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THE M_α AND C-INTEGRALS

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Abstract. In this paper, we define the M_α-integral of real-valued functions defined on an interval $[a,b]$ and investigate important properties of the M_α-integral. In particular, we show that a function $f\colon [a,b] \to R$ is M_α-integrable on $[a,b]$ if and only if there exists an ACG_α function F such that $F' = f$ almost everywhere on $[a,b]$. It can be seen easily that every McShane integrable function on $[a,b]$ is M_α-integrable and every M_α-integrable function on $[a,b]$ is Henstock integrable. In addition, we show that the M_α-integral is equivalent to the C-integral.

Keywords: M_α-integral, ACG_α function

MSC 2010: 26A39

1. Introduction and Preliminaries

It is well-known [3] that a function $f\colon [a,b] \to \mathbb{R}$ is C-integrable on $[a,b]$ if and only if there exists an ACG_c function F such that $F' = f$ almost everywhere on $[a,b]$.

In this paper, for a fixed positive real number α we define the M_α-integral and prove that a function $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$ if and only if there exists an ACG_α function F such that $F' = f$ almost everywhere on $[a,b]$.

In particular, we show that a function $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$ if and only if f is C-integrable on $[a,b]$, and the integrals are equal.

A gauge on the interval $[a,b] \subset \mathbb{R}$ is a positive function defined on $[a,b]$. Given a gauge δ, a δ-fine division of $[a,b]$ is a collection $\{(I_i, x_i)\colon i = 1,2,\ldots,n\}$ of pairwise non-overlapping intervals $I_i \subset [a,b]$ such that $\bigcup_{i=1}^n I_i = [a,b]$, $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $x_i \in [a,b]$. If $\bigcup_{i=1}^n I_i \subset [a,b]$, then the collection $\{(I_i, x_i)\colon i = 1,2,\ldots,n\}$

is called a δ-fine partial division of $[a,b]$ and the points $\{x_i\}$ are called the tags of the partial division $\{(I_i,x_i)\}$.

Given a function $f\colon\ [a,b]\to\mathbb{R}$ and a partial division $D=\{(I_i,x_i)\colon\ 1\leqslant i\leqslant n\}$, we use the following notation:

$$f(D)=\sum_{i=1}^{n} f(x_i)|I_i| \quad\text{and}\quad \varrho(D)=\sum_{i=1}^{n}\operatorname{dist}(x_i,I_i),$$

where $|I_i|$ is the Lebesgue measure of the interval I_i and $\operatorname{dist}(x_i,I_i)=\inf\{|t-x_i|\colon t\in I_i\}$.

2. The M_α-integral

We now present the definition of the M_α-integal.

Definition 2.1. Let $\alpha>0$ be a constant. A function $f\colon\ [a,b]\to\mathbb{R}$ is M_α-integrable if there exists a real number A such that for each $\varepsilon>0$ there exists a positive function $\delta\colon\ [a,b]\to\mathbb{R}^+$ such that

$$|f(D)-A|<\varepsilon$$

for each $\delta-$fine division $D=\{(I_i,x_i)\}_{i=1}^{n}$ of $[a,b]$ satisfying the condition $\varrho(D)<\alpha$. The number A is called the M_α-integral of f on $[a,b]$, and we write $A=\int_a^b f$ or $A=(M_\alpha)\int_a^b f$.

The function f is M_α-integrable on the set $E\subset[a,b]$ if the function $f\chi_E$ is M_α-integrable on $[a,b]$, and we write $\int_E f=\int_a^b f\chi_E$.

We can easily get some basic properties of the M_α-integral.

Theorem 2.2. *Let $f\colon\ [a,b]\to\mathbb{R}$. Then*

(1) *If f is M_α-integrable on $[a,b]$, then f is M_α-integrable on every subinterval of $[a,b]$.*
(2) *If f is M_α-integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is M_α-integrable on $[a,b]$ and $\int_a^c f+\int_c^b f=\int_a^b f$.*

The following theorem shows the linearity properties of the M_α-integral.

Theorem 2.3. *Let f and g be M_α-integrable functions on $[a,b]$. Then*

(1) *kf is M_α-integrable on $[a,b]$ and $\int_a^b kf=k\int_a^b f$ for each $k\in\mathbb{R}$,*
(2) *$f+g$ is M_α-integrable on $[a,b]$ and $\int_a^b(f+g)=\int_a^b f+\int_a^b g$.*

The following lemma is used frequently in the theory of the M_α-integral.

Lemma 2.4 (Saks-Henstock Lemma). *Let $f\colon [a,b] \to \mathbb{R}$ be M_α-integrable on $[a,b]$ and let $\varepsilon > 0$. Suppose that δ is a gauge on $[a,b]$ such that*

$$\left| f(D) - \int_a^b f \right| < \varepsilon$$

for each δ-fine division $D = \{(I_i, x_i)\}$ of $[a,b]$ satisfying the condition $\varrho(D) < \alpha$. If $D' = \{(I_i, x_i)\}_{i=1}^m$ is a δ-fine partial division of $[a,b]$ satisfying the condition $\varrho(D') < \alpha$, then

$$\left| f(D') - \sum_{i=1}^m \int_{I_i} f \right| \leqslant \varepsilon.$$

Proof. Assume that $D' = \{(I_i, x_i)\}_{i=1}^m$ is an arbitrary δ-fine partial division of $[a,b]$ satisfying the condition $\varrho(D') < \alpha$. Let $[a,b] - \bigcup\limits_{i=1}^m I_i = \bigcup\limits_{j=1}^k I'_j$.

Let $\eta > 0$. Since f is M_α-integrable on each I'_j, there exists a gauge $\delta_j\colon I'_j \to \mathbb{R}^+$ such that

$$\left| f(D_j) - \int_{I'_j} f \right| < \frac{\eta}{k}$$

for each δ_j-fine division D_j of I'_j satisfying the condition $\varrho(D_j) < \alpha$.

We may assume that $\delta_j(x) \leqslant \delta(x)$ for all $x \in I'_j$. For each j, choose a δ_j-fine division D_j of I'_j with $\varrho(D_j) < (\alpha - \varrho(D'))/k$. Let $D_0 = D' \cup D_1 \cup \ldots \cup D_k$. Then D_0 is a δ-fine division of $[a,b]$ satisfying $\varrho(D_0) < \alpha$ and we have

$$\left| f(D_0) - \int_a^b f \right| < \varepsilon.$$

Consequently, we have

$$\begin{aligned}\left| f(D') - \sum_{i=1}^m \int_{I_i} f \right| &= \left| f(D_0) - \sum_{j=1}^k f(D_j) - \left(\int_a^b f - \sum_{j=1}^k \int_{I'_j} f \right) \right| \\ &\leqslant \left| f(D_0) - \int_a^b f \right| + \sum_{j=1}^k \left| f(D_j) - \int_{I'_j} f \right| \\ &< \varepsilon + k \cdot \frac{\eta}{k} = \varepsilon + \eta.\end{aligned}$$

Since $\eta > 0$ was arbitrary, we have $|f(D') - \sum\limits_{i=1}^m \int_{I_i} f| \leqslant \varepsilon$. □

If $F\colon [a,b] \to \mathbb{R}$, then F can be treated as a function of intervals by defining $F([c,d]) = F(d) - F(c)$ for each subinterval $[c,d] \subset [a,b]$.

Theorem 2.5. *If the function $F\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ with $f(x) = F'(x)$ for each $x \in [a,b]$, then $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable.*

P r o o f. Let $\varepsilon > 0$. By the definition of derivative, for each $x \in [a,b]$ there exists a positive function $\delta\colon [a,b] \to \mathbb{R}^+$ such that

$$\Bigl|\frac{F(y)-F(x)}{y-x} - f(x)\Bigr| < \frac{\varepsilon}{2(\alpha+b-a)}$$

for all $y \in [a,b]$ with $0 < |y-x| < \delta(x)$. Assume that $D = \{(I_i, x_i)\}_{i=1}^n$ is a δ-fine division of $[a,b]$ satisfying the condition $\varrho(D) < \alpha$. Then we have

$$\begin{aligned}\Bigl|\sum_{i=1}^n [f(x_i)|I_i| - F(I_i)]\Bigr| &\leqslant \sum_{i=1}^n |f(x_i)|I_i| - F(I_i)| \\ &< \frac{\varepsilon}{\alpha+b-a}\sum_{i=1}^n (\operatorname{dist}(x_i, I_i) + |I_i|) \\ &< \frac{\varepsilon}{\alpha+b-a}(\alpha+b-a) = \varepsilon.\end{aligned}$$

Hence, $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$. □

Let F be a function defined on the subintervals of $[a,b]$. For a given partial division $D = \{(I_i, x_i)\colon i = 1, 2, \ldots, n\}$, we write

$$F(D) = \sum_{i=1}^n F(I_i).$$

Definition 2.6. Let $\alpha > 0$ be a constant. Let $F\colon [a,b] \to \mathbb{R}$ and let E be a subset of $[a,b]$.

a) F is said to be AC_α on E if for each $\varepsilon > 0$ there exist a constant $\eta > 0$ and a gauge $\delta\colon [a,b] \to \mathbb{R}^+$ such that $|F(D)| < \varepsilon$ for each δ-fine partial division $D = \{(I_i, x_i)\}$ of $[a,b]$ satisfying $x_i \in E$, $\sum\limits_i |I_i| < \eta$ and $\varrho(D) < \alpha$.

b) F is said to be ACG_α on E if E can be expressed as a countable union of sets on each of which F is AC_α.

Theorem 2.7. *If a function $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$ with the primitive F, then F is ACG_α on $[a,b]$.*

P r o o f. By the definition of the M_α-integral and by the Saks-Henstock Lemma, for each $\varepsilon > 0$ there exists a gauge $\delta\colon [a,b] \to \mathbb{R}^+$ such that

$$\Bigl|\sum_{i=1}^n [f(x_i)|I_i| - F(I_i)]\Bigr| \leqslant \varepsilon$$

for each δ-fine partial division $D = \{(I_i, x_i)\}$ of $[a, b]$ satisfying the condition $\varrho(D) < \alpha$.

Assume that $E_n = \{x \in [a, b] \colon n - 1 \leqslant |f(x)| < n\}$ for each $n \in \mathbb{N}$. Then we have $[a, b] = \bigcup E_n$. To show that F is AC_α on each E_n, fix n and take a δ-fine partial division $D_0 = \{(I_i, x_i)\}$ of $[a, b]$ satisfying $x_i \in E_n$ for all i and $\varrho(D) < \alpha$. If $\sum\limits_i |I_i| < \varepsilon/n$, then

$$\begin{aligned} |F(D_0)| &\leqslant \Big|\sum_i [F(I_i) - f(x_i) \cdot |I_i|]\Big| + \Big|\sum_i f(x_i)|I_i|\Big| \\ &\leqslant \Big|\sum_i [F(I_i) - f(x_i)|I_i|]\Big| + \sum_i |f(x_i)| \cdot |I_i| \\ &\leqslant \varepsilon + n\sum_i |I_i| < 2\varepsilon. \end{aligned}$$

□

Now we recall the definitions of the McShane and Henstock integrals.

A function $f \colon [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$ if there exists a real number A such that for each $\varepsilon > 0$ there exists a gauge $\delta \colon [a, b] \to \mathbb{R}^+$ such that

$$|f(D) - A| < \varepsilon$$

for each δ-fine division $D = \{(I_i, x_i)\}_{i=1}^n$ of $[a, b]$.

A function $f \colon [a, b] \to \mathbb{R}$ is Henstock integrable if there exists a real number A such that for each $\varepsilon > 0$ there exists a gauge $\delta \colon [a, b] \to \mathbb{R}^+$ such that

$$|f(D) - A| < \varepsilon$$

for each δ-fine division $D = \{(I_i, x_i)\}_{i=1}^n$ of $[a, b]$ with $x_i \in I_i$.

From the definitions of the two integrals, we easily get the following theorem.

Theorem 2.8. *Let $f \colon [a, b] \to \mathbb{R}$ be a function.*

a) *If f is McShane integrable on $[a, b]$, then f is M_α-integrable on $[a, b]$.*

b) *If f is M_α-integrable on $[a, b]$, then f is Henstock integrable on $[a, b]$.*

A function $f \colon [a, b] \to \mathbb{R}$ is M_α-integrable on $[a, b]$ if and only if there exists an ACG_α function F on $[a, b]$ such that $F' = f$ almost everywhere on $[a, b]$. To prove this fact, we need the following two lemmas.

Lemma 2.9. *Suppose that $f\colon [a,b]\to\mathbb{R}$ and let $E\subset[a,b]$. If $\mu(E)=0$, then for each $\varepsilon>0$ there exists a positive function δ on E such that $|f(D)|<\varepsilon$ for every δ-fine partial division $D=\{(I_i,x_i)\}_{i=1}^n$ of $[a,b]$ satisfying $x_i\in E$ for all $i=1,2,\dots,n$ and $\varrho(D)<\alpha$.*

P r o o f. For each n, let $E_n=\{x\in E\colon\ n-1\leqslant|f(x)|<n\}$ and let $\varepsilon>0$. Then $E=\bigcup E_n$. Since $\mu(E_n)=0$ for each n, we can choose an open set $O_n\supset E_n$ with $\mu(O_n)<\varepsilon/n\cdot 2^n$.

For $x\in E_n$, define $\delta(x)=\operatorname{dist}(x,O_n^c)$. Suppose that D is a δ-fine partial division of $[a,b]$ with tags in E satisfying the condition $\varrho(D)<\alpha$. Let D_n be a subset of D that has tags in E_n and let $\pi=\{n\in\mathbb{Z}^+\colon\ D_n\neq\varphi\}$. Then

$$|f(D)|\leqslant\sum_{n\in\pi}|f(D_n)|\leqslant\sum_{n\in\pi}|f|(D_n)<\sum_{n\in\pi}n\mu(O_n)<\sum_{n\in\pi}n\cdot\frac{\varepsilon}{n\cdot 2^n}=\varepsilon.$$

□

Lemma 2.10. *Suppose that $F\colon [a,b]\to\mathbb{R}$ is ACG_α on $[a,b]$ and let $E\subset[a,b]$. If $\mu(E)=0$, then for each $\varepsilon>0$ there exists a gauge δ on E such that $|F(D)|<\varepsilon$ for every δ-fine partial division $D=\{(I_i,x_i)\}_{i=1}^n$ of $[a,b]$ satisfying $x_i\in E$ for all $i=1,2,\dots,n$ and $\varrho(D)<\alpha$.*

P r o o f. Let $E=\bigcup_{n=1}^{\infty}E_n$ where the E_n's are pairwise disjoint and F is AC_α on each E_n. Let $\varepsilon>0$. For each n, there exist a gauge $\delta_n\colon E_n\to\mathbb{R}^+$ and a positive number $\eta_n>0$ such that $|F(D)|<\varepsilon/2^n$ for each δ_n-fine partial division $D=\{(I_i,x_i)\}$ of $[a,b]$ satisfying $x_i\in E_n$, $\sum|I_i|<\eta_n$ and $\varrho(D)<\alpha$. For each n, choose an open set $O_n\supset E_n$ with $\mu(O_n)<\eta_n$. Define $\delta(x)=\min\{\delta_n(x),\varrho(x,{O_n}^c)\}$ for $x\in E_n$. Suppose that $D=\{(I_i,x_i)\}_{i=1}^n$ is a δ-fine partial division of $[a,b]$ satisfying $x_i\in E$ and $\varrho(D)<\alpha$. Let D_n be subset of D that has tags in E_n and note that $(D_n)\sum|I_i|<\mu(O_n)<\eta_n$. Hence,

$$|F(D)|\leqslant\sum_{n=1}^{\infty}|F(D_n)|<\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}=\varepsilon.$$

□

Theorem 2.11. *A function $f\colon [a,b]\to\mathbb{R}$ is M_α-integrable on $[a,b]$ if and only if there exists an ACG_α function F on $[a,b]$ such that $F'=f$ almost everywhere on $[a,b]$.*

P r o o f. Suppose that f is M_α-integrable on $[a,b]$ and let $F(x)=\int_a^x f$ for each $x\in[a,b]$. Then by Theorem 2.7, F is ACG_α on $[a,b]$. Since f is Henstock integrable on $[a,b]$, $F'=f$ almost everywhere on $[a,b]$ by [4, Theorem 9.12].

Conversely, suppose that there exists an ACG_α function F such that $F' = f$ almost everywhere on $[a,b]$. Let $E = \{x \in [a,b]\colon F'(x) \neq f(x)\}$ and let $\varepsilon > 0$. Then $\mu(E) = 0$. For each $x \in [a,b] - E$, choose $\delta(x) > 0$ such that

$$|F(y) - F(x) - f(x)(y-x)| < \frac{\varepsilon}{6(\alpha + b - a)}|y - x|$$

whenever $|y - x| < \delta(x)$ and $y \in [a,b]$. By Lemma 2.9 and 2.10, we can find $\delta(x) > 0$ on E such that $|f(D)| < \varepsilon/3$ and $|F(D)| < \varepsilon/3$, whenever $D = \{(I_i, x_i)\}$ is a δ-fine partial division of $[a,b]$ satisfying $x_i \in E$ and $\varrho(D) < \alpha$.

Suppose that $D = \{(I_i, x_i)\}$ is a δ-fine partial division of $[a,b]$ satisfying $\varrho(D) < \alpha$. Let D_1 be the subset of D that has tags in E and let $D_2 = D - D_1$. Then

$$\begin{aligned}
|f(D) - F(D)| &= |f(D_2) - F(D_2)| + |f(D_1)| + |F(D_1)| \\
&\leqslant (D_2)\sum \big|f(x_i)|I_i| - F(I_i)\big| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leqslant \frac{\varepsilon}{3(\alpha + b - a)} \sum(\operatorname{dist}(x_i, I_i) + |I_i|) + \frac{2}{3}\varepsilon \\
&\leqslant \frac{\varepsilon}{3(\alpha + b - a)}(\alpha + b - a) + \frac{2}{3}\varepsilon = \varepsilon.
\end{aligned}$$

Hence f is M_α-integrable on $[a,b]$. □

The following examples show that the converse of Theorem 2.8 is not true.

Example 2.12. (1) Let f be a function defined by

$$f(x) = \begin{cases} 2x\sin\dfrac{1}{x^2} - \dfrac{2}{x}\cos\dfrac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then it is easy to show that the primitive of f is

$$F(x) = \begin{cases} x^2\sin\dfrac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Since F is differentiable and $F' = f$ everywhere on $[0,1]$, f is M_α-integrable due to Theorem 2.5. But F is not absolutely continuous on $[0,1]$ and therefore f is not McShane integrable on $[0,1]$.

(2) The function F defined by

$$F(x) = \begin{cases} x\sin\dfrac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable almost everywhere on $[0,1]$. By [3, Theorem 9.6], F' is Henstock integrable on $[0,1]$. But we can show that F is not ACG_α on $[0,1]$.

To show this, suppose that F is ACG_α. Then there exists a set $E \subset [0,1]$ such that $0 \in E$ and F is AC_α on E. Hence, there exist a gauge $\delta\colon [0,1] \to \mathbb{R}^+$ and a positive number $\eta > 0$ such that $|F(D)| < \alpha/2$ whenever $D = \{(I_i, x_i)\}$ is a δ-fine partial division of $[0,1]$ satisfying the conditions $x_i \in E$, $\sum |I_i| < \eta$ and $\varrho(D) < \alpha$.

Let $a_n = 1/\sqrt{(2n+\frac{1}{2})\pi}$ and $b_n = 1/\sqrt{2n\pi}$ for each positive integer n. Then $a_n < b_n < 1$ and $\sum\limits_{n=1}^{\infty} a_n = \infty$. Choose a δ-fine partial division $D = \{([a_i, b_i], 0)\colon N \leqslant i \leqslant M\}$ such that $\alpha/2 < \sum\limits_{i=N}^{M} a_i < \alpha$ and $b_N < \min\{\delta(0), \eta\}$. Then $0 \in E$, $\sum\limits_{i=N}^{M} (b_i - a_i) < \eta$, and $\sum\limits_{i=N}^{M} \operatorname{dist}(0, [a_i, b_i]) = \sum\limits_{i=N}^{M} a_i < \alpha$.

Hence, D is a δ-fine partial division of $[0,1]$ satisfying the condition $\varrho(D) < \alpha$. But we have

$$|F(D)| = \left| \sum_{i=N}^{M} [F(b_i) - F(a_i)] \right| = \sum_{i=N}^{M} a_i > \alpha/2.$$

This contradiction shows that F is not ACG_α on $[0,1]$. Hence, F' is not M_α-integrable on $[0,1]$.

3. Equivalence of the M_α and C-integrals

Recall [1], [2] that a function $f\colon [a,b] \to \mathbb{R}$ is C-integrable on $[a,b]$ if there exists a real number A such that for each $\varepsilon > 0$ there exists a gauge δ such that

$$|f(D) - A| < \varepsilon$$

for each δ-fine division $D = \{(I_i, x_i)\colon i = 1, 2, \ldots, n\}$ of $[a,b]$ satisfying the condition $\varrho(D) < 1/\varepsilon$.

To show that the M_α-integral is equivalent to the C-integral, we need the following lemma.

Lemma 3.1. *Let $\alpha > 0$ be a constant and let $\delta\colon [a,b] \to \mathbb{R}^+$ be a gauge with $\delta(x) < \alpha/4$ for each $x \in [a,b]$. If D is a δ-fine division of $[a,b]$ with $\varrho(D) < n\alpha$ for some positive integer n, then there exist δ-fine pairwise disjoint partial divisions $D_1, D_2, \ldots, D_m$ of intervals in D such that $D = \bigcup\limits_{i=1}^{m} D_i$, $\varrho(D_i) < \alpha$ for each $i = 1, 2, \ldots, m$ and $m < 2n$.*

P r o o f. Let $D = \{(I_i, x_i)\}_{i=1}^{p}$ be a δ-fine division of $[a,b]$ with $\varrho(D) < n\alpha$ for some positive integer n. Choose the greatest positive integer n_1 such that $\sum\limits_{i=1}^{n_1} \operatorname{dist}(x_i, I_i) < \alpha$ and let $D_1 = \{(I_i, x_i)\}_{i=1}^{n_1}$. Next, choose the greatest positive integer n_2 such that $\sum\limits_{i=n_1+1}^{n_2} \operatorname{dist}(x_i, I_i) < \alpha$ and let $D_2 = \{(x_i, I_i)\}_{i=n_1+1}^{n_2}$. Continuing in this way, we have partial divisions $D_1, D_2, \ldots, D_m$ such that

$$D = \bigcup_{i=1}^{m} D_i \quad \text{and} \quad \varrho(D_i) < \alpha$$

for each $i = 1, 2, \ldots, m$.

From the construction of each D_i we have

$$\frac{3}{4}\alpha < \varrho(D_i) < \alpha$$

for each $i = 1, 2, \ldots, m$.

Suppose that $m \geqslant 2n$. Then

$$\varrho(D) = \sum_{i=1}^{m} \varrho(D_i) > \sum_{i=1}^{m} \frac{3}{4}\alpha = \frac{3}{4}\alpha m \geqslant \frac{3}{4}\alpha \cdot 2n = \frac{3}{2}\alpha n.$$

This contradicts the fact that $\varrho(D) < n\alpha$. Hence, $m < 2n$. $\square$

Theorem 3.2. *Let $\alpha > 0$ be a constant. A function $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$ if and only if f is C-integrable on $[a,b]$. The value of the integral is the same in both cases.*

P r o o f. Suppose that f is C-integrable on $[a,b]$ and let $F(x) = (C)\int_a^x f$. Let $\varepsilon > 0$. Choose $\varepsilon_1 > 0$ such that $\alpha < 1/\varepsilon_1$ and $\varepsilon_1 < \varepsilon$. Since f is C-integrable on $[a,b]$, there exists a gauge $\delta\colon [a,b] \to \mathbb{R}^+$ such that

$$\left| f(D) - (C)\int_a^b f \right| < \varepsilon_1$$

for each δ-fine division D of $[a,b]$ with $\varrho(D) < 1/\varepsilon_1$.

If D is a δ-fine division of $[a,b]$ with $\varrho(D) < \alpha$, then

$$\left| f(D) - (C)\int_a^b f \right| < \varepsilon_1 < \varepsilon.$$

Hence, f is M_α-integrable on $[a,b]$ and

$$(M_\alpha)\int_a^b f = (C)\int_a^b f.$$

Conversely, suppose that f is M_α-integrable on $[a,b]$ and let $F(x) = (M_\alpha)\int_a^x f$ for each $x \in [a,b]$. Let $\varepsilon > 0$. Choose a positive integer n such that $1/\varepsilon < n\alpha$. Since f is M_α-integrable on $[a,b]$, there exists a gauge $\delta_1\colon [a,b] \to \mathbb{R}^+$ such that

$$|f(D) - F([a,b])| < \frac{\varepsilon}{2n}$$

for each δ_1-fine division D of $[a,b]$ with $\varrho(D) < \alpha$. Define $\delta(x) = \min\{\delta_1(x), \alpha/4\}$ for each $x \in [a,b]$. Let D be a δ-fine division of $[a,b]$ with $\varrho(D) < 1/\varepsilon$. By Lemma 3.1, we can decompose D into pairwise disjoint δ-fine partial divisions $D_1, D_2, \ldots, D_m$ such that $D = \bigcup\limits_{i=1}^{m} D_i$, $\varrho(D_i) < \alpha$ for each $i = 1, 2, \ldots, m$ and $m < 2n$.

By the Saks-Henstock Lemma we have

$$|f(D) - F([a,b])| \leqslant \sum_{i=1}^{m} |f(D_i) - F(D_i)| \leqslant \sum_{i=1}^{m} \frac{\varepsilon}{2n} = \frac{m\varepsilon}{2n} < \varepsilon.$$

Hence, f is C-integrable on $[a,b]$. □

For any constant $\alpha > 0$, the M_α-integral is equivalent to the C-integral by Theorem 3.2. Hence, we have the following corollary.

Corollary 3.3. *Let α and β be positive constants. A function $f\colon [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$ if and only if f is M_β-integrable on $[a,b]$.*

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