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Muhuo Liu

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SOME GRAPHS DETERMINED BY THEIR (SIGNLESS) LAPLACIAN SPECTRA

Muhuo Liu, Guangzhou

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Abstract. Let $W_n = K_1 \vee C_{n-1}$ be the wheel graph on n vertices, and let $S(n, c, k)$ be the graph on n vertices obtained by attaching $n - 2c - 2k - 1$ pendant edges together with k hanging paths of length two at vertex v_0, where v_0 is the unique common vertex of c triangles. In this paper we show that $S(n, c, k)$ ($c \geqslant 1$, $k \geqslant 1$) and W_n are determined by their signless Laplacian spectra, respectively. Moreover, we also prove that $S(n, c, k)$ and its complement graph are determined by their Laplacian spectra, respectively, for $c \geqslant 0$ and $k \geqslant 1$.

Keywords: Laplacian spectrum, signless Laplacian spectrum, complement graph

MSC 2010: 05C50, 15A18

1. Introduction

Throughout this paper, $G = (V, E)$ is an undirected simple graph. Let $N(u)$ be the neighbor set of a vertex u, and let $d(u)$ be the degree of the vertex u, namely, $d(u) = |N(u)|$. If $d(u) = 1$, then u is called a *pendant vertex* of G. Suppose the degree of the vertex v_i equals d_i for $i = 1, 2, \ldots, n$. In the sequel, we enumerate the degrees in non-increasing order, i.e., $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Sometimes we write $d_i(G)$ in place of d_i, in order to indicate the dependence on G. As usual, $K_{1,n-1}$, P_n and C_n denote the star, path and cycle of order n, respectively. In particular, K_1 denotes an isolated vertex. The join $G_1 \vee G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph having the vertex set $V(G_1 \vee G_2) = V(G_1 \cup G_2)$ and the edge

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set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv\colon\ u \in V(G_1), v \in V(G_2)\}$. Let W_n be the wheel graph on n vertices, i.e., $W_n = K_1 \vee C_{n-1}$. A graph is a *cactus*, or a *treelike* graph, if any pair of its cycles has at most one common vertex [1], [26]. If all cycles of the cactus G have exactly one common vertex, then G is called a *bundle* [1]. Let $S(n, c, k)$ be the bundle graph obtained by attaching $n - 2c - 2k - 1$ pendant edges together with k hanging paths of length two at the vertex v_0, where v_0 is the unique common vertex of c triangles. For instance, the bundle graph $S(15, 3, 2)$ is shown in Figure 1.

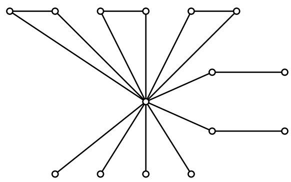

Figure 1. The bundle $S(15, 3, 2)$.

$S(n, c, k)$ have been investigated in many papers. For instance, $S(n, c, 0)$ is the unique graph with the maximal spectral radius [1] (or the Merrifield-Simmons index [19]), the minimal Hosoya index (or the Wiener index [19], the Randić index [19]) in the set of all connected cacti on n vertices with c cycles, and $S(n, 0, \beta - 1)$ is the unique tree with the maximum Laplacian Estrada index [8], and the minimum Laplacian-energy-like invariant [15], (or the Wiener index [9], the hyper-Wiener index [28]) in the class of trees with n vertices and the matching number β, where $2 \leqslant \beta \leqslant \lfloor \frac{1}{2} n \rfloor$. Moreover, $S(n, \frac{1}{2}(n-k) - 1, 1)$ is also an extremal graph [17] with the maximum signless Laplacian spectral radius in the class of connected cacti with n vertices and k pendant vertices. Let $A(G)$ be the adjacency matrix, and $D(G)$ the diagonal matrix of G. Then the *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. Since $L(G)$ is positive semidefinite, its eigenvalues can be arranged as

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G) = 0,$$

where $\lambda_{n-1}(G) > 0$ if and only if G is connected and $\lambda_{n-1}(G)$ is called the *algebraic connectivity* of the graph G [10]. It is easy to see that $Q(G)$ is also positive semidefinite [2] and hence its eigenvalues can be arranged as

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) \geqslant 0.$$

If there is no confusion, sometimes we write $\lambda_i(G)$ as λ_i, and $\mu_i(G)$ as μ_i. Moreover, we sometimes abbreviate $\lambda_1(G)$ and $\mu_1(G)$ as $\mu(G)$ and $\lambda(G)$, respectively, and call $\mu(G)$ and $\lambda(G)$ the signless Laplacian and the Laplacian spectral radius of G, respectively. In the following, let $SL(G)$ and $SQ(G)$ denote the spectra, i.e., the eigenvalues of $L(G)$ and $Q(G)$, respectively.

Two graphs are said to be Q-cospectral (resp. A-cospectral, L-cospectral) if they have the same signless Laplacian (resp. adjacency, Laplacian) spectra. A graph G is said to *be determined by its signless Laplacian spectrum* (*resp. adjacency spectrum, Laplacian spectrum*) if there does not exist other non-isomorphic graph H such that H and G are Q-cospectral (resp. A-cospectral, L-cospectral).

Which graphs are determined by their spectra? This question was proposed by Dam and Haemers in [4]. This research has drawn much attention recently, and more and more results on this item have been reported. For instance, the path, the complement graph of path, the complete graph, the cycle were proved to be determined by their adjacency spectra [4], [7] respectively, and the path, the complete graph, the cycle, the star and the multi-fan graphs, together with their complement graphs were shown to be determined by their Laplacian spectra [4], [7], [22], respectively.

Let K_n^m be the graph obtained by attaching m pendant vertices to a vertex of the complete graph K_{n-m}, and let $U_{n,p}$ be the graph obtained by attaching $n-p$ pendant vertices to a vertex of C_p. Let G^c be the complement graph of G. Recently, Zhang et al. in [29] proved that K_n^m, $U_{n,p}$, $(K_n^m)^c$, $(U_{n,p})^c$ are determined by their Laplacian spectra, respectively. Moreover, they proved that K_n^m, $(K_n^m)^c$ and $U_{n,p}$ are determined by their adjacency spectra if p is odd. The wheel graph W_n was shown to be determined by its Laplacian spectrum [30] except for the case of $n=7$, and the unicyclic graph $G_{r,p}$ on n vertices, obtained by joining a vertex of a cycle C_r and the center of a star $K_{1,p-1}$ to each of the two end vertices of a path P_{n-p-r}, was proved to be determined by its Laplacian spectrum when $r \neq 4$ and r is even [27].

However, only a few families of graphs were shown to be determined by their spectra, most of which were restricted to the cases of adjacency or Laplacian spectra. Thus, it seems rather interesting to consider the problem: Which graphs are determined by their signless Laplacian spectra? Recently, the lollipop graph was proved to be determined by its signless Laplacian spectrum [31], and the bundle graph $S(n,c,k)$ and its complement graph were shown to be determined by their signless Laplacian and Laplacian spectra [21], respectively, for $k=0$ and $c \geqslant 0$.

This article is organized in the following way. In Section 2, we introduce the weak interlacing theorems of Laplacian and signless Laplacian spectra by deleting a vertex. By employing the weak interlacing theorem of the signless Laplacian spectrum and some (new) lower bounds for $\mu_2(G)$, we prove that W_n is determined by its signless Laplacian spectrum in Section 3. By a similar method, we verify that all the $S(n,c,k)$ are determined by their signless Laplacian spectra for $c \geqslant 1$ and $k \geqslant 1$ in Section 4, and $S(n, c, k)$ and its complement graph are also determined by their Laplacian spectra, respectively, for $k \geqslant 1$ and $c \geqslant 0$ in Section 5.

2. The weak interlacing theorem of the (signless) Laplacian spectrum by deleting a vertex

Consider two sequences of real numbers: $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$, and $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_m$ with $m < n$. The latter sequence is said to *interlace* the former whenever $\alpha_i \geqslant \beta_i \geqslant \alpha_{n-m+i}$ for $i = 1, 2, \ldots, m$.

Theorem 2.1 ([13]). *Let G be a graph of order n, and let $G - e$ be the graph obtained from G by deleting the edge e of G. Then*

$$0 \leqslant \mu_n(G-e) \leqslant \mu_n(G) \leqslant \mu_{n-1}(G-e) \leqslant \mu_{n-1}(G) \leqslant \ldots \leqslant \mu_1(G-e) \leqslant \mu_1(G),$$
$$0 = \lambda_n(G-e) = \lambda_n(G) \leqslant \lambda_{n-1}(G-e) \leqslant \lambda_{n-1}(G) \leqslant \ldots \leqslant \lambda_1(G-e) \leqslant \lambda_1(G).$$

Moreover, it is well-known that the adjacency eigenvalues of G and $G - v$ also interlace (see [18], Theorem 1.4.8). Recently, the relation between the Laplacian eigenvalues of G and $G - v$ were considered in [23], and the following result was proved:

Theorem 2.2 ([23]). *Let G be a graph of order n, and let $G - v$ be the graph obtained from G by deleting the vertex v of G. Then $\lambda_{i+1}(G) - 1 \leqslant \lambda_i(G-v) \leqslant \lambda_i(G)$ for each $i = 1, \ldots, n-1$.*

Lotker [23] called Theorem 2.2 the weak interlacing theorem between the Laplacian eigenvalues of G and $G-v$. Now it is natural for us to consider the following problem: What is the relation between the signless Laplacian eigenvalues of G and $G - v$? In the sequel, we shall prove that the weak interlacing theorem also holds for the signless Laplacian eigenvalues between G and $G - v$.

Lemma 2.1 ([12]). *Suppose B is the principal submatrix of a symmetric matrix A. Then the eigenvalues of B interlace the eigenvalues of A.*

Let M be a Hermitian matrix of order n. Denote by $\varrho_1(M) \geqslant \varrho_2(M) \geqslant \ldots \geqslant \varrho_n(M)$ the eigenvalues of M.

Lemma 2.2 ([14] Weyl). *Let A, B be two Hermitian matrices of order n with eigenvalues $\varrho_i(A)$, $\varrho_i(B)$ and $\varrho_i(A+B)$. For each $k = 1, 2, \ldots, n$, we have*

$$\varrho_k(A) + \varrho_n(B) \leqslant \varrho_k(A+B) \leqslant \varrho_k(A) + \varrho_1(B). \tag{2.1}$$

Theorem 2.3. *Let G be a graph of order n, and let $G-v$ be the graph obtained from G by deleting the vertex v of G.*

(1) *If $d(v)=n-1$, then $\mu_n(G)-1\leqslant\mu_{n-1}(G-v)\leqslant\mu_{n-1}(G)-1\leqslant\ldots\leqslant\mu_2(G)-1\leqslant\mu_1(G-v)\leqslant\mu_1(G)-1$.*

(2) *If $d(v)\leqslant n-2$, then $\mu_{i+1}(G)-1\leqslant\mu_i(G-v)\leqslant\mu_i(G)$ for each $i=1,\ldots,n-1$.*

Proof. Let P be the principal submatrix after we delete the row and column that correspond to the vertex v of $Q(G)$. By Lemma 2.1, we have

$$\mu_1(G)\geqslant\varrho_1(P)\geqslant\mu_2(G)\geqslant\varrho_2(P)\geqslant\ldots\geqslant\mu_{n-1}(G)\geqslant\varrho_{n-1}(P)\geqslant\mu_n(G). \tag{2.2}$$

Let $I_v=P-Q(G-v)$. Then I_v is a $(0,1)$ diagonal matrix whose jth diagonal entry is 1 if and only if v_j is connected to v in G.

If $d(v)=n-1$, then I_v is the identity matrix of order $n-1$, and hence $\varrho_{n-1}(I_v)=\varrho_1(I_v)=1$. By inequality (2.1), we have $\varrho_i(P)=\mu_i(G-v)+1$. Then (1) follows from inequality (2.2).

If $d(v)\leqslant n-2$, then $\varrho_{n-1}(I_v)=0$ and $\varrho_1(I_v)=1$. By inequality (2.1), we have $\mu_i(G-v)\leqslant\varrho_i(P)\leqslant\mu_i(G-v)+1$. Then (2) follows from inequality (2.2). □

Remark 2.1. Actually, when $d(v)=n-1$, the result of Theorem 2.2 can be improved to [5]: $\lambda_n(G)=0$, $\lambda_1(G)=n$ and $\lambda_i(G-v)+1=\lambda_{i+1}(G)$ for $n-2\leqslant i\leqslant 1$. Moreover, the proof in [23] seems too difficult, and Theorem 2.2 can be proved analogously to Theorem 2.3.

3. W_n is determined by its signless Laplacian spectrum

In [30] it was proved that W_n, except for W_7, is determined by its Laplacian spectrum. In this section, we shall show that W_n is also determined by its signless Laplacian spectrum.

Lemma 3.1. *For any graph G, if $d_2(G)=d_3(G)$, then*

$$\mu_2(G)\geqslant\min\Big\{d_2(G),\frac{1}{2}\big(d_1(G)+d_2(G)+1-\sqrt{\big(d_1(G)-d_2(G)-2\big)\big(d_1(G)-d_2(G)\big)+9}\big)\Big\}.$$

Moreover, if $d_3(G)=d_2(G)\leqslant d_1(G)-2$, then $\mu_2(G)\geqslant d_2(G)$.

Proof. Suppose $d(v_i)=d_i$ for $1\leqslant i\leqslant 3$. By $u\sim v$, we mean that u is adjacent to v in G. If $v_2\nsim v_3$, then $Q(G)$ has $B=\left(\begin{smallmatrix}d_2&0\\0&d_2\end{smallmatrix}\right)$ as its principal submatrix. By

Lemma 2.1, $\mu_2(G) \geqslant \varrho_2(B) = d_2$. If $v_1 \not\sim v_2$ or $v_1 \not\sim v_3$, then $Q(G)$ has $B = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ as its principal submatrix. By Lemma 2.1, $\mu_2(G) \geqslant \varrho_2(B) = d_2$. If $v_1 \sim v_2$, $v_1 \sim v_3$ and $v_2 \sim v_3$, then $Q(G)$ has $B = \begin{pmatrix} d_1 & 1 & 1 \\ 1 & d_2 & 1 \\ 1 & 1 & d_2 \end{pmatrix}$ as its principal submatrix. By Lemma 2.1, $\mu_2(G) \geqslant \varrho_2(B) = \frac{1}{2}\big(d_1 + d_2 + 1 - \sqrt{(d_1 - d_2 - 2)(d_1 - d_2) + 9}\big)$.

If $d_3 = d_2 \leqslant d_1 - 2$, it is easy to see that $\mu_2 \geqslant d_2$. □

Lemma 3.2 ([2]). *In any bipartite graph G, $L(G)$ and $Q(G)$ have the same eigenvalues. Moreover, the least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.*

Lemma 3.3. $1 \leqslant \mu_n(W_n) \leqslant \mu_2(W_n) < 5$, *and*

$$\mu_1(W_n) = \frac{1}{2}\big(n + 4 + \sqrt{(n-4)^2 + 16}\big).$$

Proof. Suppose a is an eigenvalue of $Q(W_n)$, and $d(v_1) = n - 1$. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to a, and let x_i correspond to v_i, where $1 \leqslant i \leqslant n$. By the equalities $Q(W_n)\mathbf{X} = a\mathbf{X}$ corresponding to $v_2, \ldots, v_n$, we have

$$\tag{3.1} \begin{cases} x_1 + x_3 + x_n = (a-3)x_2, \\ x_1 + x_2 + x_4 = (a-3)x_3, \\ x_1 + x_3 + x_5 = (a-3)x_4, \\ \qquad\vdots \\ x_1 + x_{n-2} + x_n = (a-3)x_{n-1}, \\ x_1 + x_2 + x_{n-1} = (a-3)x_n. \end{cases}$$

Now suppose $a = 5$, then $x_1 = 0$ follows from equalities (3.1). Suppose that $x_k = \max\{x_i \colon 2 \leqslant i \leqslant n\}$. If $k = 2$, then $2x_2 = x_3 + x_n \leqslant 2x_2$, and hence $x_2 = x_3 = x_n$. By equalities (3.1), we have $x_2 = x_3 = \ldots = x_n$. It can be proved similarly that $x_2 = x_3 = \ldots = x_n$ for $k \neq 2$. Moreover, by the equality $Q(W_n)\mathbf{X} = 5\mathbf{X}$ corresponding to v_1 we have $(n-1)x_1 + x_2 + \ldots + x_n = 5x_1$. Thus, $x_1 = x_2 = \ldots = x_n = 0$, a contradiction. So, $a \neq 5$.

By Theorem 2.3, we have $\mu_2(W_n) \leqslant \mu_1(W_n - v_1) + 1 = \mu_1(C_{n-1}) + 1 = 5$. Thus, $\mu_2(W_n) < 5$. Next we shall show that $\mu_n(W_n) \geqslant 1$. On the contrary, suppose $\mu_n(W_n) = a < 1$.

Case 1. $x_1 \neq 0$. Then $(n-1)x_1 = (a-5)(x_2 + x_3 + \ldots + x_n)$ follows from equalities (3.1). Moreover, by the equality $Q(W_n)\mathbf{X} = 5\mathbf{X}$ corresponding to v_1 we have $(n-1)x_1 + x_2 + \ldots + x_n = ax_1$. Thus, $a = \frac{1}{2}\bigl(n + 4 - \sqrt{(n-4)^2 + 16}\bigr)$ because $x_1 \neq 0$ and $a < 1$. But $\frac{1}{2}\bigl(n + 4 - \sqrt{(n-4)^2 + 16}\bigr) \geqslant 1$, a contradiction.

Case 2. $x_1 = 0$. Let $x_j = \min\{x_i\colon\ 2 \leqslant i \leqslant n\}$ and $x_k = \max\{x_i\colon\ 2 \leqslant i \leqslant n\}$. Note that $0 < a < 1$ by Lemma 3.2. Then $x_j < x_k$ by equalities (3.1). If k, $j \in \{3, 4, \ldots, n-1\}$, by the equalities $Q(W_n)\mathbf{X} = a\mathbf{X}$ corresponding to v_j and v_k we have

$$\begin{cases} x_{k-1} + x_{k+1} = (a-3)x_k, \\ x_{j-1} + x_{j+1} = (a-3)x_j. \end{cases}$$

Thus, $x_{k-1} + x_{k+1} - x_{j-1} - x_{j+1} = (a-3)(x_k - x_j) < -2(x_k - x_j)$, which implies that $0 \leqslant (x_{k-1} + x_{k+1} - 2x_j) + (2x_k - x_{j-1} - x_{j+1}) < 0$, a contradiction. This can also yield a contradiction for the other cases.

By combining Case 1 and Case 2, $\mu_n(W_n) \geqslant 1$. From the Perron-Frobenius Theorem on non-negative matrices, $\mu_1(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\mu_1(G)$. Thus, it can be proved analogously to Case 1 that $\mu_1(W_n) = \frac{1}{2}\bigl(n + 4 + \sqrt{(n-4)^2 + 16}\bigr)$. $\square$

Lemma 3.4 ([6])**.** *If G is a graph on n vertices with vertex degrees $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ and signless Laplacian eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$, then $\mu_2 \geqslant \frac{1}{2}\bigl(d_1 + d_2 - \sqrt{(d_1 - d_2)^2 + 4}\bigr) \geqslant d_2 - 1$. Moreover, if G is connected, then $\mu_n < d_n$.*

Lemma 3.5 ([5])**.** *Let G be a connected graph on n vertices. Then $\mu_1(G) \leqslant \max\{d(v) + m(v)\colon\ v \in V\}$, where $m(v) = \sum\limits_{u \in N(v)} d(u)/d(v)$. Moreover, $\mu_1(G) \leqslant d_1(G) + d_2(G)$, where equality holds if and only if G is regular or $G \cong K_{1,n-1}$.*

Lemma 3.6 ([2])**.** *Let G be a graph with n vertices, m edges and t triangles. Then $\sum\limits_{i=1}^{n} \mu_i = \sum\limits_{i=1}^{n} d_i = 2m$, $\sum\limits_{i=1}^{n} \mu_i^2 = 2m + \sum\limits_{i=1}^{n} d_i^2$ and $\sum\limits_{i=1}^{n} \mu_i^3 = 6t + 3\sum\limits_{i=1}^{n} d_i^2 + \sum\limits_{i=1}^{n} d_i^3$.*

Lemma 3.7. *If G and W_n are Q-cospectral, then G is connected with $2 \leqslant d_n(G) \leqslant d_2(G) \leqslant 5$.*

P r o o f. By Lemmas 3.3 and 3.4 we have $d_2(G) - 1 \leqslant \mu_2(G) = \mu_2(W_n) < 5$. So, $d_2(G) \leqslant 5$. Now, we assume that G is disconnected. By Lemmas 3.3 and 3.5, $n < \mu_1(W_n) = \mu_1(G) \leqslant d_1(G) + d_2(G)$. Thus, $n - 4 \leqslant d_1(G) \leqslant n - 2$. Note that $1 \leqslant \mu_n(W_n) = \mu_n(G)$ by Lemma 3.3. So, $d_1(G) = n - 4$ and $G = G_1 \cup C_3$, where $d_1(G_1) = n - 4$. Clearly, $n \geqslant 8$ and hence $\mu_1(G) = \mu_1(G_1)$. Next we shall show that $\mu_1(G_1) < \mu_1(W_n)$. Suppose $\max\{d(v) + m(v)\colon\ v \in V(G_1)\}$ occurs at the vertex u_0. Since $d_2(G_1) \leqslant 5$, we consider the following two cases.

Case 1. $1 \leqslant d(u_0) \leqslant 4$. Then $d(u_0) + m(u_0) \leqslant d(u_0) + d_1(G_1) \leqslant d(u_0) + n - 4 \leqslant n < \mu_1(W_n)$.

Case 2. $d(u_0) = 5$ or $d(u_0) = n - 4$. By Lemma 3.6, G_1 has $2n - 5$ edges. Then

$$\begin{aligned} d(u_0) + m(u_0) &\leqslant d(u_0) + \frac{2(2n-5) - d(u_0)}{d(u_0)} \\ &= d(u_0) - 1 + \frac{4n-10}{d(u_0)} \\ &\leqslant \max\left\{4 + \frac{4n-10}{5}, n - 5 + \frac{4n-10}{n-4}\right\} \\ &< \frac{1}{2}\left(n + 4 + \sqrt{(n-4)^2 + 16}\right). \end{aligned}$$

By Lemmas 3.3 and 3.5, $\mu_1(W_n) = \mu_1(G) = \mu_1(G_1) < \mu_1(W_n)$, a contradiction. Thus, G is connected. Then $1 \leqslant \mu_n(W_n) = \mu_n(G) < d_n(G)$ by Lemmas 3.3 and 3.4, which implies that $d_n(G) \geqslant 2$. □

Lemma 3.8. *If $d_1(G) \leqslant n - 3$, then G and W_n are not Q-cospectral.*

Proof. Next we assume that $d_1(G) \leqslant n - 3$ but $SQ(G) = SQ(W_n)$. By Lemmas 3.3 and 3.7, G is connected with $2 \leqslant d_n(G) \leqslant d_2(G) \leqslant 5$ and $\mu_1(W_n) = \frac{1}{2}\left(n + 4 + \sqrt{(n-4)^2 + 16}\right)$. Suppose $\max\{d(v) + m(v)\colon v \in V(G)\}$ occurs at the vertex u_0.

Case 1. $2 \leqslant d(u_0) \leqslant 3$. Then $d(u_0) + m(u_0) \leqslant d(u_0) + d_1(G) \leqslant d(u_0) + n - 3 \leqslant n < \mu_1(W_n)$, a contradiction.

Case 2. $d(u_0) = 4$. Then $n \geqslant 7$, since $d(u_0) = 4 \leqslant n - 3$.

When $n = 7$, then $d_1(G) = 4$. Note that $\mu_1(G) = \mu_1(W_n) = 8 = 2d_1(G)$. By Lemma 3.5, G is regular and hence G has 14 edges. But W_7 has 12 edges, a contradiction to Lemma 3.6.

When $n = 8$, by Lemmas 3.6–3.7, G also has 14 edges, and $d_n(G) \geqslant 2$. Then $d(u_0) + m(u_0) \leqslant 4 + \frac{28-4-3\times 2}{4} = 8.5 < 6 + 2\sqrt{2} = \mu_1(W_n)$, a contradiction.

Now we suppose that $n \geqslant 9$. Since $d_2(G) \leqslant 5$ by Lemma 3.7, we have $d(u_0) + m(u_0) \leqslant 4 + \frac{n-3+3\times 5}{4} < \mu_1(W_n)$, a contradiction.

Case 3. $5 \leqslant d(u_0) \leqslant n - 3$. By Lemmas 3.6–3.7, G also has $2n - 2$ edges, and $d_n(G) \geqslant 2$. Then

$$d(u_0) + m(u_0) \leqslant d(u_0) + \frac{2m - d(u_0) - 2\times 2}{d(u_0)} = d(u_0) - 1 + \frac{4n-8}{d(u_0)}.$$

Let $f(x) = x - 1 + (4n-8)/x$, where $5 \leqslant x \leqslant n - 3$. It is easy to see that

$$f(x) \leqslant \max\left\{4 + \frac{4n-8}{5}, n - 4 + \frac{4n-8}{n-3}\right\} < \frac{1}{2}\left(n + 4 + \sqrt{(n-4)^2 + 16}\right) = \mu_1(W_n).$$

Thus, $\mu_1(G) < \mu_1(W_n)$, a contradiction.

By combining the above arguments, we can conclude that G and W_n are not Q-cospectral. $\square$

Lemma 3.9. *If G and W_n are Q-cospectral, then $d_1(G) = n-1$.*

Proof. Assume that G has n_i vertices of degree i for $i = 2, \ldots, n-1$. If $SQ(G) = SQ(W_n)$, by Lemmas 3.6–3.8 G is connected with $2 \leqslant d_n(G) \leqslant d_2(G) \leqslant 5$, $n-2 \leqslant d_1(G) \leqslant n-1$ and

$$\sum_{i=2}^{n-1} n_i = n, \ \sum_{i=2}^{n-1} i n_i = 4(n-1), \ \sum_{i=2}^{n-1} i^2 n_i = n^2 + 7n - 8. \tag{3.2}$$

Now assume that $d_1(G) = n-2$.

Case 1. $n_{n-2} \geqslant 2$. Then $4 \leqslant n \leqslant 7$, since $n-2 = d_1(G) = d_2(G) \leqslant 5$. Clearly, $n = 4$ and $n = 5$ are impossible by equalities (3.2).

If $n = 7$, then $d_1(G) = d_2(G) = 5$. By equalities (3.2), we have $n_5 = 2$, $n_3 = 4$, $n_2 = 1$. It is easily checked with the aid of a computer that G and W_n are not Q-cospectral, a contradiction.

If $n = 6$, then $d_1(G) = d_2(G) = 4$. By equalities (3.2), we have $n_4 = 3$, $n_3 = 2$, $n_2 = 1$. It is easily checked with the aid of a computer that G and W_n are not Q-cospectral, a contradiction.

Case 2. $n_{n-2} = 1$.

Subcase 1. $n_5 \geqslant 2$. Then $n \geqslant 8$ because $d_1(G) = n-2 > 5 = d_2(G)$. If $n = 8$, by equalities (3.2) we have $n_4 + 3n_5 = 4$, a contradiction to $n_5 \geqslant 2$. Thus, $n \geqslant 9$. Note that $n-2 \geqslant 7 = d_2(G) + 2$. By Lemmas 3.1 and 3.3, we have $5 > \mu_2(W_n) = \mu_2(G) \geqslant 5$, a contradiction.

Subcase 2. $n_5 = 1$. If $n-2 = 5 > d_2(G)$, then $n = 7$, and hence G is a connected graph with $n_5 = 1$, $n_4 = 3$, $n_3 = 1$ and $n_2 = 2$ by equalities (3.2). It is easily checked that G and W_7 are not Q-cospectral.

If $n-2 > 5$, by equalities (3.2) we have $n_{n-2} = n_5 = 1$, $n_4 = n-7$, $n_3 = 11-n$, $n_2 = n-6$. Thus, $8 \leqslant n \leqslant 11$. If $n = 8$, then $d_1(G) = 6$ and $d_2(G) = 5$. By Lemma 3.4, $4.25 > \mu_2(W_8) = \mu_2(G) > 4.38$, a contradiction.

It can be proved similarly that $9 \leqslant n \leqslant 11$ is also impossible.

Subcase 3. $n_5 = 0$. Since $n_5 = 0$, it is easy to see that $n > 7$ by equalities (3.2). By equalities (3.2), we have $n_{n-2} = 1$, $n_4 = n-4$, $n_3 = 8-n$, $n_2 = n-5$. Then, $n = 8$, and hence G is a connected graph on eight vertices with $n_6 = 1$, $n_4 = 4$, $n_2 = 3$. It is easily checked with the aid of a computer that G and W_n are not Q-cospectral, a contradiction.

By combining the above arguments, we can conclude that $d_1(G) = n-1$. $\square$

Theorem 3.1. *W_n is determined by its signless Laplacian spectrum.*

Proof. If $n = 4$, it is easily checked that the result holds. Thus, we may suppose $n \geqslant 5$ in the sequel. Suppose $SQ(G) = SQ(W_n)$. By Lemmas 3.7 and 3.9, then $2 \leqslant d_n(G) \leqslant d_2(G) \leqslant 5$, $d_1(G) = n - 1$. If $d_2(G) = n - 1$, then $5 \leqslant n \leqslant 6$, and this will yield a contradiction by equalities (3.2). Thus, $d_2(G) < n - 1$, and hence $n_{n-1} = 1$.

By equalities (3.2), we can conclude that G and W_n share the same degree sequences. Thus, $G = K_1 \vee (C_{k_1} \cup C_{k_2} \cup \ldots \cup C_{k_t})$, where $k_1 + k_2 + \ldots + k_t = n - 1$. Now we only need to prove that $t = 1$. On the contrary, assume that $t \geqslant 2$. Choose $v \in V(G)$ with $d(v) = n-1$. By Theorem 2.3, we have $4 = \mu_2(G-v) \leqslant \mu_2(G) - 1 \leqslant \mu_1(G - v) = 4$, and hence $\mu_2(G) = 5$. On the other hand, Lemma 3.3 implies that $\mu_2(G) = \mu_2(W_n) < 5$, a contradiction.

Thus, $t = 1$ and hence $G \cong W_n$. □

4. $S(n, c, k)$ is determined by its signless Laplacian spectrum

In [21], $S(n, c, k)$ was proved to be determined by its signless Laplacian spectrum for $c \geqslant 0$ and $k = 0$. In this section, we shall show that $S(n, c, k)$ is also determined by its signless Laplacian spectrum for $c \geqslant 1$ and $k \geqslant 1$.

Suppose v is a vertex of a connected graph G with at least two vertices. Let $G_{k,l}$ ($l \geqslant k \geqslant 1$) be the graph obtained from G by attaching two new paths P: $v(= v_0)v_1v_2 \ldots v_k$ and Q: $v(= u_0)u_1u_2 \ldots u_l$ of length k and l, respectively, at v, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_l$ are distinct new vertices. Let $G_{k-1,l+1} = G_{k,l} - v_{k-1}v_k + u_lv_k$. The following results have been proved:

Lemma 4.1 ([3], [20])**.** *Let G be a connected graph on $n \geqslant 2$ vertices. If $l \geqslant k \geqslant 1$, then $\mu(G_{k,l}) > \mu(G_{k-1,l+1})$.*

Lemma 4.2 ([24], [25])**.** *If G is a graph on n vertices with at least one edge, then $\mu(G) \geqslant \lambda(G) \geqslant d_1 + 1$. If G is connected, the former equality holds if and only if G is bipartite, the latter holds if and only if $d_1 = n - 1$.*

Lemma 4.3. *For $k \geqslant 1$ and $n \geqslant 4$, $\mu_2(S(n, c, k)) \leqslant 3$ and $n-k < \mu_1(S(n, c, k)) < n - k + 1$. Moreover, if $0 \leqslant c \leqslant 1$, then $\mu_2(S(n, c, k)) < 3$.*

Proof. Let v_1 be the vertex of $S(n, c, k)$ such that $d(v_1) = n - k - 1$. By Theorem 2.3, $\mu_2(S(n, c, k)) \leqslant \mu_1(S(n, c, k) - v_1) + 1 = \mu_1(P_2) + 1 = 3$. Thus, $\mu_2(S(n, c, k)) \leqslant 3$.

Since $n \geqslant 4$, we have $2 + \frac{1}{2}(n-k+1) \leqslant n-k+1$ because $n-k \geqslant 3$. Thus, $n-k < \mu_1(S(n,c,k)) < n-k+1$ follows from Lemmas 3.5 and 4.2.

Assume that $\mu_2(S(n,c,k)) = 3$. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to 3, and x_i let correspond to v_i, where $1 \leqslant i \leqslant n$.

If $c = 1$, suppose $V(C_3) = \{v_1, v_2, v_3\}$. By the equalities $Q(S(n,1,k))\mathbf{X} = 3\mathbf{X}$ corresponding to v_2, v_3, we have $x_1 + x_3 = x_2$, $x_1 + x_2 = x_3$, and hence $x_1 = 0$, $x_2 = x_3$. From the equalities $Q(S(n,1,k))\mathbf{X} = 3\mathbf{X}$ corresponding to $v_4, \ldots, v_n$, we have $x_4 = \ldots = x_n = 0$ because $x_1 = 0$. Moreover, from the equality $Q(S(n,1,k))\mathbf{X} = 3\mathbf{X}$ corresponding to v_1, we have $x_2 = x_3 = 0$. Thus, $\mathbf{X} = (0,0,\ldots,0)^T$, a contradiction. So, $\mu_2(S(n,1,k)) < 3$.

If $c = 0$, suppose $N(v_1) = \{v_2, v_4, \ldots, v_{2k}, v_{2k+2}, v_{2k+3}, \ldots, v_n\}$ and $v_2 \sim v_3$, $v_4 \sim v_5, \ldots, v_{2k} \sim v_{2k+1}$. By the equalities $Q(S(n,0,k))\mathbf{X} = 3\mathbf{X}$ corresponding to $v_2, \ldots, v_n$ we have $x_1 = x_3 = \frac{1}{2}x_2$, $x_1 = x_5 = \frac{1}{2}x_4, \ldots, x_1 = x_{2k+1} = \frac{1}{2}x_{2k}$, $x_1 = 2x_{2k+2}, \ldots, x_1 = 2x_n$. From the equalities $Q(S(n,0,k))\mathbf{X} = 3\mathbf{X}$ corresponding to v_1, we have $\frac{1}{2}(3n-3)x_1 = 3x_1$, and hence $x_1 = 0$. Thus, $\mathbf{X} = (0,0,\ldots,0)^T$, a contradiction. So, $\mu_2(S(n,0,k)) < 3$. $\square$

Lemma 4.4. *Suppose $n \geqslant 4$, $k \geqslant 1$ and $SQ(G) = SQ(S(n,c,k))$. (1) If $c \geqslant 2$, then G is connected with $d_2(G) \leqslant 4$. Moreover, $d_2(G) = 4$ implies that $d_1(G) = d_2(G)$. (2) If $0 \leqslant c \leqslant 1$, then $d_2(G) \leqslant 3$. Moreover, if $c = 1$, then G is connected.*

Proof. (1) By Lemmas 3.4 and 4.3, it follows that

$$d_2(G) - 1 \leqslant \frac{1}{2}\big(d_1 + d_2 - \sqrt{(d_1-d_2)^2+4}\big) \leqslant \mu_2(G) = \mu_2(S(n,c,k)) \leqslant 3.$$

Thus, $d_2(G) \leqslant 4$, and $d_2(G) = 4$ implies that $d_1(G) = d_2(G)$.

By Lemma 3.2, $\mu_n(G) = \mu_n(S(n,c,k)) > 0$. If G is disconnected, then no connected component of G is a tree by Lemma 3.2. Hence, G has at least two connected components, which contain at least one cycle. By Theorem 2.1, we have $\mu_2(S(n,c,k)) = \mu_2(G) \geqslant 4$, a contradiction to Lemma 4.3. So, G is connected.

(2) By Lemma 4.3, (2) can be proved similarly to (1). $\square$

Lemma 4.5. *Suppose $k \geqslant 1$, $n \geqslant 2c + 2k + 3$ and let G be a connected graph with n vertices and $n + c - 1$ edges. If $d_1(G) \leqslant n - k - 2$, then $\mu_1(G) \leqslant n - k$.*

Proof. Suppose $\max\{d(v) + m(v)\colon v \in V\}$ occurs at the vertex u_0 of G.

Case 1. $1 \leqslant d(u_0) \leqslant 2$. Then $d(u_0) + m(u_0) \leqslant d(u_0) + d_1(G) \leqslant n - k$.

Case 2. $3 \leqslant d(u_0) \leqslant n-k-2$. Note that $3 \leqslant d(u_0) \leqslant n-k-2$ and G has $n+c-1$ edges. Since $d_n(G) \geqslant 1$, we have

$$\begin{aligned} d(u_0)+m(u_0) &\leqslant d(u_0)+\frac{2(n+c-1)-d(u_0)-k-1}{d(u_0)} \\ &= d(u_0)-1+\frac{2n+2c-k-3}{d(u_0)} \\ &\leqslant \max\Big\{2+\frac{2n+2c-k-3}{3}, n-k-3+\frac{2n+2c-k-3}{n-k-2}\Big\} \\ &\leqslant n-k. \end{aligned}$$

By Lemma 3.5, the result follows. □

Lemma 4.6. *Suppose $k \geqslant 1$ and $n = 2c+2k+2$. If $d_1(G) \leqslant n-k-2$ and G is connected, then G and $S(n,c,k)$ are not Q-cospectral.*

Proof. We assume that $SQ(G) = SQ(S(n,c,k))$. By Lemma 3.5 and Lemmas 4.3–4.4 we can conclude that G is connected with $d_2(G) \leqslant 4$ and $n-k-3 \leqslant d_1(G) \leqslant n-k-2$.

Case 1. $d_1(G) = n-k-3$. If $d_2(G) \leqslant 3$, then Lemma 3.5 implies that $\mu_1(G) \leqslant n-k < \mu_1(S(n,c,k))$, a contradiction. Thus, $d_2(G) = 4$. So Lemma 4.4 implies that $d_1(G) = d_2(G)$ and $c \geqslant 2$. Thus, $n = k+7$. Since $2+2c+2k = n = k+7$, we have $5 = 2c+k$. Then $c = 2$, $k = 1$ and $n = 8$. By Lemma 3.6, we can conclude that either $n_1 = 6$, $n_2 = -4$, $n_3 = 4$, $n_4 = 2$, or $n_1 = 5$, $n_2 = -1$, $n_3 = 1$, $n_4 = 3$, a contradiction.

Case 2. $d_1(G) = n-k-2$. By Lemmas 3.5 and 4.3, either $d_2(G) = 4$ or $d_2(G) = 3$.

If $d_2(G) = 4$, by Lemma 4.4 we have $d_1(G) = d_2(G) = n-k-2$ and $c \geqslant 2$. Thus, $2+2c+2k = n = k+6$ and hence $4 = 2c+k$, which contradicts $k \geqslant 1$ and $c \geqslant 2$.

Thus, $d_2(G) = 3$. If $d_1(G) = 3$, then $2+2c+2k = n = k+5$. Thus, either $c = 1$, $k = 1$ and $n = 6$ or $c = 0$, $k = 3$ and $n = 8$. By Lemma 3.6, either G is a unicyclic graph with $n_1 = n_3 = 3$ or G is a tree with $n_1 = 5, n_3 = 3$. It is easily checked with the aid of a computer that G and $S(n,c,k)$ are not Q-cospectral. If $d_1(G) \geqslant 4$, by Lemma 3.6 and $n = 2c+2k+2$ it follows that

$$\begin{cases} n_1+n_2+n_3 = 2k+2c+1, \\ n_1+2n_2+3n_3 = 3k+4c+2, \\ n_1+4n_2+9n_3 = 7k+12c+2. \end{cases} \tag{4.1}$$

By equalities (4.1), we have $n_1 = 2k+2c-1$, $n_2 = 3-2c-k$ and $n_3 = k+2c-1$. Since $2c+k = n-k-2 \geqslant 4$, it follows that $n_2 < 0$, a contradiction.

By combining the above arguments, we complete the proof of this result. □

Lemma 4.7. *Suppose $n \geqslant 4$, $k \geqslant 1$ and $n = 2c + 2k + 1$. If $d_1(G) \leqslant n - k - 2$ and G is connected, then G and $S(n, c, k)$ are not Q-cospectral.*

Proof. We assume that $SQ(G) = SQ(S(n, c, k))$. By Lemma 3.5 and Lemmas 4.3–4.4 we can conclude that G is connected with $d_2(G) \leqslant 4$ and $n - k - 3 \leqslant d_1(G) \leqslant n - k - 2$.

Case 1. $d_1(G) = n - k - 3$. By Lemmas 3.5 and 4.3, it follows that $d_2(G) = 4$. Thus, by Lemma 4.4 we have $d_1(G) = d_2(G)$ and $c \geqslant 2$. Hence, $n = k + 7$. Since $1 + 2c + 2k = n = k + 7$, we have $6 = 2c + k$. Then $c = 2$, $k = 2$ and $n = 9$. By Lemma 3.6, we can conclude that $n_1 = 5$, $n_3 = 1$, $n_4 = 3$. By Lemmas 3.5 and 4.3, $\mu_1(G) \leqslant 4 + \frac{8+3+1}{4} = 7 < \mu_1(S(9, 2, 2))$, a contradiction.

Case 2. $d_1(G) = n-k-2$. By Lemmas 3.5 and 4.3, either $d_2(G) = 4$ or $d_2(G) = 3$. If $d_2(G) = 4$, by Lemma 4.4 we have $d_1(G) = d_2(G) = n - k - 2$ and $c \geqslant 2$. Thus, $1 + 2c + 2k = n = k + 6$ and hence $5 = 2c + k$. Then $c = 2$, $k = 1$ and $n = 7$. By Lemma 3.6 we can conclude that $n_1 = n_4 = 2$, and $n_2 = 3$. It is easily checked with the aid of a computer that G and $S(n, c, k)$ are not Q-cospectral, a contradiction.

Thus, $d_2(G) = 3$. If $d_1(G) = 3$, then $1 + 2c + 2k = n = k + 5$ and hence either $c = 1$, $k = 2$ and $n = 7$ or $c = 0$, $k = 4$ and $n = 9$. By Lemma 3.6, G is a unicyclic graph with $n_1 = n_3 = 3$ and $n_2 = 1$ or G is a tree with $n_1 = 5$, $n_2 = 1$ and $n_3 = 3$. It is easily checked with the aid of a computer, a contradiction that G and $S(n, c, k)$ are not Q-cospectral. If $d_1(G) \geqslant 4$, by Lemma 3.6 and $n = 2c + 2k + 1$ we have $n_1 = 2k + 2c - 3$, $n_2 = 5 - 2c - k$ and $n_3 = k + 2c - 2$. Note that $0 \leqslant n_2 = 5 - 2c - k$ and $4 \leqslant d_1(G) = n - k - 2 = 2c + k - 1$. Then $2c + k = 5$. Either $c = 0$, $k = 5$ and $n = 11$ or $c = 1$, $k = 3$ and $n = 9$ or $c = 2$, $k = 1$ and $n = 7$.

If $c = 0$, $k = 5$ and $n = 11$, then G is a tree with $n_1 = 7$, $n_3 = 3$ and $n_4 = 1$. Thus, $Q(G)$ contains $B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ as its principal submatrix. By Lemma 2.1, $\mu_2(G) \geqslant \varrho_2(B) = 3$, which contradicts Lemma 4.3.

If $c = 1$, $k = 3$ and $n = 9$, then G is a unicyclic graph with $n_1 = 5$, $n_3 = 3$ and $n_4 = 1$. Thus, $Q(G)$ contains $B = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ or $B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ as its principal submatrix. By Lemma 2.1, $\mu_2(G) \geqslant \varrho_2(B) = 3$, which contradicts Lemma 4.3.

If $c = 2$, $k = 1$ and $n = 7$, then G is a bicyclic graph with $n_1 = 3$, $n_3 = 3$ and $n_4 = 1$. It is easily checked with the aid of a computer, a contradiction that G and $S(n, c, k)$ are not Q-cospectral.

By combining the above arguments we complete the proof of this result. □

Theorem 4.1. *If $k \geqslant 1$, then $S(n, c, k)$ is determined by its signless Laplacian spectrum for $c \geqslant 1$, and there exists no other tree T such that T and $S(n, 0, k)$ are Q-cospectral.*

P r o o f. If $n \leqslant 4$, it is easily checked that the result holds. Thus, we may suppose $n \geqslant 5$ in the sequel. Now suppose that there exists a graph G such that $SQ(G) = SQ(S(n,c,k))$. By Lemmas 4.3–4.7, we can conclude that G is a connected graph with $d_1(G) = n-k-1$ and $d_2(G) \leqslant 4$.

Case 1. $d_2(G) = d_1(G)$. If $n-k-1 = d_1(G) = d_2(G) \leqslant 3$, then $2c+1+2(n-4) \leqslant 2c+1+2k \leqslant n$. If $c \geqslant 1$, then $n=5$ and $k=c=1$. By Lemma 3.6, we have $n_3 = 1$, a contradiction. If $c=0$, since $d_2(G) = d_1(G)$, by Lemma 3.6 we can conclude that $n=5$, $k=2$, and $G \cong S(5,0,2) = P_5$.

If $n-k-1 = d_1(G) = d_2(G) = 4$, then $n = k+5$. By Lemma 3.6, it follows that

$$\begin{cases} n_1+n_2+n_3+n_4 = n, \\ n_1+2n_2+3n_3+4n_4 = 2(n+c-1), \\ n_1+4n_2+9n_3+16n_4 = 4n+6c. \end{cases} \tag{4.2}$$

By equalities (4.2), we have $n_3+3n_4 = 3$, a contradiction to $n_4 \geqslant 2$.

Case 2. $d_2(G) < d_1(G)$. Then $d_2(G) \leqslant 3$ by Lemma 4.4. By Lemma 3.6, it follows that

$$\begin{cases} n_1+n_2+n_3 = n-1, \\ n_1+2n_2+3n_3 = n+2c+k-1, \\ n_1+4n_2+9n_3 = n+6c+3k-1. \end{cases} \tag{4.3}$$

By equalities (4.3) we have $n_1 = n-2c-k-1$, $n_2 = 2c+k$, and $n_{n-k-1} = 1$, i.e., G is a connected graph with the same degree sequence as $S(n,c,k)$.

By Lemma 3.6, G has exactly c triangles. Let $\mathbb{R}(n,c,k)$ be the set of connected bundle graphs obtained by attaching $n-2c-k-1$ paths to v_0, where v_0 is the unique common vertex of c cycles. Since $n-k-1 \geqslant 3$, G is a graph of $\mathbb{R}(n,c,k)$. By Lemma 4.1, $S(n,c,k)$ is the unique graph with the maximum signless Laplacian spectral radius in $\mathbb{R}(n,c,k)$. Thus, $G \cong S(n,c,k)$ because $\mu_1(G) = \mu_1(S(n,c,k))$.

□

5. $S(n,c,k)$ IS DETERMINED BY ITS LAPLACIAN SPECTRUM

In [21], it was proved that $S(n,c,k)$ and its complement graph are determined by their Laplacian spectra for $k=0$ and $c \geqslant 0$. In this section, we shall show that $S(n,c,k)$ and its complement graph are also determined by their Laplacian spectra for $k \geqslant 1$ and $c \geqslant 0$.

Lemma 5.1. *Let G be a graph with n vertices, m edges and t triangles. Then* $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i = 2m$, $\sum_{i=1}^{n} \lambda_i^2 = 2m + \sum_{i=1}^{n} d_i^2$ *and* $\sum_{i=1}^{n} \lambda_i^3 = \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2 - 6t$.

P r o o f. By $\sum_{i=1}^{n} \lambda_i = \operatorname{Tr}(L)$ and $\sum_{i=1}^{n} \lambda_i^2 = \operatorname{Tr}(L^2) = \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} d_i^2$, the first two equalities hold. Since $L = D - A$, we have $L^3 = D^3 - D^2A - AD^2 - DAD + A^2D + DA^2 + ADA - A^3$. Note that $\operatorname{Tr}(D^2A) = 0$. Then

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^3 = \operatorname{Tr}(L^3) &= \operatorname{Tr}(D^3) - 3\operatorname{Tr}(D^2A) + 3\operatorname{Tr}(A^2D) - \operatorname{Tr}(A^3)\\ &= \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2 - 6t.\end{aligned}$$

Thus, the third equality holds. □

Lemma 5.2. *For $k \geqslant 1$ and $n \geqslant 4$ we have $\lambda_2(S(n,c,k)) \leqslant 3$ and $n - k < \lambda_1(S(n,c,k)) < n - k + 1$.*

P r o o f. By Theorem 2.2 and Lemmas 3.5 and 4.2, this can be proved similarly to Lemma 4.3. □

Lemma 5.3. *If $n \geqslant 4$, $k \geqslant 1$ and $SL(G) = SL(S(n,c,k))$, then G is connected and $d_2(G) \leqslant 3$. Moreover, if $c = 0$, then $d_2(G) \leqslant 2$.*

P r o o f. Since $S(n,c,k)$ is connected, we have $\lambda_{n-1}(G) = \lambda_{n-1}(S(n,c,k)) > 0$ and hence G is connected. It is well known that $d_2(G) \leqslant \lambda_2(G)$ for a connected graph (see [16]). Thus, $d_2(G) \leqslant 3$ by Lemma 5.2. If c=0, by Lemma 3.2 we have $SL(G) = SL(S(n,0,k)) = SQ(S(n,0,k))$. Thus, $d_2(G) \leqslant 2$ by Lemma 4.3. □

Lemma 5.4. *If $n \geqslant 4$, $k \geqslant 1$ and $SL(G) = SL(S(n,c,k))$, then $d_1(G) = n-k-1$.*

P r o o f. Suppose $SQ(G) = SQ(S(n,c,k))$. By Lemmas 4.2 and 5.2, $d_1(G) \leqslant n - k - 1$. Next we assume that $d_1(G) \leqslant n - k - 2$. By Lemma 3.5, Lemma 4.2, and Lemmas 5.2–5.3, we can conclude that G is connected with $d_2(G) = 3$ and $d_1(G) = n - k - 2$, and hence $c \geqslant 1$. Moreover, by Lemma 4.2, Lemma 4.5 and Lemmas 5.1–5.2, we can conclude that either $n = 1 + 2c + 2k$ or $n = 2 + 2c + 2k$.

Case 1. $n = 2 + 2c + 2k$. If $d_1(G) = 3$, since $2 + 2c + 2k = n = k + 5$, we have $3 = 2c + k$. Then, $c = 1$, $k = 1$ and $n = 6$. By Lemma 5.1, G is a unicyclic graph on 6 vertices with $n_1 = n_3 = 3$. It is easily checked with the aid of a computer that G and $S(6,1,1)$ are not L-cospectral, a contradiction.

If $d_1(G) > 3$, by Lemma 5.1 and $n = 2c + 2k + 2$ we have $n_1 = 2k + 2c - 1$, $n_2 = 3 - 2c - k$ and $n_3 = k + 2c - 1$. Since $2c + k = n - k - 2 \geqslant 4$, we have $n_2 < 0$, a contradiction.

Case 2. $n = 1 + 2c + 2k$. It can be proved similarly to Case 1 (or Lemma 4.7).

By combining the above arguments, we complete the proof of this result. $\square$

Lemma 5.5 ([11]). *Let v be a vertex of a connected graph G and suppose that $v_1, \ldots, v_s$ are pendant vertices of G which are adjacent to v. Let G^* be the graph obtained from G by adding any b ($1 \leqslant b \leqslant \frac{1}{2}s(s-1)$) edges between $v_1, \ldots, v_s$. Then $\lambda(G) = \lambda(G^*)$.*

Theorem 5.1. *If $k \geqslant 1$, then $S(n, c, k)$ is determined by its Laplacian spectrum for $c \geqslant 0$.*

P r o o f. If $n \leqslant 4$, it is easily checked that the result holds. Thus, we may suppose $n \geqslant 5$ in the sequel. Now suppose that there exists a graph G such that $SL(G) = SL(S(n, c, k))$. By Lemmas 5.3–5.4, G is a connected graph with $d_1(G) = n - k - 1$ and $d_2(G) \leqslant 3$.

Case 1. $d_1(G) = d_2(G)$. Since $n - k - 1 = d_1(G) = d_2(G) \leqslant 3$, we have $n - 7 \leqslant 2c + 1 + 2(n - 4) \leqslant 2c + 1 + 2k \leqslant n$. Thus, $5 \leqslant n \leqslant 7$. It is easily checked that the result follows by Lemma 5.1.

Case 2. $d_2(G) < d_1(G)$. Since $d_2(G) < d_1(G)$, by Lemma 5.1 we have $n_1 = n - 2c - k - 1$, $n_2 = 2c + k$, and $n_{n-k-1} = 1$, i.e., G is a connected graph with the same degree sequence as $S(n, c, k)$. By Lemma 5.1, G has exactly c triangles. Then G is a bundle graph of $\mathbb{R}(n, c, k)$. Let $\mathbb{T}(n, k)$ denote the set of trees on n vertices obtained by attaching t paths to t pendant vertices of $K_{1,n-k-1}$, where $1 \leqslant t \leqslant k$. Let T be the tree obtained from G by deleting the c edges, the end vertices of which are of degrees two, of c triangles. Then $T \in \mathbb{T}(n, k)$. By Lemmas 3.2 and 5.5, $\lambda(G) = \lambda(T) = \mu(T)$.

Let T^* be the tree obtained from $S(n, c, k)$ by deleting the c edges, the end vertices of which are of degrees two, of c triangles. By Lemma 3.2, Lemma 4.1 and Lemma 5.5, $\mu(T) \leqslant \mu(T^*) = \lambda(T^*) = \lambda(S(n, c, k))$, where $\mu(T) = \mu(T^*)$ if and only if $T \cong T^*$. Thus, if $\lambda(G) = \lambda(S(n, c, k))$, then $T \cong T^*$, which implies that $G \cong S(n, c, k)$. $\square$

Lemma 5.6 ([24]). *Let G be a graph with n vertices. If $\lambda_i(G)$, $i = 1, 2, \ldots, n$ are the eigenvalues of $L(G)$, then the eigenvalues of $L(G^c)$ are $n - \lambda_i(G)$, $i = 1, 2, \ldots, n - 1$ and 0.*

By Theorem 5.1 and Lemma 5.6, we have

Corollary 5.1. *If $k \geqslant 1$, then the complement graph of $S(n,c,k)$ is determined by its Laplacian spectrum for $c \geqslant 0$.*

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References

[1] *B. Borovićanin, M. Petrović*: On the index of cactuses with n vertices. Publ. Inst. Math., Nouv. Sér. *79(93)* (2006), 13–18.

[2] *D. Čvetković, P. Rowlinson, S. K. Simić*: Signless Laplacians of finite graphs. Linear Algebra Appl. *423* (2007), 155–171.

[3] *D. Cvetković, S. K. Simić*: Towards a spectral theory of graphs based on the signless Laplacian II. Linear Algebra Appl. *432* (2010), 2257–2272.

[4] *E. R. van Dam, W. H. Haemers*: Which graphs are determined by their spectrum? Linear Algebra Appl. *373* (2003), 241–272.

[5] *K. Ch. Das*: The Laplacian spectrum of a graph. Comput. Math. Appl. *48* (2004), 715–724.

[6] *K. Ch. Das*: On conjectures involving second largest signless Laplacian eigenvalue of graphs. Linear Algebra Appl. *432* (2010), 3018–3029.

[7] *M. Doob, W. H. Haemers*: The complement of the path is determined by its spectrum. Linear Algebra Appl. *356* (2002), 57–65.

[8] *Z. B. Du, Z. Z. Liu*: On the Estrada and Laplacian Estrada indices of graphs. Linear Algebra Appl. *435* (2011), 2065–2076.

[9] *Z. B. Du, B. Zhou*: Minimum on Wiener indices of trees and unicyclic graphs of the given matching number. MATCH Commun. Math. Comput. Chem. *63* (2010), 101–112.

[10] *M. Fiedler*: Algebraic connectivity of graphs. Czech. Math. J. *23(98)* (1973), 298–305.

[11] *J. M. Guo*: The effect on the Laplacian spectral radius of a graph by adding or grafting edges. Linear Algebra Appl. *413* (2006), 59–71.

[12] *W. H. Haemers*: Interlacing eigenvalues and graphs. Linear Algebra Appl. *226–228* (1995), 593–616.

[13] *J. van den Heuvel*: Hamilton cycles and eigenvalues of graphs. Linear Algebra Appl. *226–228* (1995), 723–730.

[14] *R. A. Horn, C. R. Johnson*: Matrix Analysis. Cambridge University Press XIII, Cambridge, 1985.

[15] *A. Ilić*: Trees with minimal Laplacian coefficients. Comput. Math. Appl. *59* (2010), 2776–2783.

[16] *J. S. Li, Y. L. Pan*: A note on the second largest eigenvalue of the Laplacian matrix of a graph. Linear Multilinear Algebra *48* (2000), 117–121.

[17] *S. C. Li, M. J. Zhang*: On the signless Laplacian index of cacti with a given number of pendant vertices. Linear Algebra Appl. *436* (2012), 4400–4411.

[18] *B. L. Liu*: Combinatorial Matrix Theory. Science Press, Beijing, 2005. (In Chinese.)

[19] *H. Q. Liu, M. Lu*: A unified approach to extremal cacti for different indices. MATCH Commun. Math. Comput. Chem. *58* (2007), 183–194.

[20] *M. H. Liu, X. Z. Tan, B. L. Liu*: The (signless) Laplacian spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices. Czech. Math. J. *60* (2010), 849–867.

[21] *M. H. Liu, B. L. Liu, F. Y. Wei*: Graphs determined by their (signless) Laplacian spectra. Electron. J. Linear Algebra *22* (2011), 112–124.

[22] *X. G. Liu, Y. P. Zhang, X. Q. Gui*: The multi-fan graphs are determined by their Laplacian spectra. Discrete Math. *308* (2008), 4267–4271.

[23] *Z. Lotker*: Note on deleting a vertex and weak interlacing of the Laplacian spectrum. Electron. J. Linear Algebra. *16* (2007), 68–72.

[24] *R. Merris*: Laplacian matrices of graphs: A survey. Linear Algebra Appl. *197–198* (1994), 143–176.

[25] *Y. L. Pan*: Sharp upper bounds for the Laplacian graph eigenvalues. Linear Algebra Appl. *355* (2002), 287–295.

[26] *Z. Radosavljević, M. Rašajski*: A class of reflexive cactuses with four cycles. Publ. Elektroteh. Fak., Univ. Beogr., Ser. Mat. *14* (2003), 64–85.

[27] *X. L. Shen, Y. P. Hou*: A class of unicyclic graphs determined by their Laplacian spectrum. Electron. J. Linear Algebra. *23* (2012), 375–386.

[28] *G. H. Yu, L. H. Feng, A. Ilić*: The hyper-Wiener index of trees with given parameters. Ars Comb. *96* (2010), 395–404.

[29] *X. L. Zhang, H. P. Zhang*: Some graphs determined by their spectra. Linear Algebra Appl. *431* (2009), 1443–1454.

[30] *Y. P. Zhang, X. G. Liu, X. R. Yong*: Which wheel graphs are determined by their Laplacian spectra? Comput Math. Appl. *58* (2009), 1887–1890.

[31] *Y. P. Zhang, X. G. Liu, B. Y. Zhang, X. R. Yong*: The lollipop graph is determined by its Q-spectrum. Discrete Math. *309* (2009), 3364–3369.

Author's address: M. Liu, Department of Applied Mathematics, South China Agricultural University, Guangzhou, 510642, P. R. China, and School of Mathematical Science, Nanjing Normal University, Nanjing, 210097, P. R. China, e-mail: `liumuhuo@163.com`.