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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 4, 937–949

Persistent URL: `http://dml.cz/dmlcz/143037`

Czechoslovak Mathematical Journal, 62 (137) (2012), 937–949

NATURAL DIAGONAL RIEMANNIAN ALMOST PRODUCT AND PARA-HERMITIAN COTANGENT BUNDLES

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(Received February 28, 2011)

Abstract. We obtain the natural diagonal almost product and locally product structures on the total space of the cotangent bundle of a Riemannian manifold. Studying the compatibility and the anti-compatibility relations between the determined structures and a natural diagonal metric, we find the Riemannian almost product (locally product) and the (almost) para-Hermitian cotangent bundles of natural diagonal lift type. Finally, we prove the characterization theorem for the natural diagonal (almost) para-Kählerian structures on the total space of the cotangent bundle.

Keywords: natural lift, cotangent bundle, almost product structure, para-Hermitian structure, para-Kähler structure

MSC 2010: 53C05, 53C15, 53C55

1. Introduction

Some new interesting geometric structures on the total space T^*M of the cotangent bundle of a Riemannian manifold (M, g) were obtained for example in [7], [19], [21]–[23] by considering the natural lifts of the metric from the base manifold to T^*M. Extensive literature, concerning the cotangent bundles of natural bundles, may be found in [12].

The fundamental differences between the geometry of the cotangent bundle and that of the tangent bundle, its dual, are due to the construction of the lifts to T^*M, which is not similar to the definition of the lifts to TM (see [27]).

In a few papers such as [2]–[6], [9]–[11], [17], [24], and [26], some almost product structures and almost para-Hermitian structures (called also almost hyperbolic Her-

The author was supported by the Program POSDRU/89/1.5/S/49944, Universitatea "Alexandru Ioan Cuza" din Iaşi, Romania.

mitian structures) were constructed on the total spaces of the tangent and cotangent bundles.

In 1965, K. Yano initiated in [26] the study of the Riemannian almost product manifolds. A. M. Naveira gave in 1983 a classification of these manifolds with respect to the covariant derivative of the almost product structure (see [20]). In the paper [25] in 1992, M. Staikova and K. Gribachev obtained a classification of the Riemannian almost product manifolds, for which the trace of the almost product structure vanishes, the basic class being that of the almost product manifolds with nonintegrable structure (see [16]).

A classification of the almost para-Hermitian manifolds was made in 1988 by C. Bejan, who obtained in [3] 36 classes, up to duality, and the characterizations of some of them. P. M. Gadea and J. Muñoz Masqué gave in 1991 a classification à la Gray-Hervella, obtaining 136 classes, up to duality (see [8]). Maybe the best known class of (almost) para-Hermitian manifolds are the (almost) para-Kähler manifolds, characterized by the closure of the associated 2-form, and studied for example in [1] and [15].

In the present paper we consider a $(1,1)$-tensor field P obtained as a natural diagonal lift of the metric g from the base manifold M to the total space T^*M of the cotangent bundle. This tensor field depends on four coefficients which are smooth functions of the energy density t. We first determine the conditions under which the tensor field constructed in this way is an almost product structure on T^*M. We obtain some simple relations between the coefficients of P. From the study of the integrability conditions of the determined almost product structure, it follows that the base manifold must be a space form, and two coefficients may be expressed as simple rational functions of the other two coefficients, their first order derivatives, the energy density, and the constant sectional curvature of the base manifold. Then we prove characterization theorems for the cotangent bundles which are Riemannian almost product (locally product) manifolds, or (almost) para-Hermitian manifolds, with respect to the obtained almost product structure, and a natural diagonal lifted metric G. Finally, we obtain the (almost) para-Kähler cotangent bundles of natural diagonal lift type.

Throughout this paper, the manifolds, tensor fields and other geometric objects are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used, the range of the indices h, i, j, k, l, m, r being always $\{1,\dots,n\}$.

2. Preliminary results

The cotangent bundle of a smooth n-dimensional Riemannian manifold may be endowed with a structure of $2n$-dimensional smooth manifold, induced by the structure on the base manifold. If (M,g) is a smooth Riemannian manifold of dimension n, we denote its cotangent bundle by $\pi\colon T^*M \to M$. Every local chart on M, $(U,\varphi) = (U,x^1,\dots,x^n)$, induces a local chart $(\pi^{-1}(U),\Phi) = (\pi^{-1}(U),q^1,\dots,q^n, p_1,\dots,p_n)$ on T^*M, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1,\dots,q^n$ are the local coordinates of its base point $x = \pi(p)$ in the local chart (U,φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi$, $i = 1,\dots n$). The last n local coordinates $p_1,\dots,p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the basis $(\mathrm{d}x^1_{\pi(p)},\dots,\mathrm{d}x^n_{\pi(p)})$, defined by the local chart (U,φ), i.e. $p = p_i\,\mathrm{d}x^i_{\pi(p)}$.

The concept of M-tensor field on the cotangent bundle of a Riemannian manifold was defined by the present author in [7], in the same manner as the M-tensor fields were introduced on the tangent bundle (see [18]).

We recall the splitting of the tangent bundle to T^*M into the vertical distribution $VT^*M = \operatorname{Ker}\pi_*$ and the horizontal one determined by the Levi Civita connection $\dot\nabla$ of the metric g:

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

If $(\pi^{-1}(U),\Phi) = (\pi^{-1}(U),q^1,\dots,q^n,p_1,\dots,p_n)$ is a local chart on T^*M, induced from the local chart $(U,\varphi) = (U,x^1,\dots,x^n)$, the local vector fields $(\partial/\partial p_1),\dots, (\partial/\partial p_n)$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $(\delta/\delta q^1),\dots,(\delta/\delta q^n)$ define a local frame for HT^*M over $\pi^{-1}(U)$, where $(\delta/\delta q^i) = (\partial/\partial q^i) + \Gamma^0_{ih}(\partial/\partial p_h)$, $\Gamma^0_{ih} = p_k\Gamma^k_{ih}$, and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $\{(\partial/\partial p_i),(\delta/\delta q^j)\}_{i,j=\overline{1,n}}$, denoted by $\{\partial^i,\delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on T^*M, adapted to the direct sum decomposition (2.1).

We consider the energy density defined by g in the cotangent vector p:

$$t = \frac{1}{2}\|p\|^2 = \frac{1}{2}g^{-1}_{\pi(p)}(p,p) = \frac{1}{2}g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U).$$

We have $t \in [0,\infty)$ for all $p \in T^*M$.

In the sequel we shall use the following lemma, which may be proved easily.

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on T^*M such that*

$$ug_{ij} + vp_ip_j = 0, \quad ug^{ij} + vg^{0i}g^{0j} = 0, \quad \text{or} \quad u\delta^i_j + vg^{0i}p_j = 0$$

on the domain of any induced local chart on T^*M, then $u = 0$, $v = 0$. We have used the notation $g^{0i} = p_h g^{hi}$.

3. Almost product structures of natural diagonal lift type on the cotangent bundle

In this section we shall find the almost product structures on the (total space of the) cotangent bundle, which are natural diagonal lifts of the metric from the base manifold M to T^*M. Then we shall study the integrability conditions for the determined structures, obtaining the natural diagonal locally product structures on T^*M.

An *almost product structure* J on a differentiable manifold M is a $(1,1)$-tensor field on M such that $J^2 = I$. The pair (M, J) is called an *almost product manifold.* When the almost product structure J is integrable, it is called a *locally product structure*, and the manifold (M, J) is a *locally product manifold.*

An *almost paracomplex manifold* is an almost product manifold (M, J), such that the two eigenbundles associated with the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. Equivalently, a splitting of the tangent bundle TM into the Whitney sum of two subbundles $T^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure* on M.

V. Cruceanu presented in [6] two simple almost product structures on the total space T^*M of the cotangent bundle, obtained by considering on the base manifold M a linear connection ∇ and a non-degenerate $(0,2)$-tensor field g. If α is a differentiable 1-form and X is a vector field on M, α^V denotes the vertical lift of α and X^H the horizontal lift of X to T^*M, one can consider

$$P(X^H) = -X^H, \quad P(\alpha^V) = \alpha^V, \tag{3.1}$$

$$Q(X^H) = (X^\flat)^V, \quad Q(\alpha^V) = (\alpha^\sharp)^H, \tag{3.2}$$

where $X^\flat = g_X$ is the 1-form on M defined by $X^\flat(Y) = g_X(Y) = g(X, Y)$, for all $Y \in \mathcal{T}_0^1(M)$, $\alpha^\sharp = g_\alpha^{-1}$ is a vector field on M defined by $g(\alpha^\sharp, Y) = \alpha(Y)$, for all $Y \in \mathcal{T}_0^1(M)$. P is a paracomplex structure if and only if ∇ has vanishing curvature, while Q is paracomplex if and only if the curvature of ∇ and the exterior covariant differential Dg of g, given by

$$(Dg)(X, Y) = \nabla_X(Y^\flat) - \nabla_Y(X^\flat) - [X, Y]^\flat,$$

vanish.

The results from [13] and [14] concerning the natural lifts allow us to introduce a $(1,1)$-tensor field P on T^*M, which is a natural diagonal lift of the metric g from the base manifold to the total space T^*M of the cotangent bundle. Using the adapted frame $\{\partial^i, \delta_j,\}_{i,j=\overline{1,n}}$ to T^*M, we define P by the relations

$$P\delta_i = P^{(1)}_{ij}\partial^j, \quad P\partial^i = P^{ij}_{(2)}\delta_j, \tag{3.3}$$

where the M-tensor fields involved as coefficients have the forms

$$P^{(1)}_{ij} = a_1(t)g_{ij} + b_1(t)p_ip_j, \quad P^{ij}_{(2)} = a_2(t)g^{ij} + b_2(t)g^{0i}g^{0j}, \tag{3.4}$$

a_1, b_1, a_2, and b_2 being smooth functions of the energy density t.

The invariant expression of the defined structure is

$$\begin{aligned} PX^H_p &= a_1(t)(X^\flat)^V_p + b_1(t)p(X)p^V_p, \\ P\alpha^V_p &= a_2(t)(\alpha^\sharp)^H_p + b_2(t)g^{-1}_{\pi(p)}(p,\alpha)(p^\sharp)^H_p, \end{aligned} \tag{3.5}$$

at every point p of the induced local card $(\pi^{-1}(U),\Phi)$ on T^*M, for all $X \in \mathcal{T}^1_0(M)$, for all $\alpha \in \mathcal{T}^0_1(M)$. The vector $p^\sharp$ is tangent to M in $\pi(p)$, $p^V = p_i\partial^i$ is the Liouville vector field on T^*M, and $(p^\sharp)^H = g^{0i}\delta_i$ is the geodesic spray on T^*M.

Example 3.1. When $a_1 = a_2 = 1$, b_1 and b_2 vanish, we have the structure given by (3.2).

The next theorems present the conditions under which the above tensor field P defines an almost product (locally product) structure on the total space of the cotangent bundle.

Theorem 3.2. *The tensor field P, given by* (3.3) *or* (3.5)*, defines an almost product structure of natural diagonal lift type on T^*M, if and only if its coefficients satisfy the relations*

$$a_1 = \frac{1}{a_2}, \qquad a_1 + 2tb_1 = \frac{1}{a_2 + 2tb_2}. \tag{3.6}$$

Proof. The condition $P^2 = I$ in the definition of the almost product structure may be written in the following form, by using (3.3):

$$P^{(1)}_{ij}P^{il}_{(2)} = \delta^l_j, \qquad P^{ij}_{(2)}P^{(1)}_{il} = \delta^j_l,$$

and substituting (3.4) it becomes

$$(a_1a_2 - 1)\delta^l_j + [b_1(a_2 + 2tb_2) + a_1b_2]g^{0l}p_j = 0.$$

Using Lemma 2.1, we have that the above expression vanishes if and only if

$$a_1 = \frac{1}{a_2}, \qquad b_1 = -\frac{a_1 b_2}{a_2 + 2tb_2},$$

which implies also the second relation in (3.6). $\square$

Remark 3.3. If $a_1 = \beta^{-1}$, $a_2 = \beta$, $b_1 = u(\alpha\beta)^{-1}$ and $b_2 = -u\beta(\alpha + 2tu)^{-1}$, where α and β are real constants and u is a smooth function of t, the statements of Theorem 3.2 are satisfied, so the structure considered in [24] is an almost product structure on the total space T^*M of the cotangent bundle.

Theorem 3.4. *The natural diagonal almost product structure P on the total space of the cotangent bundle of an n-dimensional connected Riemannian manifold (M, g), with $n > 2$, is a locally product structure on T^*M (i.e. P is integrable) if and only if the base manifold is of constant sectional curvature c, and the coefficients b_1, b_2 are given by:*

$$b_1 = \frac{a_1 a_1' + c}{a_1 - 2ta_1'}, \qquad b_2 = \frac{a_1 a_2' - a_2^2 c}{a_1 + 2cta_2}. \tag{3.7}$$

P r o o f. The almost product structure P on T^*M is integrable if and only if the vanishing condition for the Nijenhuis tensor field N_P,

$$N_P(X, Y) = [PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y], \quad \forall\, X, Y \in \mathcal{T}_0^1(T^*M),$$

is satisfied.

Studying the components of N_P with respect to the adapted frame on T^*M, $\{\partial^i, \delta_j\}_{i,j=\overline{1,n}}$, we first obtain

$$N_P(\partial^i, \partial^j) = [P^{(1)}_{km}(\partial^j P^{mi}_{(2)} - \partial^i P^{mj}_{(2)}) + \mathrm{Rim}^0_{kml} P^{mi}_{(2)} P^{lj}_{(2)}]\partial^k.$$

Substituting the values (3.4), after some tensorial computations the above expression becomes

$$a_1(a_2' - b_2)(\delta^h_i p_j - \delta^h_j p_i) - a_2^2 \mathrm{Rim}^h_{kij} + a_2 b_2(\mathrm{Rim}^h_{kjl} p_i - \mathrm{Rim}^h_{kil} p_j) g^{0k} g^{0l} = 0. \tag{3.8}$$

We differentiate (3.8) with respect to p_h. Since the curvature of the base manifold does not depend on p, we take the value of this derivative at $p = 0$, and we obtain

$$R^h_{kij} = c(\delta^h_i g_{kj} - \delta^h_j g_{ki}), \tag{3.9}$$

where

$$c = \frac{a_1(0)}{a_2^2(0)}(a_2'(0) - b_2(0))$$

is a function depending on $q^1, \dots, q^n$ only. Schur's theorem implies that c must be a constant when M is connected, of dimension $n > 2$.

Moreover, by virtue of (3.9), the relation (3.8) becomes

$$[a_1 a_2' - a_2^2 c - b_2(a_1 + 2ta_2 c)](\delta_i^h g_{kj} - \delta_j^h g_{ki}) = 0. \tag{3.10}$$

Solving (3.10) with respect to b_2, we obtain the second relation in (3.7).

The Nijenhuis tensor field computed for both horizontal arguments is

$$N_P(\delta_i, \delta_j) = (P_{li}^{(1)} \partial^l P_{hj}^{(1)} - P_{lj}^{(1)} \partial^l P_{hi}^{(1)} + \mathrm{Rim}_{hij}^0)\partial^h,$$

which vanishes if and only if

$$[b_1(2ta_1' - a_1) + a_1' a_1 + c](g_{hj} p_i - g_{hi} p_j) = 0,$$

namely, when b_1 has the form in (3.7).

The mixed components of the Nijenhuis tensor field have the form

$$N_P(\delta_i, \partial^j) = -N_P(\partial^j, \delta_i) = (P_{mi}^{(1)} \partial^m P_{(2)}^{hj} + P_{(2)}^{hl} \partial^j P_{li}^{(1)} - P_{(2)}^{lh} P_{(2)}^{jm} \mathrm{Rim}_{lim}^0)\delta_h,$$

which after substituting (3.4) and (3.9) become

$$\begin{aligned}
&(a_1 a_2' + a_2 b_1 - a_2^2 c + 2ta_2' b_1) g^{hj} p_i \\
&\quad + (a_2 b_1 + a_1 b_2 + 2tb_1 b_2)\delta_i^j g^{0h} + (a_1' a_2 + a_1 b_2 + a_2^2 c + 2cta_2 b_2)\delta_i^h g^{0j} \\
&\quad + (a_2 b_1' + a_1' b_2 + 3b_1 b_2 + a_1 b_2' - a_2 b_2 c + 2tb_1' b_2 + 2tb_1 b_2') p_i g^{0h} g^{0j}.
\end{aligned}$$

Taking (3.6) into account, the above expression takes the form

$$\begin{aligned}
&\frac{(a_1 - 2a_1' t)b_1 - a_1 a_1' - c}{a_1^2} g^{hj} p_i + \frac{a_1 a_1' + c - (a_1 - 2a_1' t)b_1}{a_1(a_1 + 2tb_1)} \delta_i^h g^{0j} \\
&\qquad + \frac{(a_1 a_1' + c)b_1 - (a_1 - 2ta_1')b_1^2}{a_1^2(a_1 + 2tb_1)} p_i g^{0h} g^{0j}
\end{aligned}$$

and it vanishes if and only if b_1 is expressed by the first relation in (3.7).

One can verify that all the components of the Nijenhuis tensor field vanish under the same conditions, so the almost product structure P on T^*M is integrable, i.e. P is a locally product structure on T^*M. □

Remark 3.5. If the coefficients involved in the definition of P have the values presented in Remark 3.3, the relations (3.7) take the form $u = c\alpha\beta^2$, so Theorem 3.4 implies the results stated in [24, Theorem 4.2].

4. Natural diagonal Riemannian almost product and almost para-Hermitian structures on T^*M

Authors like M. Anastasiei, C. Bejan, V. Cruceanu, H. Farran, A. Heydari, S. Ishihara, I. Mihai, G. Mitric, C. Nicolau, V. Oproiu, L. Ornea, N. Papaghiuc, E. Peyghan, K. Yano, and M. S. Zanoun considered almost product structures and almost para-Hermitian structures (called also almost hyperbolic Hermitian structures) on the total spaces of the tangent and cotangent bundles.

A Riemannian manifold (M,g), endowed with an almost product structure J satisfying the relation

$$g(JX,JY)=\varepsilon g(X,Y),\quad \forall\, X,Y\in \mathcal{T}_0^1(M), \tag{4.1}$$

is called a *Riemannian almost product manifold* if $\varepsilon=1$, or an *almost para-Hermitian manifold* (called also an *almost hyperbolic Hermitian manifold*) if $\varepsilon=-1$.

In the following we shall find the Riemannian almost product (locally product) and the (almost) para-Hermitian cotangent bundles of natural diagonal lift type.

To this aim, we consider a natural diagonal lifted metric on the total space T^*M of the cotangent bundle, defined by

$$\begin{cases} G_p(X^H,Y^H)=c_1(t)g_{\pi(p)}(X,Y)+d_1(t)p(X)p(Y),\\ G_p(\alpha^V,\omega^V)=c_2(t)g^{-1}_{\pi(p)}(\alpha,\omega)+d_2(t)g^{-1}_{\pi(p)}(p,\alpha)g^{-1}_{\pi(p)}(p,\omega),\\ G_p(X^H,\alpha^V)=G_p(\alpha^V,X^H)=0,\end{cases} \tag{4.2}$$

for every $X,Y\in\mathcal{T}_0^1(M)$, $\alpha,\omega\in\mathcal{T}_1^0(M)$, $p\in T^*M$, where the coefficients c_1, c_2, d_1, d_2 are smooth functions of the energy density.

The conditions for G to be nondegenerate are enssured if

$$c_1c_2\neq 0,\quad (c_1+2td_1)(c_2+2td_2)\neq 0.$$

The metric G is positive definite if

$$c_1+2td_1>0,\quad c_2+2td_2>0.$$

Using the adapted frame $\{\partial^i,\delta_j\}_{i,j=\overline{1,n}}$ on T^*M, (4.2) becomes

$$\begin{cases} G(\delta_i,\delta_j)=G^{(1)}_{ij}=c_1(t)g_{ij}+d_1(t)p_ip_j,\\ G(\partial^i,\partial^j)=G^{ij}_{(2)}=c_2(t)g^{ij}+d_2(t)g^{0i}g^{0j},\\ G(\partial^i,\delta_j)=G(\delta_i,\partial^j)=0,\end{cases} \tag{4.3}$$

where c_1, c_2, d_1, d_2 are smooth functions of the density energy on T^*M.

Next we shall prove the following characterization theorem:

Theorem 4.1. *Let (M, g) be an n-dimensional connected Riemannian manifold, with $n > 2$, and T^*M the total space of its cotangent bundle. Let G be a natural diagonal lifted metric on T^*M, defined by (4.2), and P an almost product structure on T^*M, characterized by Theorem 3.2. Then (T^*M, G, P) is a Riemannian almost product manifold, or an almost para-Hermitian manifold if and only if the following proportionality relations between the coefficients hold:*

$$\frac{c_1}{a_1} = \varepsilon\frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \varepsilon\frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu, \tag{4.4}$$

where ε takes the corresponding values from the definition (4.1), and the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are some functions depending on the energy density t.

If, moreover, the relations stated in Theorem 3.4 are fulfilled, then (T^*M, G, P) is a Riemannian locally product manifold for $\varepsilon = 1$, or a para-Hermitian manifold for $\varepsilon = -1$.

P r o o f. With respect to the adapted frame $\{\partial^i, \delta_j\}_{i,j=\overline{1,n}}$, the relation (4.1) becomes:

$$G(P\delta_i, P\delta_j) = \varepsilon G(\delta_i, \delta_j), \quad G(P\partial^i, P\partial^j) = \varepsilon G(\partial^i, \partial^j), \quad G(P\partial^i, P\delta_j) = 0, \tag{4.5}$$

and using (3.3) and (4.3) we have

$$(-\varepsilon c_1 + a_1^2 c_2)g_{ij} + [-\varepsilon d_1 + a_1^2 d_2 + 2b_1c_2(a_1 + tb_1) + 4tb_1d_2(a_1 + tb_1)]p_ip_j = 0,$$
$$(a_2^2 c_1 - \varepsilon c_2)g^{ij} + [-\varepsilon d_2 + a_2^2 d_1 + 2b_2c_1(a_2 + tb_2) + 4tb_2d_1(a_2 + tb_2)]g^{0i}g^{0j} = 0.$$

Taking into account Lemma 2.1, the coefficients which appear in the above expressions vanish. Due to the first relation in (3.6), we get by equalizing to zero the coefficients of g_{ij} and g^{ij} the first relation in (4.4).

Moreover, multiplying by $2t$ the coefficients of p_ip_j and $g^{0i}g^{0j}$ and adding them to the coefficients of g_{ij} and g^{ij}, respectively, we obtain

$$\begin{aligned} &-\varepsilon(c_1 + 2td_1) + (a_1 + 2tb_1)^2(c_2 + 2td_2) = 0,\\ &(a_2 + 2tb_2)^2(c_1 + 2tb_1) - \varepsilon(c_2 + 2td_2) = 0. \end{aligned} \tag{4.6}$$

Using the second relation in (3.6), (4.6) leads to the second relation in (4.4). □

Remark 4.2. When the coefficients of the almost product structure P have the expressions in Remark 3.3, and the coefficients of the metric G on T^*M are $c_1 = a_1$, $d_1 = b_1$, $c_2 = -a_2$ and $d_2 = -b_2$, Theorem 4.1 implies that T^*M endowed with the almost product structure and with the metric considered in [24] is an almost para-Hermitian manifold.

5. Natural diagonal para-Kähler structures on T^*M

In the sequel we shall study the cotangent bundles endowed with para-Kähler structures of natural diagonal lift type. This class of almost para-Hermitian structures, studied for example in [1] and [15], is characterized by the closedness of the associated 2-form Ω.

The 2-form Ω associated with the almost para-Hermitian structure (G,P) of natural diagonal lift type on the total space of the cotangent bundle is given by the relation

$$\Omega(X,Y) = G(X,PY), \quad \forall\, X,Y \in \mathcal{T}_0^1(T^*M).$$

Studying the closedness of Ω, we may prove the following theorem:

Theorem 5.1. *The almost para-Hermitian structure (G,P) of natural diagonal lift type on the total space T^*M of the cotangent bundle of a Riemannian manifold (M,g) is almost para-Kählerian if and only if*

$$\mu = \lambda'.$$

Proof. The 2-form Ω on T^*M has the following expression with respect to the local adapted frame $\{\partial^i, \delta_j\}_{i,j=1,\dots,n}$:

$$\Omega(\partial^i,\partial^j) = \Omega(\delta_i,\delta_j) = 0, \quad \Omega(\partial^j,\delta_i) = G^{jh}_{(2)} P^{(1)}_{hi}, \quad \Omega(\delta_i,\partial^j) = G^{(1)}_{ih} P^{hj}_{(2)}.$$

By substituting (3.4) and (4.3) in the above expressions, and taking into account the conditions for (T^*M,G,P) to be an almost para-Hermitian manifold (see Theorem 4.1), we have

$$\Omega(\delta_i,\partial^j) = -\Omega(\partial^j,\delta_i) = \lambda\delta_i^j + \mu p_i g^{0j}, \tag{5.1}$$

which has the invariant expression

$$\Omega(X^H_p,\alpha^V_p) = \lambda\alpha(X) + \mu p(X) g^{-1}_{\pi(p)}(p,\alpha)$$

for every $X \in \mathcal{T}_0^1(M)$, $\alpha \in \mathcal{T}_1^0(M)$, $p \in T^*M$.

Taking into account the relation (5.1) we have that the 2-form Ω associated to the natural diagonal para-Hermitian structure has the form

$$\Omega = (\lambda\delta_i^j + \mu p_i g^{0j})\,\mathrm{d}q^i \wedge \mathrm{D}p_j, \tag{5.2}$$

where $\mathrm{D}p_j = \mathrm{d}p_j - \Gamma_{jh}^0\,\mathrm{d}q^h$ is the absolute differential of p_i.

Moreover, the differential of Ω will be

$$\begin{aligned}\mathrm{d}\Omega = (\mathrm{d}\lambda\delta_i^j + \mathrm{d}\mu g^{0j} p_i + \mu\,\mathrm{d}g^{0j} p_i + \mu g^{0j}\,\mathrm{d}p_i) \wedge \mathrm{d}q^i \\ \wedge \mathrm{D}p_j - (\lambda\delta_i^j + \mu p_i g^{0j})\,\mathrm{d}q^i \wedge \mathrm{d}\mathrm{D}p_j.\end{aligned}$$

Let us compute the expressions of $\mathrm{d}\lambda$, $\mathrm{d}\mu$, $\mathrm{d}g^{0i}$ and $\mathrm{d}\mathrm{D}p_i$:

$$\mathrm{d}\lambda = \lambda' g^{0h}\mathrm{D}p_h, \quad \mathrm{d}\mu = \mu' g^{0h}\mathrm{D}p_h, \quad \mathrm{d}g^{0i} = g^{hi}\mathrm{D}p_h - \Gamma_{j0}^i\,\mathrm{d}q^j,$$
$$\mathrm{d}\mathrm{D}p_i = \frac{1}{2}R_{ijh}^0\,\mathrm{d}q^h \wedge \mathrm{d}q^j - \Gamma_{ij}^h\mathrm{D}p_h \wedge \mathrm{d}q^j.$$

Then, by substituting these relations into the expression of $\mathrm{d}\Omega$, taking into account the properties of the external product, the symmetry of g^{ij} and Γ_{ij}^h and the Bianchi identities, we obtain

$$\mathrm{d}\Omega = (\mu - \lambda')p_k g^{kh}\delta_j^i\mathrm{D}p_h \wedge \mathrm{D}p_i \wedge \mathrm{d}q^j,$$

which, due to the antisymmetry of $\delta_j^i\mathrm{D}p_i \wedge \mathrm{d}q^j$, may be written as

$$\mathrm{d}\Omega = \frac{1}{2}(\mu - \lambda')p_k(g^{kh}\delta_i^j - g^{kj}\delta_i^h)\mathrm{D}p_h \wedge \mathrm{D}p_j \wedge \mathrm{d}q^i,$$

and it vanishes if and only if $\mu = \lambda'$. □

Using Theorems 3.2, 3.4 and 5.1, we immediately prove

Theorem 5.2. *An almost para-Hermitian structure (G, P) of natural diagonal lift type on the total space T^*M of the cotangent bundle of a Riemannian manifold (M, g) is para-Kählerian if and only if P is a locally product structure (see Theorem 3.4) and $\mu = \lambda'$.*

Remark 5.3. The almost para-Kählerian structures of natural diagonal lift type on T^*M depend on three essential coefficients a_1, b_1 and λ, while the natural diagonal para-Kählerian structures on T^*M depend on two essential coefficients a_1 and λ, which in both cases must satisfy the supplementary conditions $a_1 > 0$, $a_1 + 2tb_1 > 0$, $\lambda > 0$, $\lambda + 2t\lambda' > 0$, where b_1 is given by (3.7).

Remark 5.4. Taking into account Remark 4.2, we have that the constant λ is equal to 1 and μ vanishes, so Theorem 5.1 leads to the statements in [24, Theorem 3.1], namely, the structure constructed in [24] is almost para-Kählerian on T^*M. Moreover, taking into account Remark 3.5, it follows that the relations (14) in [24] are fulfilled in the case when the constructed structure is para-Kähler.

Acknowledgement. The author would like to thank Professor V. Oproiu for the techniques learned and Professor M. I. Munteanu for the critical remarks during the preparation of this paper.

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