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POSITIVE SOLUTIONS AND EIGENVALUE INTERVALS OF A NONLINEAR SINGULAR FOURTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract. We consider the classical nonlinear fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t), u'(t), u''(t)), & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u'''(1) = 0. \end{cases}$$

In this problem, the nonlinear term $h(t)f(t, u(t), u'(t), u''(t))$ contains the first and second derivatives of the unknown function, and the function $h(t)f(t, x, y, z)$ may be singular at $t = 0$, $t = 1$ and at $x = 0$, $y = 0$, $z = 0$. By introducing suitable height functions and applying the fixed point theorem on the cone, we establish several local existence theorems on positive solutions and obtain the corresponding eigenvalue intervals.

Keywords: nonlinear ordinary differential equation, singular nonlinearity, positive solution, eigenvalue interval

MSC 2010: 34B15, 34B16, 34B18

1. Introduction

It is well known that the deflection of elastic beams can be described by some fourth-order boundary value problems, for example, see [3], [11]. Consequently, fourth-order boundary value problems play a very important role for ordinary differential equations in both theory and applications.

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Let λ be a positive parameter. In this paper, we consider the classical nonlinear fourth-order two-point boundary value problem

$$\text{(P1)} \qquad \begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, \end{cases}$$

and its simplified form

$$\text{(P2)} \qquad \begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u'''(1) = 0. \end{cases}$$

In applied mathematics, the problems (P1) and (P2) can model the deflection of an elastic beam simply supported at the left end and fastened with a sliding clamp at the right end.

In the last decade or so, several papers have been devoted to the boundary value problems (P1) and (P2), for example, see [5], [7]–[9], [11], [15], [21]. However, in the literature, the problems (P1) and (P2) have not received as much attention as the fourth-order boundary value problems with boundary condition $u(0) = u(1) = u''(0) = u''(1) = 0$, which were considered, for example, in [4], [6], [12], [14], [16], [19].

This paper focuses on the positive eigenvalue intervals for which there exist one or two positive solutions. Here, the solution $u^*(t)$ of (P1) or (P2) is called positive if $u^*(t) > 0$, $0 < t \leqslant 1$. Throughout this paper, let

$$\alpha(t) = \frac{1}{2}(3t - t^3), \quad \beta(t) = 2t - t^2, \quad \gamma(t) = \frac{1}{2}(1 - t^2).$$

In 2000, Graef and Yang [9] proved the following existence theorem for the problem (P2).

Theorem 1.1. *Assume that*

(a1) $h\colon [0,1] \to [0,\infty)$ *is a continuous function.*

(a2) $A = \frac{1}{12}\int_0^1 s^2(3-s^2)^2 h(s)\,\mathrm{d}s$, $B = \frac{1}{6}\int_0^1 s^2(2-s)(3-s^2)h(s)\,\mathrm{d}s$.

(a3) $f(t,x) = f(x)$ *and* $f\colon [0,\infty) \to [0,\infty)$ *is continuous.*

(a4) *One of the following conditions is satisfied:*

(i) $[A \lim\limits_{x\to+0} f(x)/x]^{-1} < \lambda < [B \lim\limits_{x\to\infty} f(x)/x]^{-1}$.

(ii) $[A \lim\limits_{x\to\infty} f(x)/x]^{-1} < \lambda < [B \lim\limits_{x\to+0} f(x)/x]^{-1}$.

Then problem (P2) *has at least one positive solution.*

In Theorem 1.1, the function $h(t)f(x)$ is continuous with respect to both variables t and x, and the nonlinear term $h(t)f(u(t))$ does not contain first and second derivatives of the unknown function $u(t)$.

In this paper, we generalize Theorem 1.1 under the following assumptions.

For the problem (P1), we use the assumptions:

(H1) $h\colon (0,1) \to [0,\infty)$ is continuous.

(H2) $f\colon (0,1) \times (0,\infty) \times (0,\infty) \times (-\infty,0) \to [0,\infty)$ is continuous.

(H3) For each pair of positive numbers $r_1 < r_2$, there exists a nonnegative function $j_{r_1}^{r_2} \in C(0,1)$ such that $\int_0^1 h(t) j_{r_1}^{r_2}(t)\,\mathrm{d}t < \infty$ and if

$$0 < t < 1, \quad \frac{1}{3} r_1 \alpha(t) \leqslant x \leqslant \frac{1}{2} r_2 \beta(t), \quad \frac{1}{2} r_1 \gamma(t) \leqslant y \leqslant r_2, \quad r_1 t \leqslant -z \leqslant r_2,$$

then $f(t,x,y,z) \leqslant j_{r_1}^{r_2}(t)$.

For the problem (P2), we use the assumptions:

(H1)$^\flat$ $h\colon (0,1) \to [0,\infty)$ is continuous.

(H2)$^\flat$ $f\colon (0,1) \times (0,\infty) \to [0,\infty)$ is continuous.

(H3)$^\flat$ For each pair of positive numbers $r_1 < r_2$, there exists a nonnegative function $j_{r_1}^{r_2} \in C(0,1)$ such that $\int_0^1 h(t) j_{r_1}^{r_2}(t)\,\mathrm{d}t < \infty$ and if

$$0 < t < 1, \quad r_1 \alpha(t) \leqslant x \leqslant r_2 \beta(t),$$

then $f(t,x) \leqslant j_{r_1}^{r_2}(t)$.

If (H1) holds, $\int_0^1 h(t)\,\mathrm{d}t < \infty$ and $f\colon [0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous, then (H3) holds. If (H1)$^\flat$ holds, $\int_0^1 h(t)\,\mathrm{d}t < \infty$ and $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, then (H3)$^\flat$ holds.

The assumptions (H1)–(H3) show that, in this paper, the nonlinear term $h(t)f(t, u(t), u'(t), u''(t))$ contains the first and second derivatives of the unknown function $u(t)$, and the function $h(t)f(t,x,y,z)$ may be singular at $t=0$, $t=1$ and at $x=0$, $y=0$, $z=0$. The singularities mean that $h(t)f(t,x,y,z)$ may be singular at $t=0$ and/or $t=1$ for any $(x,y,z) \in [0,\infty) \times [0,\infty) \times (-\infty,0]$, and may be singular at $x=0$, $y=0$ and/or $z=0$ for any $t \in [0,1]$.

The assumptions (H1)$^\flat$–(H3)$^\flat$ show that the function $h(t)f(t,x)$ may be singular at $t=0$, $t=1$ and at $x=0$. This implies that $h(t)f(t,x)$ may be singular at $t=0$ and/or $t=1$ for any $x \in [0,\infty)$, and may be singular at $x=0$ for any $t \in [0,1]$.

To the best of our knowledge, for the problem (P1), there are no existence results of positive solutions under the assumptions (H1)–(H3). For the problem (P2), the existence of positive solutions under (H1)$^\flat$–(H3)$^\flat$ has not been studied by any author.

Recently, various nonlinear boundary value problems with singularity have received a great deal of attention in the literature. For developments in the field, see

Agarwal and O'Regan [1], [2], Meehan and O'Regan [17], Staněk [18], Wei [19] and the references therein. The motivation of this paper comes from these papers.

In this paper, we will use the exact apriori estimation technique that came from papers [20], [22]. In [20], we presented the exact apriori estimation technique and considered the problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u''(1) = 0, \end{cases}$$

where the function $f(t, x)$ is allowed to be singular only at $x = 0$. In [22], we perfected the technique and considered the problem

$$\begin{cases} u^{(4)}(t) = h(t) f(t, u(t), u'(t)), & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $h(t) f(t, x, y)$ may be singular at $t = 0$, $t = 1$ and at $x = 0$, $y = 0$. In the present paper, we will deal with the more complicated problem (P1). The main tool is the Guo-Krasnosel'skii fixed point theorem of the cone expansion-compression type. All results are independent of the existence of upper and lower solutions.

By applying the related Green function, the problems (P1) and (P2) are changed to integral equations. In order to overcome the difficulties resulting from the above-mentioned singularities, we construct suitable cones and height functions. By estimating integrals of these height functions and considering the fixed points of the associated integral operators defined on the cones, the eigenvalue intervals for which there exist one or two positive solutions are obtained.

The rest of this paper is organized as follows. Section 2 gives some preliminaries and necessary lemmas. Section 3 is devoted to the positive solutions and positive eigenvalues of the problems (P1) and (P2). Finally, we will verify that Theorem 1.1 is a corollary of the main results, and give an example to demonstrate our results.

2. Preliminaries and lemmas

Let $G(t, s)$ be the Green function of the homogeneous linear problem

$$-u''(t) = 0, \quad 0 \leqslant t \leqslant 1, \quad u(0) = u'(1) = 0.$$

The precise expression of $G(t, s)$ is

$$G(t, s) = \begin{cases} s, & 0 \leqslant s \leqslant t \leqslant 1, \\ t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Thus $G\colon [0,1]\times[0,1]\to[0,1]$ is continuous. Let

$$J(t,s)=\int_0^1 G(t,\tau)G(\tau,s)\,\mathrm{d}\tau,\quad 0\leqslant t,s\leqslant 1.$$

Direct computations give that

$$J(t,s)=\begin{cases} ts-\frac12 t^2s-\frac16 s^3, & 0\leqslant s\leqslant t\leqslant 1,\\ ts-\frac12 ts^2-\frac16 t^3, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Computing the first and second partial derivatives of $J(t,s)$ with respect to t, one has

$$\frac{\partial}{\partial t}J(t,s)=\begin{cases} s-ts, & 0\leqslant s\leqslant t\leqslant 1,\\ s-\frac12 s^2-\frac12 t^2, & 0\leqslant t\leqslant s\leqslant 1;\end{cases}$$

$$\frac{\partial^2}{\partial t^2}J(t,s)=-G(t,s),\quad 0\leqslant t,s\leqslant 1.$$

Lemma 2.1. (1) *If* $0\leqslant t,s\leqslant 1$, *then* $ts\leqslant G(t,s)\leqslant s=\max\limits_{0\leqslant t\leqslant 1}G(t,s)$.

(2) *If* $0\leqslant t,s\leqslant 1$, *then* $\alpha(t)J(1,s)\leqslant J(t,s)\leqslant\beta(t)J(1,s)$.

(3) *If* $0\leqslant s\leqslant 1$, *then* $\max\limits_{0\leqslant t\leqslant 1}J(t,s)=J(1,s)\leqslant\frac12 s$.

(4) *If* $0\leqslant t,s\leqslant 1$, *then* $\gamma(t)s\leqslant\frac{\partial}{\partial t}J(t,s)\leqslant s$.

(5) *If* $0\leqslant t,t+\Delta t,s\leqslant 1$, *then* $|J(t+\Delta t,s)-J(t,s)|\leqslant|\Delta t|$.

(6) *If* $0\leqslant t,t+\Delta t,s\leqslant 1$, *then* $|\frac{\partial}{\partial t}J(t+\Delta t,s)-\frac{\partial}{\partial t}J(t,s)|\leqslant|\Delta t|$.

(7) *If* $0\leqslant t,t+\Delta t,s\leqslant 1$, *then* $|G(t+\Delta t,s)-G(t,s)|\leqslant|\Delta t|$.

Proof. The proofs of (1) and (7) are direct.

The proof of (2). For $0\leqslant s\leqslant t\leqslant 1$,

$$\begin{aligned} J(t,s)-\alpha(t)J(1,s)&=\frac14 ts-\frac12 t^2s-\frac16 s^3+\frac14 ts^3+\frac14 t^3s-\frac1{12}t^3s^3\\ &=\frac14 ts(1-t)-\frac14 t^2s(1-t)+\frac1{12}ts^3(1-t^2)-\frac16 s^3(1-t)\\ &=\frac1{12}s(1-t)[3t(1-t)-s^2(1-t)-s^2(1-t^2)]\\ &=\frac1{12}s(1-t)^2[3t-2s^2-ts^2]\geqslant 0,\\ \beta(t)J(1,s)-J(t,s)&=\frac16 s^3-\frac13 ts^3+\frac16 t^2s^3=\frac16 s^3(1-t)^2\geqslant 0. \end{aligned}$$

For $0\leqslant t\leqslant s\leqslant 1$, the proof is similar.

The proof of (3). Obviously, $\frac{\partial}{\partial t}J(t,s)\geqslant 0$, $0\leqslant t,s\leqslant 1$. Thus, for any $0\leqslant s\leqslant 1$,

$$\max_{0\leqslant t\leqslant 1}J(t,s)=J(1,s)=\frac12 s-\frac16 s^3\leqslant\frac12 s.$$

The proof of (4). For $0 \leqslant s \leqslant t \leqslant 1$,

$$\frac{\partial}{\partial t} J(t,s) - \gamma(t)s = s - ts - \frac{1}{2}s(1-t^2) = \frac{1}{2}s(1-t)^2 \geqslant 0.$$

For $0 \leqslant t \leqslant s \leqslant 1$,

$$\frac{\partial}{\partial t} J(t,s) - \gamma(t)s = s - \frac{1}{2}s^2 - \frac{1}{2}t^2 - \frac{1}{2}s(1-t^2) = \frac{1}{2}(1-s)(s-t^2) \geqslant 0.$$

From the expression of $\frac{\partial}{\partial t}J(t,s)$, one has $\frac{\partial}{\partial t}J(t,s) \leqslant s$, $0 \leqslant t, s \leqslant 1$.

The proof of (5). Applying the mean value theorem and (4), for any $0 \leqslant t, s$, $t + \Delta t \leqslant 1$, we obtain

$$|J(t+\Delta t, s) - J(t,s)| \leqslant \sup_{0 \leqslant t \leqslant 1} \frac{\partial}{\partial t} J(t,s)|\Delta t| \leqslant |\Delta t| s \leqslant |\Delta t|.$$

Applying (1), the proof of (6) is similar to (5). □

In order to deal with the problem (P1), the following preliminaries are necessary.

Let $C^2[0,1]$ be the Banach space of all functions twice continuously differentiable on $[0,1]$ and equipped with the norm $|\!|\!| u |\!|\!| = \max\{\|u\|, \|u'\|, \|u''\|\}$, where $\|u\| = \max\limits_{0 \leqslant t \leqslant 1} |u(t)|$. Let

$$\begin{aligned} C_0^2[0,1] &= \{u \in C^2[0,1] \colon\ u(0) = u'(1) = 0\}, \\ K &= \{u \in C_0^2[0,1] \colon\ \|u\|\alpha(t) \leqslant u(t) \leqslant \|u\|\beta(t),\ u'(t) \geqslant \|u'\|\gamma(t), \\ &\qquad -u''(t) \geqslant \|u''\|t,\ 0 \leqslant t \leqslant 1\}. \end{aligned}$$

Let $p(t) = \int_0^1 J(t,s)\,\mathrm{d}s = \frac{1}{3}t - \frac{1}{6}t^3 + \frac{1}{24}t^4$. Then $0 \neq p \in K$ by direct calculations.

If $u_1, u_2 \in K$, then

$$u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0, \quad u_1'(t) \geqslant 0, \quad u_2'(t) \geqslant 0, \quad 0 \leqslant t \leqslant 1.$$

This shows that $u_1(t)$, $u_2(t)$ are nonnegative nondecreasing functions on $[0,1]$. So, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} (u_1 + u_2)(t) &= u_1(t) + u_2(t) \leqslant \|u_1\|\beta(t) + \|u_2\|\beta(t) \\ &= u_1(1)\beta(t) + u_2(1)\beta(t) = (u_1+u_2)(1)\beta(t) = \|u_1 + u_2\|\beta(t). \end{aligned}$$

Simple verification shows that K is a nonnegative function cone in $C^2[0,1]$. Here, the set K is called a cone in $C^2[0,1]$, if K is a convex closed subset in $C^2[0,1]$ such that (i) if $u \in K$, $\varrho \geqslant 0$, then $\varrho u \in K$; (ii) if $u \in K$ and $-u \in K$, then $u = 0$. For $r > 0$, write

$$K(r) = \{u \in K \colon\ |\!|\!| u |\!|\!| < r\}, \quad \partial K(r) = \{u \in K \colon\ |\!|\!| u |\!|\!| = r\}.$$

Lemma 2.2. *If $u \in K$, then $\|u''\| = |\!|\!| u |\!|\!|$, $\frac{1}{2}|\!|\!| u |\!|\!| \leqslant \|u'\| \leqslant |\!|\!| u |\!|\!|$ and $\frac{1}{3}|\!|\!| u |\!|\!| \leqslant \|u\| \leqslant \frac{1}{2}|\!|\!| u |\!|\!|$.*

P r o o f. Since $u \in K$, one has $\|u''\|t \leqslant -u''(t) \leqslant \|u''\|$, $0 \leqslant t \leqslant 1$. Since $u(0) = u'(1) = 0$, one has

$$u(t) = \int_0^1 G(t,s)[-u''(s)]\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1.$$

It follows that

$$\|u\| = \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)[-u''(s)]\,\mathrm{d}s \leqslant \|u''\| \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)\,\mathrm{d}s = \frac{1}{2}\|u''\|,$$
$$\|u\| \geqslant \|u''\| \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)s\,\mathrm{d}s = \frac{1}{3}\|u''\|.$$

Since $u'(1) = 0$, one has $u'(t) = \int_t^1 [-u''(s)]\,\mathrm{d}s$, $0 \leqslant t \leqslant 1$. So

$$\frac{1}{2}\|u''\| = \max_{0\leqslant t\leqslant 1}\int_t^1 \|u''\|s\,\mathrm{d}s \leqslant \|u'\| \leqslant \max_{0\leqslant t\leqslant 1}\int_t^1 \|u''\|\,\mathrm{d}s = \|u''\|.$$

Hence $\|u''\| = \|\!|u|\!\|$, $\frac{1}{3}\|\!|u|\!\| \leqslant \|u\| \leqslant \frac{1}{2}\|\!|u|\!\|$ and $\frac{1}{2}\|\!|u|\!\| \leqslant \|u'\| \leqslant \|\!|u|\!\|$. □

For $u \in K \setminus \{0\}$, define the operator T as follows:

$$(Tu)(t) = \lambda \int_0^1 J(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1.$$

Lemma 2.3. *Assume that* $0 < r_1 < r_2 < \infty$ *and* (H1)–(H3) *hold. Then*
(1) $T\colon \overline{K(r_2)} \setminus K(r_1) \to C^2[0,1]$ *and for any* $u \in \overline{K(r_2)} \setminus K(r_1)$,

$$(Tu)''(t) = -\lambda \int_0^1 G(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1.$$

(2) $T\colon \overline{K(r_2)} \setminus K(r_1) \to K$ *is completely continuous.*

P r o o f. Let $u \in \overline{K(r_2)} \setminus K(r_1)$. Then $r_1 \leqslant \|\!|u|\!\| \leqslant r_2$. By Lemma 2.2, $\frac{1}{3}r_1 \leqslant \|u\| \leqslant \frac{1}{2}r_2$, $\frac{1}{2}r_1 \leqslant \|u'\| \leqslant r_2$, $r_1 \leqslant \|u''\| \leqslant r_2$. So, for $0 \leqslant t \leqslant 1$,

$$\frac{1}{3}r_1\alpha(t) \leqslant \|u\|\alpha(t) \leqslant u(t) \leqslant \|u\|\beta(t) \leqslant \frac{1}{2}r_2\beta(t),$$
$$\frac{1}{2}r_1\gamma(t) \leqslant \|u'\|\gamma(t) \leqslant u'(t) \leqslant \|u'\| \leqslant r_2,$$
$$r_1 t \leqslant \|u''\|t \leqslant -u''(t) \leqslant \|u''\| \leqslant r_2.$$

Let the function $j_{r_1}^{r_2} \in C(0,1)$ be as in (H3). Then

$$f(t,u(t),u'(t),u''(t)) \leqslant j_{r_1}^{r_2}(t), \quad 0<t<1.$$

By Lemma 2.1 (3) and the assumption (H3), we get that

$$\max_{0\leqslant t\leqslant 1}\int_0^1 J(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s \leqslant \frac{1}{2}\int_0^1 sh(s)j_{r_1}^{r_2}(s)\,\mathrm{d}s < \infty.$$

So, $(Tu)(t)$ is well defined on $[0,1]$.

Applying Lemma 2.1 (5), one has

$$\left|\frac{J(t+\Delta t,s)-J(t,s)|}{\Delta t}\right| h(s)f(s,u(s),u'(s),u''(s)) \leqslant h(s)f(s,u(s),u'(s),u''(s)).$$

By (H3), $h(s)f(s,u(s),u'(s),u''(s))$ is a nonnegative integrable function on $[0,1]$. Applying the Lebesgue dominated convergence theorem ([13, (12.24), p. 172]), we get that, for $0\leqslant t\leqslant 1$,

$$\begin{aligned}(Tu)'(t) &= \lambda \lim_{\Delta t\to 0}\frac{1}{\Delta t}[(Tu)(t+\Delta t)-(Tu)(t)]\\ &= \lambda \lim_{\Delta t\to 0}\int_0^1 \frac{J(t+\Delta t,s)-J(t,s)}{\Delta t}h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s\\ &= \lambda \int_0^1 \lim_{\Delta t\to 0}\frac{J(t+\Delta t,s)-J(t,s)}{\Delta t}h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s\\ &= \lambda \int_0^1 \frac{\partial}{\partial t}J(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s.\end{aligned}$$

Further, applying Lemma 2.1 (6) and copying the above arguments, one has

$$(Tu)''(t) = -\lambda\int_0^1 G(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s, \quad 0\leqslant t\leqslant 1.$$

Therefore, $Tu\in C^2[0,1]$ and the conclusion (1) is proved.

Since $\frac{\partial}{\partial t}J(t,s)\geqslant 0$, $0\leqslant t,s\leqslant 1$, one has $(Tu)'(t)\geqslant 0$, $0\leqslant t\leqslant 1$. Since $J(0,s)\equiv 0$, $\frac{\partial}{\partial t}J(1,s)\equiv 0$, one has $(Tu)(0)=0$, $(Tu)'(1)=0$.

According to Lemma 2.1 (1)–(3), one has, for $0 \leqslant t \leqslant 1$,

$$
\begin{aligned}
(Tu)(t) &= \lambda \int_0^1 J(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s \\
&\geqslant \lambda\alpha(t) \int_0^1 J(1,s)h(s)f(s,u(s),u'(s)u''(s))\,\mathrm{d}s \\
&\geqslant \lambda\alpha(t) \max_{0\leqslant t\leqslant 1} \int_0^1 J(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s = \|Tu\|\alpha(t), \\
-(Tu)''(t) &= \lambda \int_0^1 G(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s \\
&\geqslant \lambda t \int_0^1 sh(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s \\
&\geqslant \lambda t \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s = \|(Tu)''\|t.
\end{aligned}
$$

Similarly, $(Tu)(t) \leqslant \|Tu\|\beta(t)$, $(Tu)'(t) \geqslant \|(Tu)'\|\gamma(t)$, $0 \leqslant t \leqslant 1$. Therefore, $T\colon \overline{K(r_2)} \setminus K(r_1) \to K$.

Now we prove that the operator $T\colon \overline{K(r_2)} \setminus K(r_1) \to C^2[0,1]$ is completely continuous. Let

$$
\begin{aligned}
(T_1u)(t) &= h(t)f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1,\ u \in \overline{K(r_2)} \setminus K(r_1), \\
(T_2v)(t) &= \lambda \int_0^1 J(t,s)v(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1,\ v \in L^1[0,1].
\end{aligned}
$$

By (H1)–(H3), $T_1\colon \overline{K(r_2)} \setminus K(r_1) \to L^1[0,1]$. Let $v \in L^1[0,1]$, then

$$
\max_{0\leqslant t\leqslant 1} \left| \int_0^1 J(t,s)v(s)\,\mathrm{d}s \right| \leqslant \frac{1}{2}\int_0^1 s|v(s)|\,\mathrm{d}s \leqslant \frac{1}{2}\int_0^1 |v(s)|\,\mathrm{d}s < \infty.
$$

Hence, T_2v is well defined. Since $v(t)$ is an integrable function on $[0,1]$, by the Lebesgue dominated convergence theorem, for $0 \leqslant t \leqslant 1$,

$$
\begin{aligned}
\lim_{\Delta t\to 0} \frac{(T_2v)(t+\Delta t) - (T_2v)(t)}{\Delta t} &= \lim_{\Delta t\to 0} \frac{\lambda}{\Delta t} \int_0^1 [J(t+\Delta t,s) - J(t,s)]v(s)\,\mathrm{d}s \\
&= \lambda \lim_{\Delta t\to 0} \int_0^1 \frac{J(t+\Delta t,s) - J(t,s)}{\Delta t} v(s)\,\mathrm{d}s \\
&= \lambda \int_0^1 \frac{\partial}{\partial t} J(t,s)v(s)\,\mathrm{d}s.
\end{aligned}
$$

So, $T_2v \in C^1[0,1]$ and for $0 \leqslant t \leqslant 1$,

$$
(T_2v)'(t) = \lambda \int_0^1 \frac{\partial}{\partial t} J(t,s)v(s)\,\mathrm{d}s.
$$

Similarly, $T_2 v \in C^2[0,1]$ and for $0 \leqslant t \leqslant 1$,

$$(T_2 v)''(t) = -\lambda \int_0^1 G(t,s)v(s)\,\mathrm{d}s.$$

It follows that $T_2\colon L^1[0,1] \to C^2[0,1]$ and $T = T_2 \circ T_1$ on $\overline{K(r_2)} \setminus K(r_1)$.

From the above arguments, we see that $T_2, (T_2(\cdot))', (T_2(\cdot))''\colon L^1[0,1] \to C[0,1]$ are linear operators.

Let $r > 0$ and $U(r) = \bigl\{v \in L^1[0,1]\colon \int_0^1 |v(s)|\,\mathrm{d}s \leqslant r\bigr\}$. Then

$$\begin{aligned}\sup_{v\in U(r)} \|T_2 v\| &= \lambda \sup_{v\in U(r)} \max_{0\leqslant t\leqslant 1} \left|\int_0^1 J(t,s)v(s)\,\mathrm{d}s\right| \leqslant \frac{1}{2}\lambda \sup_{v\in U(r)} \int_0^1 s|v(s)|\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\lambda \sup_{v\in U(r)} \int_0^1 |v(s)|\,\mathrm{d}s = \frac{1}{2}\lambda r.\end{aligned}$$

This shows that $T_2\colon L^1[0,1] \to C[0,1]$ is a bounded linear operator. Hence, $T_2\colon L^1[0,1] \to C[0,1]$ is continuous and the set $T_2(U(r))$ is bounded. On the other hand, if $v \in U(r)$, by Lemma 2.1 (5) we have

$$\begin{aligned}|(T_2 v)(t_1) - (T_2 v)(t_2)| &= \lambda \left|\int_0^1 [J(t_1,s) - J(t_2,s)]v(s)\,\mathrm{d}s\right| \\ &\leqslant \lambda |t_1 - t_2| \int_0^1 |v(s)|\,\mathrm{d}s \leqslant \lambda r |t_1 - t_2|.\end{aligned}$$

So, the set $T_2(U(r))$ is equicontinuous. By the Arzela-Ascoli theorem, $T_2\colon L^1[0,1] \to C[0,1]$ is completely continuous.

Similarly, by Lemma 2.1 (6) and (7), $(T_2(\cdot))', (T_2(\cdot))''\colon L^1[0,1] \to C[0,1]$ are completely continuous.

Therefore, $T_2\colon L^1[0,1] \to C^2[0,1]$ is a completely continuous operator.

In order to prove that $T\colon \overline{K(r_2)} \setminus K(r_1) \to K$ is completely continuous, we only need to prove that $T_1\colon \overline{K(r_2)} \setminus K(r_1) \to L^1[0,1]$ is continuous.

Let $u_n, u_0 \in \overline{K(r_2)} \setminus K(r_1)$ and $\|u_n - u_0\| \to 0$. Then for any $0 < t < 1$, $u_n(t) \to u_0(t)$, $u_n'(t) \to u_0'(t)$, $u_n''(t) \to u_0''(t)$ and by (H2),

$$\lim_{n\to\infty} f(t, u_n(t), u_n'(t), u_n''(t)) = f(t, u_0(t), u_0'(t), u_0''(t)).$$

Since $u_n, u_0 \in \overline{K(r_2)} \setminus K(r_1)$, we have for $n = 1, 2, \ldots$ and $0 < t < 1$,

$$h(t)|f(t, u_n(t), u_n'(t), u_n''(t)) - f(t, u_0(t), u_0'(t), u_0''(t))| \leqslant 2h(t) j_{r_1}^{r_2}(t).$$

By the Lebesgue dominated convergence theorem, we obtain that

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^1 |(T_1u_n)(t)-(T_1u_0)(t)|\,\mathrm{d}t\\
&\qquad=\lim_{n\to\infty}\int_0^1 h(t)|f(t,u_n(t),u_n'(t),u_n''(t))-f(t,u_0(t),u_0'(t),u_0''(t))|\,\mathrm{d}t\\
&\qquad=\int_0^1 h(t)\lim_{n\to\infty}|f(t,u_n(t),u_n'(t),u_n''(t))-f(t,u_0(t),u_0'(t),u_0''(t))|\,\mathrm{d}t=0.
\end{aligned}
$$

Therefore, $T_1\colon \overline{K(r_2)}\setminus K(r_1)\to L^1[0,1]$ is continuous.

The proof is completed. □

For the problem (P2), we will use the following cone and associate integral operator.

Let $C[0,1]$ be the Banach space of all functions continuous on $[0,1]$ and equipped with the norm $\|u\|=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$. Let

$$K^\flat=\{u\in C[0,1]\colon\ \|u\|\alpha(t)\leqslant u(t)\leqslant\|u\|\beta(t),\ 0\leqslant t\leqslant 1\}.$$

Then $K^\flat$ is a cone in $C[0,1]$. In the cone, all functions are nonnegative.

For $r>0$, write $K^\flat(r)=\{u\in K^\flat\colon\ \|u\|<r\}$.

For $u\in K^\flat\setminus\{0\}$, define the operator $T^\flat$ as follows:

$$(T^\flat u)(t)=\lambda\int_0^1 J(t,s)h(s)f(s,u(s))\,\mathrm{d}s,\quad 0\leqslant t\leqslant 1.$$

Imitating and simplifying the proof of Lemma 2.3 (2), we get Lemma 2.4. The theorem concerns the complete continuity of the operator $T^\flat$.

Lemma 2.4. *Assume that $0<r_1<r_2<\infty$ and (H1)$^\flat$–(H3)$^\flat$ hold. Then $T^\flat\colon\ \overline{K^\flat(r_2)}\setminus K^\flat(r_1)\to K^b$ is completely continuous.*

For convenience of the reader, we list the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type, see [10].

Lemma 2.5. *Let X be a Banach space, let K be a cone in X, let Ω_1,Ω_2 be two bounded open subsets in K with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T\colon\ \overline{\Omega}_2\setminus\Omega_1\to K$ be a completely continuous operator. Assume that one of the following conditions is satisfied:*

(1) $\|Tx\|\leqslant\|x\|$, $x\in\partial\Omega_1$ *and* $\|Tx\|\geqslant\|x\|$, $x\in\partial\Omega_2$,

(2) $\|Tx\|\geqslant\|x\|$, $x\in\partial\Omega_1$ *and* $\|Tx\|\leqslant\|x\|$, $x\in\partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega}_2\setminus\Omega_1$.

3. Main results

Part I. On the problem (P1).

For $0 < t < 1$ and $r > 0$, define the height functions

$$\varphi(t,r) = \max\{f(t,x,y,z)\colon \tfrac{1}{3}r\alpha(t) \leqslant x \leqslant \tfrac{1}{2}r\beta(t),\ \tfrac{1}{2}r\gamma(t) \leqslant y \leqslant r,\ rt \leqslant -z \leqslant r\},$$
$$\psi(t,r) = \min\{f(t,x,y,z)\colon \tfrac{1}{3}r\alpha(t) \leqslant x \leqslant \tfrac{1}{2}r\beta(t),\ \tfrac{1}{2}r\gamma(t) \leqslant y \leqslant r,\ rt \leqslant -z \leqslant r\}.$$

If (H1)–(H3) hold, then $h(\cdot)\varphi(\cdot,r), h(\cdot)\psi(\cdot,r) \in L^1[0,1]\cap C(0,1)$.

We obtain the following local existence theorems.

Theorem 3.1. *Assume that* (H1)–(H3) *hold and there exist two positive numbers $a < b$ such that one of the following conditions is satisfied:*

(b1) $$\frac{b}{\int_0^1 sh(s)\psi(s,b)\,\mathrm{d}s} \leqslant \lambda \leqslant \frac{a}{\int_0^1 sh(s)\varphi(s,a)\,\mathrm{d}s},$$

(b2) $$\frac{a}{\int_0^1 sh(s)\psi(s,a)\,\mathrm{d}s} \leqslant \lambda \leqslant \frac{b}{\int_0^1 sh(s)\varphi(s,b)\,\mathrm{d}s}.$$

Then problem (P1) *has at least one increasing positive solution $u^* \in K$ such that $a \leqslant \|u^*\| \leqslant b$.*

Proof. Without loss of generality, we only prove the case (b1). By Lemma 2.3 (2), $T\colon \overline{K(b)}\setminus K(a) \to K$ is completely continuous.

If $u \in \partial K(a)$, then $\|u\| = a$. By Lemma 2.2, for $0 \leqslant t \leqslant 1$,

$$\tfrac{1}{3}a\alpha(t) \leqslant u(t) \leqslant \tfrac{1}{2}a\beta(t), \quad \tfrac{1}{2}a\gamma(t) \leqslant u'(t) \leqslant a, \quad at \leqslant -u''(t) \leqslant a.$$

By the definition of $\varphi(t,a)$, then

$$f(t,u(t),u'(t),u''(t)) \leqslant \varphi(t,a), \quad 0 < t < 1.$$

By Lemma 2.3 (2), $Tu \in K$. By Lemmas 2.2 and 2.1 (1), then

$$\begin{aligned}\|Tu\| = \|(Tu)''\| &= \lambda \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)f(s,u(s),u'(s),u''(s))\,\mathrm{d}s\\ &\leqslant \frac{a}{\int_0^1 sh(s)\varphi(s,a)\,\mathrm{d}s}\cdot\int_0^1 sh(s)\varphi(s,a)\,\mathrm{d}s = a = \|u\|.\end{aligned}$$

If $u \in \partial K(b)$, then $\|u\| = b$ and for $0 \leqslant t \leqslant 1$,

$$\tfrac{1}{3}b\alpha(t) \leqslant u(t) \leqslant \tfrac{1}{2}b\beta(t), \quad \tfrac{1}{2}b\gamma(t) \leqslant u'(t) \leqslant b, \quad bt \leqslant -u''(t) \leqslant b.$$

By the definition of $\psi(t,b)$, one has

$$f(t,u(t),u'(t),u''(t)) \geqslant \psi(t,b), \quad 0<t<1.$$

By Lemma 2.1 (1), one has $G(t,s) \geqslant ts$ for $0 \leqslant t,s \leqslant 1$. It follows that

$$\begin{aligned} \|Tu\| = \|(Tu)''\| &\geqslant \lambda \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)\psi(s,b)\,\mathrm{d}s \\ &\geqslant \frac{b \max\limits_{0\leqslant t\leqslant 1} t}{\int_0^1 sh(s)\psi(s,b)\,\mathrm{d}s} \cdot \int_0^1 sh(s)\psi(s,b)\,\mathrm{d}s = b = \|u\|. \end{aligned}$$

By Lemma 2.5 (1), the operator T has a fixed point $u^* \in K$ and $a \leqslant \|u^*\| \leqslant b$. Since $u^* \in K$, one has $u^*(0)=0$, $(u^*)'(1)=0$ and $u^* \in C^2[0,1]$. Since $Tu^*=u^*$ and by virtue of Lemma 2.3 (1), we get that, for $0 \leqslant t \leqslant 1$,

$$(u^*)''(t) = (Tu^*)''(t) = -\lambda \int_0^1 G(t,s)h(s)f(s,u^*(s),(u^*)'(s),(u^*)''(s))\,\mathrm{d}s.$$

Since $G(0,s) \equiv 0$, one has $(u^*)''(0)=0$. Since $h(s)f(s,u^*(s),(u^*)'(s),(u^*)''(s))$ is integrable on $[0,1]$, differentiating both sides of the above equality we get that, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} (u^*)'''(t) &= -\lambda \int_0^1 \frac{\partial}{\partial t} G(t,s)h(s)f(s,u^*(s),(u^*)'(s),(u^*)''(s))\,\mathrm{d}s \\ &= -\lambda \int_t^1 h(s)f(s,u^*(s),(u^*)'(s),(u^*)''(s))\,\mathrm{d}s. \end{aligned}$$

So, $(u^*)'''(1)=0$. Since $h(\cdot)f(\cdot,u^*(\cdot),(u^*)'(\cdot),(u^*)''(\cdot)) \in C(0,1)$, differentiating both sides of the equality we get that, for $0<t<1$,

$$(u^*)^{(4)}(t) = \lambda h(t)f(t,u^*(t),(u^*)'(t),(u^*)''(t)).$$

Therefore, u^* is a solution of the problem (P1).

Since $u^*(t) \geqslant \|u^*\|\alpha(t) \geqslant \frac{1}{3}a\alpha(t) > 0$, $0<t\leqslant 1$, we see that $u^*(t)$ is a positive solution. Since $(u^*)'(t) \geqslant \frac{1}{2}a\gamma(t) > 0$, $0<t<1$, we see that $u^*(t)$ is a strictly increasing function. □

Imitating the proof of Theorem 3.1, we can prove Theorem 3.2 concerned with twin positive solutions.

Theorem 3.2. *Assume that* (H1)–(H3) *hold and there exist three positive numbers $a_1 < b < a_2$ such that one of the following conditions is satisfied:*

(c1)
$$\frac{b}{\int_0^1 sh(s)\psi(s,b)\,\mathrm{d}s} < \lambda \leqslant \min_{i=1,2} \frac{a_i}{\int_0^1 sh(s)\varphi(s,a_i)\,\mathrm{d}s}.$$

(c2)
$$\max_{i=1,2} \frac{a_i}{\int_0^1 sh(s)\psi(s,a_i)\,\mathrm{d}s} \leqslant \lambda < \frac{b}{\int_0^1 sh(s)\varphi(s,b)\,\mathrm{d}s}.$$

Then problem (P1) *has at least two strictly increasing positive solutions $u_1^*, u_2^* \in K$ such that $a_1 \leqslant \|u_1^*\| < b < \|u_2^*\| \leqslant a_2$.*

Part II. On the problem (P2).

Now, for $0 < t < 1$ and $r > 0$, we use the height functions

$$\begin{aligned}\varphi^\flat(t,r) &= \max\{f(t,x)\colon\ r\alpha(t) \leqslant x \leqslant r\beta(t)\},\\ \psi^\flat(t,r) &= \min\{f(t,x)\colon\ r\alpha(t) \leqslant x \leqslant r\beta(t)\}.\end{aligned}$$

Since $\frac{\partial}{\partial t}J(t,s) \geqslant 0$, $0 \leqslant t,s \leqslant 1$, we see that $\int_0^1 J(t,s)h(s)\varphi^\flat(s,r)\,\mathrm{d}s$ and $\int_0^1 J(t,s)h(s)\psi^\flat(s,r)\,\mathrm{d}s$ are nondecreasing functions with respect to t under the assumptions (H1)$^\flat$–(H3)$^\flat$. And since $J(1,s) = \frac{1}{2}s - \frac{1}{6}s^3$, one has

$$\begin{aligned}\max_{0\leqslant t\leqslant 1} \int_0^1 J(t,s)h(s)\varphi^\flat(s,r)\,\mathrm{d}s &= \frac{1}{6}\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,r)\,\mathrm{d}s,\\ \max_{0\leqslant t\leqslant 1} \int_0^1 J(t,s)h(s)\psi^\flat(s,r)\,\mathrm{d}s &= \frac{1}{6}\int_0^1 s(3-s^2)h(s)\psi^\flat(s,r)\,\mathrm{d}s.\end{aligned}$$

Applying Lemma 2.4 and imitating the proofs of Theorems 3.1 and 3.2, we can prove the following local existence theorems.

Theorem 3.3. *Assume that* (H1)$^\flat$–(H3)$^\flat$ *hold and there exist two positive numbers $a < b$ such that one of the following conditions is satisfied:*

(d1)
$$\frac{6b}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,b)\,\mathrm{d}s} \leqslant \lambda \leqslant \frac{6a}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,a)\,\mathrm{d}s}.$$

(d2)
$$\frac{6a}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,a)\,\mathrm{d}s} \leqslant \lambda \leqslant \frac{6b}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,b)\,\mathrm{d}s}.$$

Then problem (P2) *has at least one strictly increasing positive solution $u^* \in K^\flat$ such that $a \leqslant \|u^*\| \leqslant b$.*

Theorem 3.4. *Assume that* $(\mathrm{H1})^\flat$–$(\mathrm{H3})^\flat$ *hold and there exist three positive numbers* $a_1 < b < a_2$ *such that one of the following conditions is satisfied:*

(e1) $$\frac{6b}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,b)\,\mathrm{d}s} < \lambda \leqslant \min_{i=1,2} \frac{6a_i}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,a_i)\,\mathrm{d}s}.$$

(e2) $$\max_{i=1,2} \frac{6a_i}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,a_i)\,\mathrm{d}s} \leqslant \lambda < \frac{6b}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,b)\,\mathrm{d}s}.$$

Then problem (P2) *has at least two strictly increasing positive solutions* $u_1^*, u_2^* \in K^\flat$ *such that* $a_1 \leqslant \|u_1^*\| < b < \|u_2^*\| \leqslant a_2$.

4. Further discussion

Proposition 4.1. *Theorem* 1.1 *is a special case of Theorem* 3.3.

Proof. We only prove the proposition with the condition (a4)(i), that is,

$$[A \lim_{x\to\infty} f(x)/x]^{-1} < \lambda < [B \lim_{x\to+0} f(x)/x]^{-1}.$$

By (a1) and (a3), the assumptions $(\mathrm{H1})^\flat$–$(\mathrm{H3})^\flat$ are satisfied. Moreover,

$$A = \frac{1}{12}\int_0^1 s^2(3-s^2)^2 h(s)\,\mathrm{d}s = \frac{1}{6}\int_0^1 s(3-s^2)h(s)\alpha(s)\,\mathrm{d}s,$$
$$B = \frac{1}{6}\int_0^1 s^2(2-s)(3-s^2)h(s)\,\mathrm{d}s = \frac{1}{6}\int_0^1 s(3-s^2)h(s)\beta(s)\,\mathrm{d}s.$$

Since $\lim\limits_{x\to+0} f(x)/x < (\lambda B)^{-1}$, there exists $a > 0$ such that $f(x) \leqslant (\lambda B)^{-1}x$, $0 \leqslant x \leqslant a$. Thus, for $0 \leqslant t \leqslant 1$,

$$\varphi^\flat(t,a) \leqslant \max\{(\lambda B)^{-1}x\colon\ a\alpha(t) \leqslant x \leqslant a\beta(t)\} = (\lambda B)^{-1}a\beta(t).$$

It follows that

$$\frac{6a}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,a)\,\mathrm{d}s} \geqslant \frac{6a\lambda B}{a\int_0^1 s(3-s^2)h(s)\beta(s)\,\mathrm{d}s} = \lambda.$$

Since $\lambda A \lim\limits_{x\to\infty} f(x)/x > 1$, there exist $\varepsilon > 0$ and $0 < \sigma < 1$ such that

$$\lambda\Big[\lim_{x\to\infty} f(x)/x - \varepsilon\Big]\Big[\frac{1}{6}\int_\sigma^1 s(3-s^2)h(s)\alpha(s)\,\mathrm{d}s\Big] \geqslant 1.$$

Let $\bar{A} = \frac{1}{6}\int_\sigma^1 s(3-s^2)h(s)\alpha(s)\,\mathrm{d}s$. Then $\lim\limits_{x\to\infty} f(x)/x \geqslant (\lambda\bar{A})^{-1} + \varepsilon$.

Since $\lim_{x\to\infty} f(x)/x > (\lambda\bar{A})^{-1}$, there exists $b_1 > a$ such that $f(x) \geqslant (\lambda\bar{A})^{-1}x$, $b_1 \leqslant x < \infty$. Let $b = b_1\sigma^{-1}$. Then

$$\min_{\sigma\leqslant t\leqslant 1}[b\alpha(t)] \geqslant b \min_{\sigma\leqslant t\leqslant 1} t = b\sigma = b_1,$$

where $\alpha(t) = \frac{1}{2}t(3-t^2) \geqslant t$ for any $0 \leqslant t \leqslant 1$. So, if $\sigma \leqslant t \leqslant 1$ and $b\alpha(t) \leqslant x \leqslant b\beta(t)$, then $x \geqslant b_1$. This shows that

$$\psi^\flat(t,b) \geqslant \min\{(\lambda\bar{A})^{-1}x\colon\ b\alpha(t) \leqslant x \leqslant b\beta(t)\} = (\lambda\bar{A})^{-1}b\alpha(t), \quad \sigma \leqslant t \leqslant 1.$$

Consequently,

$$\begin{aligned}\frac{6b}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,b)\,\mathrm{d}s} &\leqslant \frac{6b}{\int_\sigma^1 s(3-s^2)h(s)\psi^\flat(s,b)\,\mathrm{d}s}\\ &\leqslant \frac{6b\lambda\bar{A}}{b\int_\sigma^1 s(3-s^2)h(s)\alpha(s)\,\mathrm{d}s} = \lambda.\end{aligned}$$

By Theorem 3.3 (d1), the proof is completed. □

E x a m p l e 4.2. Consider the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda\Big[\dfrac{1}{5}u^2(t) + \dfrac{1}{\sqrt{u(t)}}\sin^2\dfrac{\pi}{8t(1-t)}\Big], & 0 < t < 1,\\ u(0) = u'(1) = u''(0) = u'''(1) = 0. \end{cases}$$

Here $h(t) \equiv 1$, $f(t,x) = \frac{1}{5}x^2 + \frac{1}{\sqrt{x}}\sin^2\frac{\pi}{8t(1-t)}$.

For $r_2 > r_1 > 0$, let $j_{r_1}^{r_2}(t) = \frac{1}{5}r_2^2t^2(2-t)^2 + \frac{1}{\sqrt{r_1 t}}$, then $f(t,x,y) \leqslant j_{r_1}^{r_2}(t)$ for any $0 < t < 1$, $\frac{1}{2}r_1t(3-t^2) \leqslant x \leqslant r_2t(2-t)$. So, the assumptions $(\mathrm{H1})^\flat$–$(\mathrm{H3})^\flat$ are satisfied.

Direct computation gives that

$$\begin{aligned}\varphi^\flat(t,r) &\leqslant \max\Big\{\frac{1}{5}x^2 + \frac{1}{\sqrt{x}}\colon\ \frac{1}{2}rt(3-t^2) \leqslant x \leqslant rt(2-t)\Big\}\\ &\leqslant \frac{1}{5}r^2t^2(2-t)^2 + \frac{1}{\sqrt{rt}},\\ \psi^\flat(t,r) &\geqslant \min\Big\{\frac{1}{5}x^2 + \frac{1}{\sqrt{x}}\sin^2\frac{\pi}{8t(1-t)}\colon\ \frac{1}{2}rt(3-t^2) \leqslant x \leqslant rt(2-t)\Big\}\\ &\geqslant \max\Big\{\frac{1}{20}r^2t^2(3-t)^2, \frac{1}{\sqrt{rt(2-t)}}\sin^2\frac{\pi}{8t(1-t)}\Big\}.\end{aligned}$$

Let $a_1 = \frac{1}{3}$, $b = 4$, $a_2 = 20$. Then

$$\begin{aligned}
\frac{6a_1}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,a_1)\,\mathrm{d}s} &\leqslant \frac{2}{\sqrt{3}\int_0^1 \frac{s(3-s^2)}{\sqrt{s(2-s)}}\sin^2\frac{\pi}{8t(1-t)}\,\mathrm{d}s} \\
&\leqslant \frac{8}{3\sqrt{3}\int_{1/4}^{3/4}\sqrt{s}(3-s^2)\,\mathrm{d}s} \approx 1.6757, \\
\frac{6b}{\int_0^1 s(3-s^2)h(s)\varphi^\flat(s,b)\,\mathrm{d}s} &\geqslant \frac{24}{\frac{16}{5}\int_0^1 s^3(3-s^2)(2-s)^2\,\mathrm{d}s + \frac{1}{2}\int_0^1\frac{\mathrm{d}s}{\sqrt{s}}} \\
&= \frac{24}{\frac{16}{5}\cdot\frac{739}{840}+1} \approx 6.2905, \\
\frac{6a_2}{\int_0^1 s(3-s^2)h(s)\psi^\flat(s,a_2)\,\mathrm{d}s} &\leqslant \frac{120}{20\int_0^1 s^3(3-s^2)^3\,\mathrm{d}s} = \frac{240}{131} \approx 1.8321.
\end{aligned}$$

By Theorem 3.4 (e2), the problem has two strictly increasing positive solutions $u_1^*, u_2^* \in K^\flat$ such that $\frac{1}{3} \leqslant \|u_1^*\| < 4 < \|u_2^*\| \leqslant 20$ for any $1.8322 \leqslant \lambda < 6.2904$.

Since the problem has two positive solutions and $f(t,x)$ is singular at $t=0$, $t=1$ and $x=0$, the multiplicity conclusion can not be derived from Theorem 1.1.

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