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ON THE H^p-L^q BOUNDEDNESS OF SOME FRACTIONAL INTEGRAL OPERATORS

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Abstract. Let $A_1, \ldots, A_m$ be $n \times n$ real matrices such that for each $1 \leqslant i \leqslant m$, A_i is invertible and $A_i - A_j$ is invertible for $i \neq j$. In this paper we study integral operators of the form

$$Tf(x) = \int k_1(x - A_1 y) k_2(x - A_2 y) \ldots k_m(x - A_m y) f(y) \, \mathrm{d}y,$$

$k_i(y) = \sum\limits_{j \in \mathbb{Z}} 2^{jn/q_i} \varphi_{i,j}(2^j y)$, $1 \leqslant q_i < \infty$, $1/q_1 + 1/q_2 + \ldots + 1/q_m = 1 - r$, $0 \leqslant r < 1$, and $\varphi_{i,j}$ satisfying suitable regularity conditions. We obtain the boundedness of $T\colon H^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $0 < p < 1/r$ and $1/q = 1/p - r$. We also show that we can not expect the H^p-H^q boundedness of this kind of operators.

Keywords: integral operator, Hardy space

MSC 2010: 42B20, 42B30

1. Introduction

In [4] the authors obtain the L^p boundedness, $p > 1$, for a class of maximal operators on the three dimensional Heisenberg group. The operators they consider have relevance in the analysis on $\mathrm{SL}(\mathbb{R}^3)$. Some of them actually arise in the study of the boundary behavior of Poisson integrals on the symmetric space $\mathrm{SL}(\mathbb{R}^3)/\mathrm{SO}(3)$. To obtain the principal results, they analyze the $L^2(\mathbb{R})$ boundedness of integral operators of the form

$$Tf(x) = \int |x - y|^{-\alpha} |x + y|^{\alpha - 1} f(y) \, \mathrm{d}y,$$

$0 < \alpha < 1$.

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A natural question is if these operators are also bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ for certain $1 < p, q < \infty$, and if this kind of results still hold for larger dimensions or for more general kernels. In this context, in [3] the authors study integral operators on $\mathbb{R}^n$ with kernels of the form

$$k(x,y) = k_1(x - a_1 y)k_2(x - a_2 y)\dots k_m(x - a_m y),$$

with $a_j \in \mathbb{R} \setminus \{0\},\ a_i \neq a_j$ for $i \neq j$, $1 \leqslant i, j \leqslant m$ and

$$k_i(y) = \sum_{j\in\mathbb{Z}} 2^{jn/q_i}\varphi_{i,j}(2^j y),$$

for certain functions $\varphi_{i,j}$ satisfying some regularity properties. They obtain that this operator is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < 1/r$ and $1/q = 1/p - r$.

Now we consider the following natural generalization of these operators. For $n, m \in \mathbb{N}$, let $A_1, \dots, A_m$ be real $n \times n$ matrices such that for each $1 \leqslant i \leqslant m$, A_i is invertible and $A_i - A_j$ is invertible if $i \neq j$. Let $m > 1$, $q_1, \dots, q_m$ be real numbers, $1 < q_i < \infty$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m} = 1 - r$$

for some $0 \leqslant r < 1$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, we denote $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/\partial y_1^{\alpha_1}\dots\partial y_n^{\alpha_n}$. For $1 \leqslant i \leqslant m$ let $\{\varphi_{i,j}\}_{j\in\mathbb{Z}}$ be a family of smooth and non negative real functions defined on $\mathbb{R}^n$, such that

$$\operatorname{supp}(\varphi_{i,j}) \subset \{y \in \mathbb{R}^n \colon\ 2^{-1} \leqslant |y| \leqslant 2\}$$

and such that for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ there exists M_α such that $\sup_{j\in\mathbb{Z}} \|D^\alpha \varphi_{i,j}\|_\infty \leqslant M_\alpha$.

Let

$$k(x,y) = k_1(x - A_1 y)k_2(x - A_2 y)\dots k_m(x - A_m y), \tag{1}$$

with

$$k_i(y) = \sum_{j\in\mathbb{Z}} 2^{jn/q_i}\varphi_{i,j}(2^j y),$$

and let T be the integral operator with kernel $k(x,y)$, i.e.

$$Tf(x) = \int k(x,y)f(y)\,\mathrm{d}y. \tag{2}$$

We observe that if $\varphi_{i,j} = \varphi_{i,k}$ for all $j, k \in \mathbb{Z}$ then $k_i(2^s y) = 2^{-sn/q_i}k_i(y)$. So k_i is "homogeneous" of degree $-n/q_i$ and then the "homogeneity degree" of k is $-n(1-r)$.

The Hardy-Littlewood-Sobolev theorem shows that the Riesz potential operator I_{nr}, with kernel $1/|y|^{n(1-r)}$, is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, for $0 < r < 1$, $1 < p < 1/r$ and $1/q = 1/p - r$. Also for the endpoint cases, it is known that I_{nr} is not bounded from L^1 into $L^{1/(1-r)}$ and neither from $L^{1/r}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ (See [6], p. 119). In 1960 E. Stein and G. Weiss [8] used the theory of harmonic functions of several variables to prove that these operators are bounded from $H^1(\mathbb{R}^n)$ to $L^{1/(1-r)}(\mathbb{R}^n)$ and in 1980 M. Taibleson and G. Weiss, using the molecular characterization of the real Hardy spaces, obtained the boundedness of these operators from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, where $0 < p < 1$ and $1/q = 1/p - r$ (see [9]).

Also in [1] the authors obtain the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness, $n/(n+\alpha) \leqslant p \leqslant 1$, $1/q = 1/p - \alpha/n$, for the homogeneous fractional convolution operators $T_{\Omega,\alpha}$ given by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\,\mathrm{d}y,$$

where $0 < \alpha < n$, Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$, $s \geqslant 1$.

In [5] we obtain the $H^p(\mathbb{R}^n) - L^p(\mathbb{R}^n)$ boundedness, $0 < p \leqslant 1$, of integral operators with kernels of the form

$$k(x,y) = |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}, \tag{3}$$

where $a_i \neq a_j$ for $i \neq j$, $m > 1$ and $\alpha_1 + \ldots + \alpha_m = n$ and we also show that we can not expect the $H^p(\mathbb{R}^n)$ boundedness of them. These kernels can be expresed as in (1), with $r = 0$.

In this paper we obtain the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of the operator T defined by (2), for $0 < p < 1/r$ and $1/q = 1/p - r$. By duality we obtain the corresponding $L^{1/r}(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$ boundedness. Also, in the last section, for each $0 < r < 1$ we give an example of an operator T_r on $H^p(\mathbb{R})$, having a kernel of the form (3) with $m = 2$ and $\alpha_1 + \alpha_2 = 1 - r$, that is not bounded from $H^p(\mathbb{R})$ into $H^q(\mathbb{R})$ for $0 < p \leqslant 1/(1+r)$ and $1/q = 1/p - r$.

Throughout this paper, c will denote a positive constant not necessarily the same at each occurrence.

2. Preliminary results

We note that the condition $1/q = 1/p - r$, $1 < p < 1/r$ is necessary for the boundedness from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ of certain subfamily of operators of the form (2).

Remark 1. A standard homogeneity argument shows that if an operator with general kernel k with "homogeneity degree" $-n(1-r)$ is bounded from $L^p(\mathbb{R}^n)$ into

$L^q(\mathbb{R}^n)$ for some $1 < p, q < \infty$, then $1/q = 1/p - r$. Now for $l \in \mathbb{Z}$, let T^l be the integral operator with kernel $k^l = k_1^l(x - A_1 y) \dots k_m^l(x - A_m y)$, where $k_i^l(y) = \sum_{j\in\mathbb{Z}} 2^{jn/q_i} \varphi_{i,j-l}(2^j y)$. If for each $1 \leqslant i \leqslant m$, $\varphi_{i,j} = \varphi_{i,k}$ for all $j, k \in \mathbb{Z}$ then $T^l = T$. Also, if all the operators T^l are bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for some $1 < p, q < \infty$, and $0 < \sup_l \|T^l\|_{p,q} \leqslant C < \infty$, then $1/q = 1/p - r$. Indeed for $l \in \mathbb{Z}$ we denote $f_l(x) = 2^{-ln} f(2^{-l}x)$ then

$$T(f_l)(x) = 2^{-ln(1-r)} T^l f(2^{-l}x),$$

so

$$\begin{aligned}\|Tf\|_q = \|T((f_{-l})_l)\|_q &\leqslant 2^{-ln(1-r)+nl/q} \|T^l(f_{-l})\|_q \\ &\leqslant C 2^{-ln(1-r)+l\frac{n}{q}} \|f_{-l}\|_p = C 2^{-ln(1/q-1/p+r)} \|f\|_p\end{aligned}$$

and then $1/q - 1/p + r = 0$.

With respect to the endpoint $(p, q) = (1, 1/(1-r))$ and $(p, q) = (1/r, 0)$, as in the case of the Riesz potentials, we can not expect $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness. For the first one we take $f = \chi_B$ the characteristic function of the unit ball of $\mathbb{R}^n$ and $k(x, y) = 1/|x - A_1 y|^{n/q_1} \dots 1/|x - A_m y|^{n/q_m}$. A simple computation shows that for $|x| \gg 1$, $Tf(x) \geqslant c/|x|^{n(1-r)}$ and then $Tf \notin L^{1/(1-r)}$. The second case follows by duality.

Lemma 1. *If $k(x, y)$ is the kernel defined by* (1) *and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex then*

$$\Big|\frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}} k(x, y)\Big| \leqslant c \Big(\prod_{i=1}^m |x - A_i y|^{-\frac{n}{q_i}}\Big) \Big(\sum_{l=1}^m |x - A_l y|^{-1}\Big)^{|\alpha|}$$

with c independent of x, y.

P r o o f. We denote $D_y^\alpha = \partial^{|\alpha|}/\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}$. By the Leibniz formula,

$$\begin{aligned}D_y^\alpha k(x, y) &= D_y^\alpha \Big(\prod_{1\leqslant i\leqslant m} k_i(x - A_i y)\Big) \\ &= \sum_{\Gamma_1+\dots+\Gamma_m=\alpha} c_{\Gamma_1,\dots,\Gamma_m} D_y^{\Gamma_1}(k_1(x - A_1 y)) \dots D_y^{\Gamma_m}(k_m(x - A_m y)),\end{aligned}$$

now

$$k_i(x - A_i y) = \sum_{j\in\mathbb{Z}} 2^{jn/q_i} \varphi_{i,j}(2^j(x - A_i y)).$$

For each fixed x only a finite number of j's (independent of x) are involved in the above sum, also $2^j \leqslant 2|x-A_iy|^{-1}$ for $2^j(x-A_iy) \in \operatorname{supp}\varphi_{i,j}$, also $\sup_{j\in\mathbb{Z}} \|D^\alpha \varphi_{i,j}\|_\infty < \infty$, so

$$|D_y^{\Gamma_i}(k_i(x-A_iy))| = \left|\sum_{j\in\mathbb{Z}} 2^{jn/q_i} D_y^{\Gamma_i}(\varphi_{i,j}(2^j(x-A_iy)))\right| \leqslant c|x-A_iy|^{-n/q_i-|\Gamma_i|}$$

thus

$$\begin{aligned}|D_y^\alpha k(x,y)| &\leqslant c \sum_{\Gamma_1+\ldots+\Gamma_m=\alpha} c_{\Gamma_1,\ldots,\Gamma_m} \prod_{1\leqslant i\leqslant m} |x-A_iy|^{-n/q_i-|\Gamma_i|} \\ &= c\Big(\prod_{1\leqslant i\leqslant m} |x-A_iy|^{-n/q_i}\Big)\Big(\sum_{\Gamma_1+\ldots+\Gamma_m=\alpha} c_{\Gamma_1,\ldots,\Gamma_m} \prod_{1\leqslant i\leqslant m} |x-A_iy|^{-|\Gamma_i|}\Big) \\ &\leqslant c\Big(\prod_{1\leqslant i\leqslant m} |x-A_iy|^{-n/q_i}\Big)\Big(\sum_{1\leqslant l\leqslant m} |x-A_ly|^{-1}\Big)^{|\alpha|}.\end{aligned}$$

□

3. The main results

As we have said in the introduction, in the case that A_i is a multiple of the identity, in [3] the authors obtain that T is well defined on $L^p(\mathbb{R}^n)$ and that it is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < 1/r$ and $1/q = 1/p - r$. We will show that with slight modifications on the proofs, this result still holds for A_i satisfying the above stated hypothesis.

Proposition 2. *Let T be the operator defined by* (2). *If $1 < p < 1/r$, $0 \leqslant r < 1$ and $1/q = 1/p - r$, then T is a well defined and bounded operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Proof. As in the proof of Lemma 2.1 in [3] we obtain that for $l \in \mathbb{Z}$, $1/(1-r) < p \leqslant \min_{1\leqslant i\leqslant m} p_i/q_i(1-r)$

$$\Big\| \sum_{s_1,\ldots,s_m\leqslant -l} \prod_{1\leqslant i\leqslant m} 2^{s_in/q_i}\varphi_{i,s_i}(2^{s_i}(x-A_iy))\Big\|_{L^p(\,\mathrm{d}y)} \leqslant c2^{nl/p},$$

and also as in the proof of Lemma 2.2 in the same paper,

$$\Big\| \sum_{s_i\geqslant -l} 2^{s_in/q_i}\varphi_{i,s_i}(2^{s_i}(x-A_iy)) \prod_{j\neq i} 2^{-ln/q_j}\varphi_{j,-l}(2^{-l}(x-A_jy))\Big\|_{L^p(\,\mathrm{d}y)} \leqslant c,$$

with c independent of x and l. Now we follow the proof of Theorem 3.1 in [3] with the following changes. We take

$$d = \min_{1\leqslant i\leqslant m}\Big(\min_{|y|=1}\frac{|A_i(y)|}{2}, \min_{|y|=1, j\neq i}\frac{|A_i(y)-A_j(y)|}{2}\Big)$$

and

$$D = \max_{1\leqslant i\leqslant m,\ |y|=1} |A_i(y)|,$$

for $x\in\mathbb{R}^n\backslash\{0\}$ we define $l=l(x)$ such that $2^l\leqslant |x|\leqslant 2^{l+1}$ and we set, for $1\leqslant i\leqslant m$,

$$R_i = R_i(x) = \{y\in\mathbb{R}^n\colon\ |y-A_i(x)|\leqslant 2^l d\},$$

we also set

$$R_{m+1} = \{y\in\mathbb{R}^n\colon\ |y|\leqslant 2^l D\}\cap\Big(\bigcup_{1\leqslant i\leqslant m} R_i\Big)^c \quad\text{and}\quad R_{m+2} = \Big(\bigcup_{1\leqslant i\leqslant m+1} R_i\Big)^c.$$

□

Let $0<p\leqslant 1$. We recall that a p-atom is a measurable function a supported on a ball B of $\mathbb{R}^n$ satisfying

a) $\|a\|_\infty\leqslant |B|^{-1/p}$,

b) $\int y^\beta a(y)\,\mathrm{d}y = 0$ for every multiindex β with $|\beta|\leqslant n(p^{-1}-1)$.

It is well known that for $0<p\leqslant 1$ the distributions of $H^p(\mathbb{R}^n)$ can be approximated by adequate linear combinations of p-atoms. (See Theorem 2, p. 107 in [7].)

Theorem 3.1. *Let T be the operator defined by (2). If $0\leqslant r<1$, $0<p\leqslant 1$ and $1/q=1/p-r$, then T is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

P r o o f. If $0\leqslant r<1$, $0<p\leqslant 1$, $1/q=1/p-r$ and $f\in H^p(\mathbb{R}^n)$ we write $f=\sum\limits_{j\in\mathbb{N}}\lambda_j a_j$, where a_j is a p-atom and $\sum\limits_{j\in\mathbb{N}}|\lambda_j|^p\leqslant c\|f\|^p_{H^p}$. So the theorem will be proved if we obtain that there exists $c>0$ such that $\|Ta\|_{L^q}\leqslant c$ with c independent of the p-atom a, since this estimate and the inequality $\Big(\sum\limits_{j\in\mathbb{N}}|\lambda_j|^q\Big)^{1/q}\leqslant\Big(\sum\limits_{j\in\mathbb{N}}|\lambda_j|^p\Big)^{1/p}$ give $\|Tf\|_q\leqslant c\|f\|_{H^p}$. We denote by $B(y_0,\delta)$ the closed ball centered at y_0 with radius δ. Let a be supported on a ball $B=B(y_0,\delta)$, and for each $1\leqslant i\leqslant m$ let $B_i^*=B(A_iy_0,4D\delta)$ with D defined as in the proof of Proposition 2. We decompose $\mathbb{R}^n=\bigcup\limits_{1\leqslant i\leqslant m}B_i^*\cup R$, where $R=\Big(\bigcup\limits_{1\leqslant i\leqslant m}B_i^*\Big)^c$. Proposition 2 gives that T is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ for $1/q_0 = 1/p_0 - r$, $1 < p_0 < 1/r$. Since $q < q_0$ we use the Hölder inequality with q_0/q and $q_0/(q_0 - q)$ to obtain

$$\begin{aligned}\int_{\bigcup_{1\leqslant i\leqslant m} B_i^*} |Ta(x)|^q \,\mathrm{d}x &\leqslant \sum_{1\leqslant i\leqslant m} \int_{B_i^*} |Ta(x)|^q \,\mathrm{d}x \\ &\leqslant c \sum_{1\leqslant i\leqslant m} |B_i^*|^{1-q/q_0} \|Ta\|_{q_0}^q \leqslant c\delta^{n-nq/q_0} \|a\|_{p_0}^q \\ &\leqslant c\delta^{n-nq/q_0} \Big(\int_B |a|^{p_0} \Big)^{q/p_0} \leqslant c\delta^{n-nq/q_0} \delta^{-nq/p} \delta^{nq/p_0} = c.\end{aligned}$$

To study the integral on

$$R = \{x \in \mathbb{R}^n \colon\ |x - A_i y_0| > 4\delta,\ \text{for all } 1 \leqslant i \leqslant m\},$$

we suppose $n/(n+N) < p \leqslant n/(n+N-1)$ for some $N \in \mathbb{N}$. Let $k(x,y)$ be defined by (1). The moment condition b) satisfied by the p-atom a allows us to write

$$\text{(4)} \qquad \int_R \Big| \int_B k(x,y)a(y)\,\mathrm{d}y \Big|^q \mathrm{d}x = \int_R \Big| \int_B (k(x,y) - q_N(x,y))a(y)\,\mathrm{d}y \Big|^q \mathrm{d}x$$

where $q_N(x,y)$ is the degree $N-1$ Taylor polynomial of the function $y \to k(x,y)$ expanded around y_0. By the standard estimate of the remainder term in the Taylor expansion, there exists ξ between y and y_0 such that

$$\begin{aligned}|k(x,y) - q_N(x,y)| &\leqslant c|y-y_0|^N \sum_{k_1+\ldots+k_n=N} \Big| \frac{\partial^N}{\partial y_1^{k_1} \ldots \partial y_n^{k_n}} k(x,\xi) \Big| \\ &\leqslant c|y-y_0|^N \Big(\prod_{i=1}^m |x - A_i\xi|^{-n/q_i} \Big) \Big(\sum_{l=1}^m |x - A_l\xi|^{-1} \Big)^N,\end{aligned}$$

where the last inequality follows from Lemma 1. Since $x \in R$ and $y \in B$, it follows that $|x - A_i\xi| \geqslant c|x - A_i y_0|$ for $1 \leqslant i \leqslant m$. So

$$\text{(5)} \quad |k(x,y) - q_N(x,y)| \leqslant c|y-y_0|^N \Big(\prod_{i=1}^m |x - A_i y_0|^{-n/q_i} \Big) \Big(\sum_{l=1}^m |x - A_l y_0|^{-1} \Big)^N.$$

For $1 \leqslant k \leqslant m$, let

$$R_k = \{x \in R \colon\ |x - A_k y_0| \leqslant |x - A_j y_0| \text{ for all } j \neq k\}.$$

We note that $R = \bigcup_{k=1}^{m} R_k$ and that $R_k \subseteq (B_k^*)^c$. So, from (4) and (5), we have

$$\begin{aligned}
&\int_R \left| \int_B k(x,y)a(y)\,\mathrm{d}y \right|^q \mathrm{d}x \\
&\leqslant c \int_R \left(\int_B \left(\prod_{i=1}^{m} |x - A_i y_0|^{-n/q_i} \right) \left(\sum_{l=1}^{m} |x - A_l y_0|^{-1} \right)^N |y - y_0|^N |a(y)|\,\mathrm{d}y \right)^q \mathrm{d}x \\
&\leqslant c \sum_{1 \leqslant k \leqslant m} \int_{R_k} \prod_{i=1}^{m} |x - A_i y_0|^{-qn/q_i} \left(\sum_{l=1}^{m} |x - A_l y_0|^{-1} \right)^{qN} \left(\int_B |y - y_0|^N |a(y)|\,\mathrm{d}y \right)^q \mathrm{d}x \\
&\leqslant c \sum_{1 \leqslant k \leqslant m} \int_{(B_k^*)^c} \left(\int_B |y - y_0|^N |a(y)|\,\mathrm{d}y \right)^q |x - A_k y_0|^{-qn(1-r)} (m|x - A_k y_0|^{-1})^{qN}\,\mathrm{d}x \\
&\leqslant c \sum_{1 \leqslant k \leqslant m} \delta^{qN - nq/p + nq} \int_{4D\delta}^{\infty} t^{-q(n(1-r)+N)+n-1}\,\mathrm{d}t \leqslant c,
\end{aligned}$$

with c independent of the p-atom a, since $-q(n(1-r)+N)+n<0$. □

We recall that a locally integrable function f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if the inequality

$$\frac{1}{|B|} \int_B |f(x) - f_B|\,\mathrm{d}x \leqslant A$$

holds for all balls $B \subset \mathbb{R}^n$; here $f_B = |B|^{-1} \int_B f\,\mathrm{d}x$. The dual result to the previous theorem, corresponding to the case $p = 1$, is the following.

Corollary 3. *Let T be the operator defined by* (2). *Then T is bounded from $L^{1/r}(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$ for $0 \leqslant r < 1$.*

P r o o f. Is is well known that the dual space of $H^1(\mathbb{R}^n)$ is the space $\mathrm{BMO}(\mathbb{R}^n)$. Let $\widetilde{T}$ be the integral operator with kernel $\tilde{k}(x,y) = \widetilde{k_1}(x - A_1^{-1}y) \ldots \widetilde{k_m}(x - A_m^{-1}y)$, with $\widetilde{k_i}(x) = k_i(A_i x)$. Since for each $1 \leqslant i \leqslant m$, it can be checked that A_i^{-1} is invertible and $A_i^{-1} - A_j^{-1}$ is invertible if $i \neq j$, the previous theorem gives us the boundedness of $\widetilde{T}$ from $H^1(\mathbb{R}^n)$ into $L^{1/(1-r)}$. Now it is easy to check that T is the adjoint operator of $\widetilde{T}$, so the corollary follows. □

4. A counterexample

In this section we show that we can not expect that operators of the form (2) be bounded from $H^p(\mathbb{R})$ into $H^q(\mathbb{R})$ with $0 < p \leqslant 1/(1+r)$ and $1/q = 1/p - r$.

For $n = 1$ and $0 < r < 1$ we consider the integral operator

$$T_r f(x) = \int \frac{f(y)\,\mathrm{d}y}{|x-y|^{(1-r)/2}|x+y|^{(1-r)/2}},$$

we will show that for a given 1-atom a, $\int T_r a(x)\,\mathrm{d}x \neq 0$.

We observe that $T_r a \in L^1(\mathbb{R})$ and that $\int T_r a(x)\,\mathrm{d}x = \widehat{(T_r a)}(0)$, where the Fourier transform of a integrable function f is given by $\widehat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\xi} f(x)\,\mathrm{d}x$. Thus it is enough to show that $\widehat{(T_r a)}(0) \neq 0$. Let $\varphi \in S(\mathbb{R})$ be an even function such that $\varphi(0) = 1$ and for $\varepsilon > 0$ let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Now $\widehat{(T_r a)}(0) = \lim\limits_{\varepsilon\to 0} \widehat{(\varphi_\varepsilon T_r a)}(0)$ so we will compute

$$\begin{aligned}
\widehat{(\varphi_\varepsilon T_r a)}(0) &= \int \varphi(\varepsilon x)\Big(\int |x^2-y^2|^{(r-1)/2} a(y)\,\mathrm{d}y\Big)\,\mathrm{d}x\\
&= \int a(y)\Big(\int |x^2-y^2|^{(r-1)/2}\varphi(\varepsilon x)\,\mathrm{d}x\Big)\,\mathrm{d}y\\
&= \int a(y)|y|^r\Big(\int |z^2-1|^{(r-1)/2}\varphi(\varepsilon|y|z)\,\mathrm{d}z\Big)\,\mathrm{d}y\\
&= \int a(y)|y|^r\Big(\int \widehat{(|z^2-1|^{(r-1)/2})}(\sigma)\widehat{(\varphi_{\varepsilon|y|})}(\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}y.
\end{aligned}$$

Since $-\frac{1}{2} < -\frac{1}{2}r < 0$, the Fourier transform of the function $|z^2-1|^{(r-1)/2}$ is

$$\Gamma\Big(\frac{r+1}{2}\Big)\sqrt{\pi}\Big[\Big(\frac{\sigma}{2}\Big)^{-r/2} J_{r/2}(\sigma) + \Big|\frac{\sigma}{2}\Big|^{-r/2}\Big(\frac{\cos(\pi r/2)J_{-r/2}(|\sigma|) - J_{r/2}(|\sigma|)}{\sin(\pi r/2)}\Big)\Big],$$

where

$$J_p(s) = \frac{2(s/2)^p}{\Gamma(p+\frac{1}{2})\sqrt{\pi}}\int_0^1 (1-t^2)^{p-\frac{1}{2}}\cos(st)\,\mathrm{d}t$$

is the Bessel function of order $p > -\frac{1}{2}$ (see p. 185–188 in [2]). So

$$\begin{aligned}
&\widehat{(\varphi_\varepsilon T_r a)}(0)\\
&= c_r \int a(y)\int |\varepsilon\sigma|^{-r}\Big(\int_0^1 (1-t^2)^{(r-1)/2}\cos(\varepsilon|y||\sigma|t)\,\mathrm{d}t\Big)\widehat{\varphi}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y\\
&\quad + 2\Big(1-\frac{1}{\sin(\pi r/2)}\Big)\int a(y)|y|^r\int\Big(\int_0^1 (1-t^2)^{(r-1)/2}\cos(\varepsilon|y||\sigma|t)\,\mathrm{d}t\Big)\widehat{\varphi}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y,
\end{aligned}$$

thus it is easy to check that

$$\lim_{\varepsilon\to 0}\widehat{(\varphi_\varepsilon T_r a)}(0)=2\Big(1-\frac{1}{\sin(\pi r/2)}\Big)\int_0^1(1-t^2)^{(r-1)/2}\,\mathrm{d}t\int a(y)|y|^r\,\mathrm{d}y.$$

We take the 1-atom

$$a_\delta(y)=\begin{cases}2\delta & \text{for } -\frac12\leqslant y\leqslant 0,\\ -\delta & \text{for } 0<y\leqslant 1\end{cases}$$

with $0<\delta\leqslant\frac13$. A computation shows that $\int a_\delta(y)|y|^r\,\mathrm{d}y=\delta(2^{-r}-1)/(r+1)$, so

$$\int T_r a_\delta(x)\,\mathrm{d}x=\widehat{(T_r a_\delta)}(0)=2\delta\,\frac{2^{-r}-1}{r+1}\Big(1-\frac{1}{\sin(\pi r/2)}\Big)\int_0^1(1-t^2)^{(r-1)/2}\,\mathrm{d}t\neq 0.$$

We note that

$$\lim_{r\to 0}\int T_r a_\delta(x)\,\mathrm{d}x=2\delta\ln(2)=\int T_0 a_\delta(x)\,\mathrm{d}x,$$

where the last equality is computed in [5]. Also $a_\delta\in H^p(\mathbb{R})$ for $\frac12<p\leqslant 1/(1+r)$, and $T_r a_\delta$ does not belong to $H^q(\mathbb{R})$ for $1/q=1/p-r$ since $\int T_r a_\delta\neq 0$. For $0<p\leqslant\frac12$ we take N any fixed integer with $N>p^{-1}-1$, then the set of all bounded, compactly supported functions for which $\int_{\mathbb{R}}x^\alpha f(x)\,\mathrm{d}x=0$ for all α with $0\leqslant\alpha<N$, is dense in $H^p(\mathbb{R})$ (see 5.2b), p. 128 in [7]). In particular, there exists $b\in H^p(\mathbb{R})$ such that $\|a_\delta-b\|_{H^{1/(1+r)}(\mathbb{R})}<|\widehat{(T_r a_\delta)}(0)|/2c$. Then

$$\begin{aligned}\Big|\int T_r b(x)\,\mathrm{d}x\Big|&\geqslant\Big|\int T_r a_\delta(x)\,\mathrm{d}x\Big|-\int|T_r b(x)-T_r a_\delta(x)|\,\mathrm{d}x\\&\geqslant|\widehat{(T_r a_\delta)}(0)|-c\|a_\delta-b\|_{H^{1/(1+r)}(\mathbb{R})}\geqslant\frac{|\widehat{(T_r a_\delta)}(0)|}{2},\end{aligned}$$

where the second inequality follows from Theorem 3.1 with $p=1/(1+r)$. But then T_r is not bounded on $H^p(\mathbb{R})$ into $H^q(\mathbb{R})$ for $1/q=1/p-r$, since $\int T_r b(x)\,\mathrm{d}x\neq 0$.

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