^{\otimes^n}$. Then $\delta_P^n\varphi = 0$ if and only if $\delta_C^n\varphi = 0$ and $\mathcal{L}_{1,n}\varphi = 0$.*

We find again the classical result: the associative deformations of a Poisson algebra are parametrized by the Poisson-Lichnerowciz cohomology.

Assume now that φ is a symmetric n-linear map. We have seen that:

$$\delta_P^n\varphi = 0 \Rightarrow \delta_H^n\varphi_s = \delta_H^n\varphi = 0\,.$$

Consider the operator ∇^n acting on the symmetric n-linear maps by:

$$\begin{aligned}\nabla^n\varphi_s(x_1,\dots,x_{n+1}) = {} & \{\varphi(x_1,\dots,x_n),x_{n+1}\} - \varphi(\{x_1,x_{n+1}\},x_2,\dots,x_n)\\ & - \varphi(x_1,\{x_2,x_{n+1}\},x_3,\dots,x_n)\\ & - \cdots - \varphi(x_1,x_2,\dots,x_{n-1},\{x_n,x_{n+1}\})\,.\end{aligned}$$

Then $\varphi = \varphi_s$ is a Lie n-derivation if and only if $\nabla^n\varphi_s = 0$.

Now we consider the following operator acting also on the symmetric n-linear maps by:

$$\begin{aligned}\mathcal{L}_C^n\varphi_s = {} & \mu\circ_1\varphi\circ\Phi_{-c+c^2+\cdots+(-1)^{n+1}c^{n+1}} + \mu\circ_2\varphi\circ\Phi_{Id-c+c^2+\cdots+(-1)^nc^n}\\ & + \varphi\circ_1\mu\circ\Phi_{\sum_{1\le i,j\le n+1}(-1)^{i+j+1}c_{ij}}\end{aligned}$$

where c_{ij} is the permutation $\begin{pmatrix}1&2&3&\cdots&\cdots&\cdots&\cdots&\cdots&n+1\\ i&j&1&\cdots&\check{i}&\cdots&\check{j}&\cdots&n+1\end{pmatrix}$ and $\mathcal{L}_2^n\varphi_s$ defined by:

$$\mathcal{L}_C^n\varphi_s + \mathcal{L}_2^n\varphi_s = \nabla^n\varphi_s\circ\Phi_u$$

with $u\in\mathbb{K}[\Sigma_n]$ equal to $\tau_{12}+\tau_{13}+\cdots+\tau_{1n}$. Since Φ_u is invertible, the equation $\mathcal{L}_C^n\varphi_s + \mathcal{L}_2^n\varphi_s = 0$ implies $\nabla^n\varphi_s = 0$ and we find that the Poisson-Hochschild cohomology coincides with the Poisson cohomology when $\varphi = \varphi_s$.

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