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VARIABLE LEBESGUE NORM ESTIMATES FOR BMO FUNCTIONS

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Abstract. In this paper, we are going to characterize the space $\mathrm{BMO}(\mathbb{R}^n)$ through variable Lebesgue spaces and Morrey spaces. There have been many attempts to characterize the space $\mathrm{BMO}(\mathbb{R}^n)$ by using various function spaces. For example, Ho obtained a characterization of $\mathrm{BMO}(\mathbb{R}^n)$ with respect to rearrangement invariant spaces. However, variable Lebesgue spaces and Morrey spaces do not appear in the characterization. One of the reasons is that these spaces are not rearrangement invariant. We also obtain an analogue of the well-known John-Nirenberg inequality which can be seen as an extension to the variable Lebesgue spaces.

Keywords: variable exponent, Morrey space, BMO

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1. Introduction

The aim of this paper is to obtain characterizations of $\mathrm{BMO}(\mathbb{R}^n)$. Recently an attempt has been made to characterize $\mathrm{BMO}(\mathbb{R}^n)$ through various function spaces. Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function for a measurable set $S \subset \mathbb{R}^n$. All cubes are assumed to have their sides parallel to the coordinate axes. Given a function f and a measurable set S, f_S denotes the mean value of f on S, namely

$$f_S := \frac{1}{|S|} \int_S f(x)\,\mathrm{d}x.$$

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Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all measurable functions b satisfying

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \,\mathrm{d}x < \infty, \tag{1.1}$$

where the supremum is taken over all cubes Q. Recently, given a Banach function space X, we have been asking ourselves the following question.

Problem 1. Is the norm $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ equivalent to

$$\|b\|_X^* = \sup_{Q\colon \text{cube}} \frac{1}{\|\chi_Q\|_X} \|\chi_Q(b - b_Q)\|_X?$$

Here is a series of affirmative results concerning Problem 1.

(1) $X = L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$. This is well-known as the John-Nirenberg inequality (see Lemma 3.1 to follow).
(2) X is a rearrangement invariant function space [7]. By rearrangement invariant we mean that the X-norm of a function f depends only on the function $t \in (0,\infty) \mapsto |\{|f| > t\}| \in (0,\infty)$.
(3) X is a quasi-rearrangement invariant Banach function space with $p \leqslant p_Y \leqslant q_Y < \infty$ ([8]).

The aim of this paper is to show that this is the case even when X is not rearrangement invariant. First, we consider the case when X is a Morrey space.

Theorem 1.1. *Let $1 \leqslant q \leqslant p < \infty$. If we define the Morrey space $\mathscr{M}^p_q(\mathbb{R}^n)$ by*

$$\|f\|_{\mathscr{M}^p_q(\mathbb{R}^n)} = \sup_{Q\colon \text{cube}} |Q|^{1/p-1/q} \left(\int_Q |f(x)|^q \,\mathrm{d}x \right)^{1/q},$$

then the answer of Problem 1 is affirmative for $X = \mathscr{M}^p_q(\mathbb{R}^n)$.

The second (and main) spaces we take up in this paper are variable Lebesgue spaces. A measurable function $p(\cdot)\colon \mathbb{R}^n \to [1,\infty]$ is called a variable exponent. Variable exponent spaces showed up around 1990s [11]. After 2005 the theory which is fundamental in harmonic analysis is established very rapidly. For more details we refer to the recent book [6]. We now recall the definition. Given a variable exponent $p(\cdot)$, one denotes

$$\Omega_{\infty,p} := \{x \in \mathbb{R}^n \colon p(x) = \infty\} = p^{-1}(\infty),$$
$$\varrho_p(f) := \int_{\mathbb{R}^n \setminus \Omega_{\infty,p}} |f(x)|^{p(x)} \,\mathrm{d}x + \|f\|_{L^\infty(\Omega_{\infty,p})}.$$

The variable Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \{f \text{ is measurable: } \varrho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\{\lambda > 0\colon\ \varrho_p(f/\lambda) < \infty\}.$$

This is a special case of the theory developed by Luxemburg and Nakano [13], [14], [15]. We additionally set

$$p_- := \operatorname{ess\,inf}\{p(x)\colon\ x \in \mathbb{R}^n\}, \quad p_+ := \operatorname{ess\,sup}\{p(x)\colon\ x \in \mathbb{R}^n\}.$$

Theorem 1.2. *If a variable exponent $p(\cdot)$ satisfies $1 \leqslant p_- \leqslant p_+ < \infty$ and the estimates*

$$\Bigl|\frac{1}{p(x)} - \frac{1}{p(y)}\Bigr| \leqslant -\frac{C_1}{\log|x-y|} \quad (|x-y| \leqslant \tfrac{1}{2})$$

and

$$\Bigl|\frac{1}{p(x)} - \frac{1}{p(\infty)}\Bigr| \leqslant \frac{C_2}{\log(e+|x|)} \quad (x \in \mathbb{R}^n)$$

hold for some constants C_1, C_2, $p(\infty)$, then the answer to Problem 1 is affirmative for $X = L^{p(\cdot)}(\mathbb{R}^n)$, that is,

$$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

holds for all $b \in \mathrm{BMO}(\mathbb{R}^n)$.

One of the most interesting problems on spaces with variable exponent is to give conditions on the boundedness of the Hardy-Littlewood maximal operator. The important sufficient conditions called "log-Hölder" have been obtained by Cruz-Uribe, Fiorenza, and Neugebauer [2] and Diening [3]. Under the conditions many results on spaces with variable exponent have been obtained now.

About Theorems 1.1 and 1.2, $L^{p(\cdot)}(\mathbb{R}^n)$ is not rearrangement invariant. Examples in [17] show that $\mathscr{M}^p_q(\mathbb{R}^n)$ is rearrangement invariant only when $p = q$. Theorem 1.1 is considerably easy to prove. Indeed, from the definition of the Morrey norm, we have

$$\begin{aligned}\frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^q(\mathbb{R}^n)} &\leqslant \frac{1}{\|\chi_Q\|_{\mathscr{M}^p_q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{\mathscr{M}^p_q(\mathbb{R}^n)}\\ &\leqslant \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

So the matters are reduced to the case when $X = L^p(\mathbb{R}^n)$.

However, a similar argument does not seem to work for Theorem 1.2. Especially the estimate which corresponds to

$$\frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^q(\mathbb{R}^n)} \leqslant \frac{1}{\|\chi_Q\|_{\mathscr{M}^p_q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{\mathscr{M}^p_q(\mathbb{R}^n)}$$

is hard to obtain.

We organize the remaining part of this paper as follows: Section 2 is intended as a review of variable Lebesgue spaces. We prove Theorem 1.2 in Section 3. Section 4 contains another characterization of $\mathrm{BMO}(\mathbb{R}^n)$ related to the variable exponent Lebesgue norms.

Finally we give a convention which we use throughout the rest of this paper. The symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Some basic facts on variable Lebesgue spaces

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x.

One of the key developments of the theory of variable Lebesgue spaces is that we obtained a good criterion for the boundedness of the Hardy-Littlewood maximal operators [3], [4], [5].

Before we proceed, let us recall some key terminology about the continuity and variable Lebesgue spaces. Let $r(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ be a measurable function.

(1) The function $r(\cdot)$ is said to be locally log-Hölder continuous if

$$|r(x)-r(y)| \leqslant \frac{C}{-\log(|x-y|)} \quad (|x-y| \leqslant \tfrac{1}{2}) \tag{2.1}$$

holds. The set LH_0 consists of all locally log-Hölder continuous functions.

(2) The function $r(\cdot)$ is said to be log-Hölder continuous at infinity if there exists a constant $r(\infty)$ such that

$$|r(x)-r(\infty)| \leqslant \frac{C}{\log(\mathrm{e}+|x|)}. \tag{2.2}$$

The set LH_∞ consists of all functions log-Hölder continuous at infinity.

(3) Define $LH := LH_0 \cap LH_\infty$ and say that each function belonging to LH is globally log-Hölder continuous.

(2.3) below is initially proved by Cruz-Uribe et al. [2], when $p_+ < \infty$. Later Cruz-Uribe et al. [1] and Diening et al. [5] have independently extended the result even to the case of $p_+ = \infty$. Suppose that a variable exponent $p(\cdot)$ satisfies $1 < p_- \leqslant p_+ \leqslant \infty$ and that $1/p(\cdot) \in LH$. Then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, namely

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{2.3}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

We note that $p(\cdot)$ always satisfies $p_- > 1$ whenever (2.3) is true [5]. In the case of $p_- = 1$, the weak $(p_-(\cdot), p_-(\cdot))$ type inequality for M holds. The following inequality has been also proved by Cruz-Uribe et al. [1]. If a variable exponent $p(\cdot)$ satisfies $1 = p_- \leqslant p_+ \leqslant \infty$ and $1/p(\cdot) \in LH$, then we have that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{t>0} t\|\chi_{\{Mf(x)>t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{2.4}$$

Lemma 2.1. *If a variable exponent $p(\cdot)$ satisfies the weak $(p(\cdot), p(\cdot))$ type inequality (2.4) for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then*

$$|f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all cubes Q.

Proof. Take $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and a cube Q arbitrarily. We may assume $|f|_Q > 0$. Let $t = |f|_Q/2$. From $|f|_Q\chi_Q(x) \leqslant M(f\chi_Q)(x)$, we obtain $M(f\chi_Q)(x) > t$ whenever $x \in Q$. Thus we have

$$\begin{aligned} |f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant |f|_Q\|\chi_{\{M(f\chi_Q)(x)>t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant |f|_Q \cdot Ct^{-1}\|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= C\|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

□

Remark 2.1. Lerner [12] has proved the converse of Lemma 2.1, provided that $p(\cdot)$ is radial decreasing and satisfies $p_- > 1$.

The next lemma is due to Diening [4, Lemmas 3.2, 5.3 and 5.5].

Lemma 2.2. *If a variable exponent $p(\cdot)$ satisfies $1 < p_- \leqslant p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a constant $0 < \delta_1 < 1$ such that for all $0 < \delta < \delta_1$, all families of pairwise disjoint cubes Y, all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $|f|_Q > 0$ ($Q \in Y$) and all positive sequence $\{t_Q\}_{Q\in Y} \subset (0,\infty)$,*

$$\Big\| \sum_{Q\in Y} t_Q \Big|\frac{f}{f_Q}\Big|^{\delta} \chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \Big\| \sum_{Q\in Y} t_Q \chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Here the constant is independent of Y, $\{t_Q\}_{Q\in Y}$ and f but is dependent on δ. In particular

$$\||f|^{\delta}\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C(|f|_Q)^{\delta} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{2.5}$$

holds.

3. Main results

We describe some known facts before we state the main results.

Lemma 3.1. *If $1 \leqslant q < \infty$, then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \sup_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q \,\mathrm{d}x \right)^{1/q} \leqslant C_0 q \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{3.1}$$

where $C_0 > 0$ is a constant independent of q.

The left hand-side inequality of (3.1) directly follows from the Hölder inequality. The right one is a famous consequence of an application of the John-Nirenberg inequality (cf. [10]).

Proposition 3.1. *There exist two positive constants C_1, C_2 depending only on n such that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$, all cubes Q and all $t \geqslant 0$,*

$$|\{x \in Q\colon\ |b(x) - b_Q| > t\}| \leqslant C_1 |Q| \exp(-C_2 t/\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}).$$

Lemma 3.1 can additionally be generalized to the case of variable exponents. Now we are going to prove Theorem 1.2. Recall that we announced that we are going to prove the following.

Theorem 3.1. *If a variable exponent $p(\cdot)$ satisfies $1 < p_- \leqslant p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \sup_Q \frac{\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{3.2}$$

The first author [9] has initially proved Theorem 3.1. Later we will give another proof of it.

In view of Lemma 3.1, it may be a natural question to prove (3.2) for the case of $p_- = 1$.

Now we shall prove Theorem 1.2. Take a cube Q and $b \in \mathrm{BMO}(\mathbb{R}^n)$ arbitrarily. By virtue of Lemma 2.1 we have

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q|\,\mathrm{d}x \cdot \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This gives us the left hand side inequality of the theorem. Next we shall prove the right hand side one. Let us fix a number $r \geqslant 1$ so that $rp_- > 1$ and write $u(\cdot) := rp(\cdot)$. Then the variable exponent $u(\cdot)$ satisfies $1 < u_-$ and $1/u(\cdot) \in LH$. Hence the boundedness of M on $L^{u(\cdot)}(\mathbb{R}^n)$ holds by (2.3). Using Lemma 2.2, we can take a constant $\delta \in (0, 1/r)$ so that

$$\|f^\delta \chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)} \leqslant C|f_Q|^\delta \|\chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that f is positive a.e. Now we obtain

$$\begin{aligned} \|f^{r\delta}\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|f^\delta \chi_Q\|^r_{L^{u(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C|f_Q|^{r\delta}\|\chi_Q\|^r_{L^{u(\cdot)}(\mathbb{R}^n)} = C|f_Q|^{r\delta}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.3}$$

If we put $f := |b - b_Q|^{1/(r\delta)}$ and apply Lemma 3.1 with $q = 1/(r\delta) > 1$, then we get

$$|f_Q|^{r\delta} = \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{1/(r\delta)}\,\mathrm{d}x\right)^{r\delta} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This leads us to the desired inequality and completes the proof.

For the proof of Theorem 3.1, we have only to follow the same argument as the proof of Theorem 1.2 with $r = 1$.

According to Lemma 3.1, we have

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^q\,\mathrm{d}x\right)^{1/q} \leqslant C_0 q\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C_0 > 0$ is independent of $q \in [1,\infty)$. This can be rephrased as

$$\frac{1}{|Q|}\int_Q \Big(\frac{|b(x)-b_Q|}{C_0 q\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^q\,\mathrm{d}x \leqslant 1$$

for all cubes Q. Observe that the estimate above is uniform over $1 \leqslant q < \infty$. Therefore, the following inequality is seen to hold

$$\frac{1}{|Q|}\int_Q \Big(\frac{|b(x)-b_Q|}{16C_0 p(x)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1.$$

See the inequality of the second line of the next page. Suppose that $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ is a variable exponent which is not necessarily continuous or bounded. Then define

$$\|b\|^{\dagger}_{p(\cdot)} = \sup_Q \left(\inf\left\{\lambda>0\colon \frac{1}{|Q|}\int_Q \Big(\frac{|b(x)-b_Q|}{p(x)\lambda}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1\right\}\right)$$

for measurable functions b. Now we are going to prove the following

Theorem 4.1. *If a variable exponent $p(\cdot)$ satisfies $p(x) < \infty$ for almost every $x \in \mathbb{R}^n$, then we have*

$$\|b\|^{\dagger}_{p(\cdot)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Furthermore, if $p(\cdot)$ is bounded, then the norms $\|\cdot\|^{\dagger}_{p(\cdot)}$ and $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ are mutually equivalent.

Proof. The classical John-Nirenberg inequality asserts that

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q|^p\,\mathrm{d}x \leqslant C_1\Big(\frac{p\Gamma(p)}{C_2^p}\Big)\|b\|^p_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{4.1}$$

Observe also that the function $\lambda \mapsto \sum\limits_{k=1}^{\infty} \lambda^k k\Gamma(k)/k!C_2^k$ is a holomorphic function in a neighborhood of the origin. Consequently, by the Taylor expansion, we have

$$\frac{1}{|Q|}\int_Q \Big\{\exp\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)-1\Big\}\,\mathrm{d}x = \sum_{k=1}^{\infty}\frac{1}{k!|Q|}\int_Q \Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^k\,\mathrm{d}x$$

if $(0<)\ \lambda \ll 1$. Note also that

$$\frac{1}{(16p)^p} \leqslant (n+1)^{-n-1}$$

for all $1 \leqslant n = [p] \leqslant p < n+1$, where $[p]$ denotes the integer part of $p \in \mathbb{R}$. Indeed, we have

$$\frac{1}{(16p)^p} \leqslant \frac{1}{4^p(n+1)^p} \leqslant \frac{1}{4^p(n+1)^{p-n-1}(n+1)^{n+1}} \leqslant \frac{1}{(n+1)^{n+1}}.$$

Here for the last inequality we have used $4^p \geqslant 4^n \geqslant n+3 \geqslant p+2 \geqslant n+1 \geqslant (n+1)^{n+1-p}$. Using $(16p)^{-p} \leqslant n^{-n-1}$ for all $1 \leqslant n \leqslant p < n+1$, we obtain

$$\begin{aligned}\Big(\frac{\lambda|b(x)-b_Q|}{16p(x)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)} &= \Big(\frac{1}{16p(x)}\Big)^{p(x)}\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)}\\ &\leqslant \Big(\frac{1}{[p(x)+1]}\Big)^{[p(x)+1]}\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)}.\end{aligned}$$

If we use $t^{p(x)} \leqslant t^{[p(x)]} + t^{[p(x)+1]}$ and

$$\min\Big\{\Big(\frac{1}{[p(x)]}\Big)^{[p(x)]}, \Big(\frac{1}{[p(x)+1]}\Big)^{[p(x)+1]}\Big\} = \Big(\frac{1}{[p(x)+1]}\Big)^{[p(x)+1]},$$

then we obtain

$$\begin{aligned}\Big(\frac{\lambda|b(x)-b_Q|}{16p(x)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)} &\leqslant \min\Big\{\Big(\frac{1}{[p(x)]}\Big)^{[p(x)]}, \Big(\frac{1}{[p(x)+1]}\Big)^{[p(x)+1]}\Big\}\\ &\quad \times \max\Big\{\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{[p(x)]}, \Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{[p(x)+1]}\Big\}\\ &\leqslant \Big(\frac{1}{[p(x)]}\Big)^{[p(x)]}\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{[p(x)]}\\ &\quad + \Big(\frac{1}{[p(x)+1]}\Big)^{[p(x)+1]}\Big(\frac{\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{[p(x)+1]}.\end{aligned}$$

If we use the Taylor expansion again, then we have

$$\Big(\frac{\lambda|b(x)-b_Q|}{16p(x)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)} \leqslant \exp\Big(\frac{2\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big) - 1.$$

In view of (4.1), we obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q \Big(\frac{\lambda|b(x)-b_Q|}{16p(x)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)^{p(x)}\mathrm{d}x &\leqslant \frac{1}{|Q|}\int_Q \Big\{\exp\Big(\frac{2\lambda|b(x)-b_Q|}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big) - 1\Big\}\mathrm{d}x\\ &\leqslant C_1\sum_{k=1}^{\infty}\frac{(2\lambda)^k k\Gamma(k)}{k!C_2^k}.\end{aligned}$$

Hence, if we choose $\lambda > 0$ so small that we have $C_1 \sum\limits_{k=1}^{\infty} (2\lambda)^k k\Gamma(k)/k!C_2^k \leqslant 1$, then it follows that

$$\|b\|_{p(\cdot)}^{\dagger} \leqslant 32\lambda^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

If $p(\cdot)$ is bounded, then

$$\begin{aligned}\|b\|_{p(\cdot)}^{\dagger} &\geqslant \sup_Q \biggl(\inf \biggl\{ \lambda > 0\colon \frac{1}{|Q|}\int_Q \Bigl(\frac{|b(x)-b_Q|}{p_+\lambda}\Bigr)^{p(x)} \,\mathrm{d}x \leqslant 1 \biggr\}\biggr)\\ &= \sup_Q \biggl(\inf \biggl\{ \lambda > 0\colon \frac{1}{|Q|}\int_Q \Bigl\{\frac{1}{2} + \frac{1}{2}\Bigl(\frac{|b(x)-b_Q|}{p_+\lambda}\Bigr)^{p(x)}\Bigr\} \,\mathrm{d}x \leqslant 1 \biggr\}\biggr).\end{aligned}$$

Note that

$$\frac{1}{2} + \frac{1}{2}t^{p(x)} \geqslant \frac{t}{2}$$

for all $t > 0$ and hence we have

$$\begin{aligned}\|b\|_{p(\cdot)}^{\dagger} &\geqslant \sup_Q \biggl(\inf \biggl\{ \lambda > 0\colon \frac{1}{|Q|}\int_Q \frac{|b(x)-b_Q|}{2p_+\lambda} \,\mathrm{d}x \leqslant 1 \biggr\}\biggr)\\ &= (2p_+)^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, these norms are mutually equivalent. □

Remark 4.1. Let Φ be a Young function. Namely, $\Phi\colon [0,\infty) \to [0,\infty)$ is a homeomorphism which is convex. If we assume that $\Phi(t) \leqslant t^a$ $(t \geqslant 2)$ for some $a > 1$ and define

$$\|b\|_{\Phi}^{\dagger} = \sup_Q \biggl(\inf \biggl\{ \lambda > 0\colon \frac{1}{|Q|}\int_Q \Phi\Bigl(\frac{|b(x)-b_Q|}{p(x)\lambda}\Bigr) \,\mathrm{d}x \leqslant 1 \biggr\}\biggr)$$

for measurable functions b, then $\|b\|_{\Phi}^{\dagger}$ is equivalent to $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Indeed, as we have shown in [16], the norm $\|b\|_{\Phi}^{\dagger}$ remains unchanged if we redefine $\Phi(t) = \Phi(2)(t/2)^a$ for $0 \leqslant t \leqslant 2$. Therefore, $\|b\|_{\Phi}^{\dagger} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ by virtue of Lemma 3.1. The reverse inequality is also clear since we have $\Phi(t) \geqslant \Phi(1)t$ for $t \geqslant 1$.

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