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BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS WITH INFINITE TIME HORIZON*

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Abstract. We give a sufficient condition on the coefficients of a class of infinite horizon backward doubly stochastic differential equations (BDSDES), under which the infinite horizon BDSDES have a unique solution for any given square integrable terminal values. We also show continuous dependence theorem and convergence theorem for this kind of equations.

Keywords: infinite horizon backward doubly stochastic differential equations, filtration, backward stochastic integral

MSC 2010: 60H10

1. Introduction

Since the nonlinear backward stochastic differential equations (BSDEs in short) were introduced by Pardoux and Peng [9], the theory of BSDEs has been developed by many researchers in a series of papers (for example, Ma, Protter, and Yong [7], El Karoui, Peng, and Quenez [6] and the references therein). These papers basically study BSDEs for a fixed terminal time $T > 0$, i.e., in a finite time interval $[0, T]$. In order to investigate the case of infinite time interval, i.e., $T = \infty$, many researchers, for example, Peng [12], Darling and Pardoux [5], Peng and Shi [14] and so on, present many different assumptions. However, their results essentially require the terminal values to be decay in infinite horizon. Later Chen and Wang [4] were the first to show a kind of sufficient conditions on coefficients, under which for any square integrable random variables ξ as terminal values, BSDEs still have a unique pair of solutions

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for infinite horizon case. This result is pivotal for discussing the convergence of g-martingales which were introduced by Peng [13].

After Pardoux and Peng [9] introduced the theory of BSDEs, they [10] brought forward a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral $\mathrm{d}W_t$ and a backward stochastic integral $\mathrm{d}B_t$. They have proved the existence and uniqueness of solutions to BDSDEs under uniformly Lipschitz conditions on coefficients on a finite time interval $[0, T]$. That is, for a given terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f, g, given $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, the following BDSDE has a unique solution pair (y_t, z_t) in the interval $[0, T]$:

$$(1)\quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}B_s - \int_t^T Z_s\,\mathrm{d}W_s, \quad t \in [0, T].$$

Pardoux and Peng also showed that BDSDEs can produce a probabilistic representation for certain quasilinear stochastic partial differential equations (SPDEs). Many researchers do their work in this area (for example, V. Bally and A. Matoussi [1], R. Buckdahn and J. Ma [2], [3], E. Pardoux [11], Peng and Shi [15], Zhu and Han [17] and the references therein). Infinite horizon BDSDEs are also very interesting, since they produce a probabilistic representation of certain quasilinear stochastic partial differential equations. Recently, Zhang and Zhao [16] got stationary solutions of SPDEs and infinite horizon BDSDEs, but under the assumption that the terminal value $\lim\limits_{T\to\infty} \mathrm{e}^{-KT} Y_T = 0$. This paper intends to study the existence and uniqueness of BDSDE (1) when $T = \infty$, our method being different from Zhang and Zhao. Due to sufficiently utilizing the properties of martingales, this method is essential for the theory of BSDEs. In this paper we give a sufficient condition on coefficients f, g under which for any square integrable random variable ξ, BDSDE (1) still has a unique solution pair when $T = \infty$. It is worth noting that in our argument, we have to restrict g to be independent of z. For the case of g being dependent on z, it is still an interesting open question. Our conditions are a special kind of Lipschitz conditions, which even include some cases of unbounded coefficients. At the end we will also give a continuous dependence theorem and a convergence theorem for this class of equations.

The paper is organized as follows: in Section 2 we present the setting of problems and the main assumptions; in Section 3 we prove the existence and uniqueness theorem for BDSDEs; at the end we discuss the continuous dependence theorem and the convergence theorem in Section 4.

2. Setting of infinite horizon BDSDEs

Notation. The Euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $d \times k$ matrix A, we define $\|A\| = \sqrt{\operatorname{Tr} AA^*}$, where A^* is the transpose of A.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{W_t\}_{t \geqslant 0}$ and $\{B_t\}_{t \geqslant 0}$ be two mutually independent standard Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, \infty)$, we define

$$\mathcal{F}_{0,t}^W \doteq \sigma\{W_r;\ 0 \leqslant r \leqslant t\} \vee \mathcal{N}, \quad \mathcal{F}_{t,\infty}^B \doteq \sigma\{B_r - B_t;\ t \leqslant r < \infty\} \vee \mathcal{N},$$
$$\mathcal{F}_{0,\infty}^W \doteq \bigvee_{0 \leqslant t < \infty} \mathcal{F}_{0,t}^W, \quad \mathcal{F}_{\infty,\infty}^B \doteq \bigcap_{0 \leqslant t < \infty} \mathcal{F}_{t,\infty}^B,$$

and

$$\mathcal{F}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,\infty}^B, \quad t \in [0, \infty].$$

Note that $\{\mathcal{F}_{0,t}^W;\ t \in [0, \infty]\}$ is an increasing filtration and $\{\mathcal{F}_{t,\infty}^B;\ t \in [0, \infty]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t,\ t \in [0, \infty]\}$ is neither increasing nor decreasing.

Suppose

$$\mathcal{F} = \mathcal{F}_\infty \doteq \mathcal{F}_{0,\infty}^W \vee \mathcal{F}_{\infty,\infty}^B.$$

For any $n \in \mathbb{N}$, let $S^2(\mathbb{R}^+; \mathbb{R}^n)$ denote the space of all $\{\mathcal{F}_t\}$-measurable n-dimensional processes v with the norm $\|v\|_S \doteq \Big[E\Big(\sup_{s \geqslant 0} |v(s)|\Big)^2\Big]^{1/2} < \infty$.

We denote similarly by $M^2(\mathbb{R}^+; \mathbb{R}^n)$ the space of all (classes of $\mathrm{d}P \otimes \mathrm{d}t$ a.e. equal) $\{\mathcal{F}_t\}$-measurable n-dimensional processes v with the norm

$$\|v\|_M \doteq \left[E \int_0^\infty |v(s)|^2 \,\mathrm{d}s\right]^{1/2} < \infty.$$

For any $t \in [0, \infty]$, let $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ denote the space of all $\{\mathcal{F}_t\}$-measurable n-valued random variables ξ satisfying $E|\xi|^2 < \infty$.

We also denote

$$B^2 \doteq \{(X, Y);\ X \in S^2(\mathbb{R}^+; \mathbb{R}^n),\, Y \in M^2(\mathbb{R}^+; \mathbb{R}^n)\}.$$

For each $(X, Y) \in B^2$, we define the norm of (X, Y) by

$$\|(X, Y)\|_B \doteq (\|X\|_S^2 + \|Y\|_M^2)^{1/2}.$$

Obviously B^2 is a Banach space.

Consider the infinite horizon backward doubly stochastic differential equation

$$y_t = \xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s - \int_t^\infty z_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant \infty, \tag{2}$$

where $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ is given. We note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral, see Naulart and Pardoux. Our aim is to find some conditions under which BDSDE (2) has a unique solution. Now we give the definition of a solution of BDSDE (2).

Definition 1. A pair of processes $(y, z)\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^k \times \mathbb{R}^{k\times d}$ is called a solution of BDSDE (2), if $(y, z) \in B^2$ and satisfies BDSDE (2).

Let

$$f\colon \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k,$$
$$g\colon \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \to \mathbb{R}^{k\times l}$$

satisfy the following assumptions:

(H1) For any $\omega \times t \in \Omega \times \mathbb{R}^+$, $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $f(\cdot, y, z)$ and $g(\cdot, y)$ are $\{\mathcal{F}_t\}$-progressively measurable processes such that

$$E\Big(\int_0^\infty f(t, 0, 0)\,\mathrm{d}t\Big)^2 < \infty; \quad g(\cdot, 0) \in M^2(\mathbb{R}^+; \mathbb{R}^{k\times l}).$$

(H2) f and g satisfy the Lipschitz condition with Lipschitz coefficients $v := \{v(t)\}$ and $u := \{u(t)\}$, that is, there exist two positive non-random functions $\{v(t)\}$ and $\{u(t)\}$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leqslant v(t)|y_1 - y_2| + u(t)\|z_1 - z_2\|,$$
$$\|g(t, y_1) - g(t, y_2)\| \leqslant u(t)|y_1 - y_2|$$

for all $(t, y_i, z_i) \in \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $i = 1, 2$.

(H3) $\int_0^\infty v(t)\,\mathrm{d}t < \infty$; $\int_0^\infty u^2(t)\,\mathrm{d}t < \infty$.

3. Existence and uniqueness theorem

The following existence and uniqueness theorem is our main result.

Theorem 1. *Under the above conditions, in particular* (H1), (H2), *and* (H3), *Eq.* (2) *has a unique solution* $(y, z) \in B^2$.

In order to prove the existence and uniqueness theorem, we first give a priori estimate.

Lemma 2. *Suppose* (H1), (H2), *and* (H3) *hold for* f *and* g. *For any* $T \in [0,\infty]$, *let* $Y_T^i \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, (Y^i, Z^i) *and* $(y^i, z^i) \in B^2$ $(i = 1, 2)$ *satisfy the equation*

$$
(3)\qquad \begin{aligned} Y_t^i = Y_T^i &+ \int_t^T f(s, y_s^i, z_s^i)\,\mathrm{d}s + \int_t^T g(s, y_s^i)\,\mathrm{d}B_s \\ &- \int_t^T Z_s^i\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T \leqslant \infty. \end{aligned}
$$

Then there is a constant $C > 0$ *such that, for any* $\tau \in [0, T]$,

$$
\begin{aligned} &\|((Y^1 - Y^2)I_{[\tau,T]}, (Z^1 - Z^2)I_{[\tau,T]})\|_B^2 \\ &\qquad \leqslant C[E|Y_T^1 - Y_T^2|^2 + l_{[\tau,T]}\|((y^1 - y^2)I_{[\tau,T]}, (z^1 - z^2)I_{[\tau,T]})\|_B^2] \end{aligned}
$$

where $l_{[\tau,T]} = (\int_\tau^T v(s)\,\mathrm{d}s)^2 + \int_\tau^T u^2(s)\,\mathrm{d}s$ *and* $I_{[\tau,T]}(\cdot)$ *is an indicator function.*

Proof. Without loss of generality, we assume that $\tau = 0$, $T = \infty$, otherwise we can replace f by $fI_{[\tau,T]}$ and g by $gI_{[\tau,T]}$.

Set

$$
\begin{aligned} \hat{Y}_t = Y_t^1 - Y_t^2, \quad \hat{Z}_t = Z_t^1 - Z_t^2, \quad \hat{y}_t = y_t^1 - y_t^2, \quad \hat{z}_t = z_t^1 - z_t^2, \\ \hat{f}_t = f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2), \quad \hat{g}_t = g(t, y_t^1) - g(t, y_t^2). \end{aligned}
$$

Then

$$
(4)\qquad \hat{Y}_t = \hat{Y}_T + \int_t^T \hat{f}_s\,\mathrm{d}s + \int_t^T \hat{g}_s\,\mathrm{d}B_s - \int_t^T \hat{Z}_s\,\mathrm{d}W_s.
$$

We define the filtration $\{\mathcal{G}_t\}_{0 \leqslant t \leqslant T}$ by

$$
\mathcal{G}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{0,\infty}^B.
$$

Obviously $\mathcal{G}_t$ is an increasing filtration. Since $(\hat{Y}, \hat{Z}) \in B^2$, $\{\int_0^t \hat{Z}_s\,\mathrm{d}W_s\}$ is a $\mathcal{G}_t$-martingale. Thus from (4) it follows that

$$
\hat{Y}_t = E^{\mathcal{G}_t}\left[\hat{Y}_T + \int_t^T \hat{f}_s\,\mathrm{d}s + \int_t^T \hat{g}_s\,\mathrm{d}B_s\right].
$$

Note that

$$
\begin{aligned}
E\Big(\int_0^\infty |\hat f_s|\,\mathrm{d}s\Big)^2 &\leqslant E\Big(\int_0^\infty (v(s)|\hat y_s| + u(s)\|\hat z_s\|)\,\mathrm{d}s\Big)^2\\
&\leqslant 2E\Big(\sup_{t\geqslant 0}|\hat y_t|\cdot\int_0^\infty v(s)\,\mathrm{d}s\Big)^2 + 2E\Big(\int_0^\infty u^2(s)\,\mathrm{d}s\cdot\int_0^\infty \|\hat z_s\|^2\,\mathrm{d}s\Big)\\
&\leqslant 2\Big[\Big(\int_0^\infty v(s)\,\mathrm{d}s\Big)^2 + \int_0^\infty u^2(s)\,\mathrm{d}s\Big]\cdot\|(\hat y,\hat z)\|_B^2 < \infty,
\end{aligned}
$$

and

$$
E\int_0^\infty \|\hat g_s\|^2\,\mathrm{d}s \leqslant E\int_0^\infty u^2(s)|\hat y_s|^2\,\mathrm{d}s \leqslant \Big(\int_0^\infty u^2(s)\,\mathrm{d}s\Big)\cdot\|\hat y\|_S^2 < \infty.
$$

Applying the Doob inequality and the B-D-G inequality, we can deduce

$$
\begin{aligned}
(6)\quad \|\hat Y\|_S^2 &= E(\sup_{t\geqslant 0}|\hat Y_t|)^2\\
&\leqslant 2E\Big(\sup_{t\geqslant 0}E^{\mathcal{G}_t}\Big[|\hat Y_T| + \int_t^T |\hat f_s|\,\mathrm{d}s\Big]\Big)^2 + 2E\Big(\sup_{t\geqslant 0}E^{\mathcal{G}_t}\Big[\Big|\int_t^T \hat g_s\,\mathrm{d}B_s\Big|\Big]\Big)^2\\
&\leqslant 8E\Big(|\hat Y_T| + \int_0^\infty |\hat f_s|\,\mathrm{d}s\Big)^2 + 2c_0E\int_0^\infty \|\hat g_s\|^2\,\mathrm{d}s\\
&\leqslant 16\Big(E|\hat Y_T|^2 + E\Big(\int_0^\infty |\hat f_s|\,\mathrm{d}s\Big)^2\Big) + 2c_0E\int_0^\infty \|\hat g_s\|^2\,\mathrm{d}s\\
&\leqslant (16+2c_0)\Big(E|\hat Y_T|^2 + E\Big(\int_0^\infty |\hat f_s|\,\mathrm{d}s\Big)^2 + E\int_0^\infty \|\hat g_s\|^2\,\mathrm{d}s\Big),
\end{aligned}
$$

where $c_0 > 0$ is a constant.

On the other hand, from (4) it follows that

$$
\begin{aligned}
(7)\qquad \|\hat Z\|_M^2 &= E\Big\langle \int_0^\cdot \hat Z_s\,\mathrm{d}W_s\Big\rangle_\infty\\
&= E\Big(\hat Y_T + \int_0^\infty \hat f_s\,\mathrm{d}s + \int_0^\infty \hat g_s\,\mathrm{d}B_s\Big)^2\\
&\quad - E\Big|E^{\mathcal{G}_0}\Big[\hat Y_T + \int_0^\infty \hat f_s\,\mathrm{d}s + \int_0^\infty \hat g_s\,\mathrm{d}B_s\Big]\Big|^2\\
&\leqslant E\Big(\hat Y_T + \int_0^\infty \hat f_s\,\mathrm{d}s + \int_0^\infty \hat g_s\,\mathrm{d}B_s\Big)^2\\
&\leqslant 3E\Big(|\hat Y_T|^2 + \Big(\int_0^\infty |\hat f_s|\,\mathrm{d}s\Big)^2 + \int_0^\infty \|\hat g_s\|^2\,\mathrm{d}s\Big),
\end{aligned}
$$

where $\langle M\rangle$ is the variation process generated by the martingale M.

Consequently, (6) and (7) imply that

$$\begin{aligned}\|(\hat{Y},\hat{Z})\|_B^2 &= \|\hat{Y}\|_S^2+\|\hat{Z}\|_M^2\\ &\leqslant (19+2c_0)\biggl(E|\hat{Y}_T|^2+E\biggl(\int_0^\infty|\hat{f}_s|\,\mathrm{d}s\biggr)^2+E\int_0^\infty|\hat{g}_s|^2\,\mathrm{d}s\biggr)\\ &\leqslant (57+6c_0)(E|\hat{Y}_T|^2+l_{[0,\infty]}\|(\hat{y},\hat{z})\|_B^2)\\ &= C(E|\hat{Y}_T|^2+l_{[0,\infty]}\|(\hat{y},\hat{z})\|_B^2),\end{aligned}$$

where $C=(57+6c_0)$ is a constant and $l_{[0,\infty]}=(\int_0^\infty v(s)\,\mathrm{d}s)^2+\int_0^\infty u^2(s)\,\mathrm{d}s$.

For any $\tau,T\in[0,\infty]$, we set $f_1(t,y_t,z_t)=f(t,y_t,z_t)I_{[\tau,T]}$, and $g_1(t,y_t)=g(t,y_t)I_{[\tau,T]}$. Then f_1, and g_1 satisfy the assumptions (H1), (H2), and (H3), and their Lipschitz constants are $vI_{[\tau,T]}$, $uI_{[\tau,T]}$. Repeating the above process, we can obtain the desired result. □

Now we give the proof of Theorem 1.

Proof. The proof of Theorem 1 is divided into two steps.

Step 1. We assume $l_{[0,\infty]}=(\int_0^\infty v(s)\,\mathrm{d}s)^2+\int_0^\infty u^2(s)\,\mathrm{d}s\leqslant 1/2C$. For any $(y,z)\in B^2$, let

$$M_t\doteq E^{\mathcal{G}_t}\biggl[\xi+\int_0^\infty f(s,y_s,z_s)\,\mathrm{d}s+\int_0^\infty g(s,y_s)\,\mathrm{d}B_s\biggr],\quad 0\leqslant t\leqslant\infty.$$

We will prove $\{M_t\}$ is a square integrable $\mathcal{G}_t$-martingale. From (H1)–(H3), it follows that

$$\begin{aligned}&E\biggl(\biggl|\xi\int_0^\infty f(s,y_s,z_s)\,\mathrm{d}s+\int_0^\infty g(s,y_s,z_s)\,\mathrm{d}B_s\biggr|^2\biggr)\\ &\quad\leqslant E\biggl(|\xi|+\int_0^\infty|f(s,y_s,z_s)|\,\mathrm{d}s+\biggl|\int_0^\infty g(s,y_s)\,\mathrm{d}B_s\biggr|\biggr)^2\\ &\quad\leqslant 3E|\xi|^2+9E\biggl(\int_0^\infty|f(s,0,0)|\,\mathrm{d}s\biggr)^2+9E\biggl(\int_0^\infty v(s)|y_s|\,\mathrm{d}s\biggr)^2\\ &\qquad+9E\biggl(\int_0^\infty u(s)\|z_s\|\,\mathrm{d}s\biggr)^2+6E\int_0^\infty(\|g(s,0)\|^2+u^2(s)|y_s|^2)\,\mathrm{d}s\\ &\quad\leqslant 3E|\xi|^2+9E\biggl(\int_0^\infty|f(s,0,0)|\,\mathrm{d}s\biggr)^2+9\biggl(\int_0^\infty v(s)\,\mathrm{d}s\biggr)^2\cdot\|y\|_S^2\\ &\qquad+9\int_0^\infty u^2(s)\,\mathrm{d}s\cdot\|z\|_M^2+6E\int_0^\infty\|g(s,0)\|^2\,\mathrm{d}s\\ &\qquad+6\int_0^\infty u^2(s)\,\mathrm{d}s\cdot\|y\|_S^2<\infty,\end{aligned}$$

which means $\{M_t\}$ is a square integrable $\mathcal{G}_t$-martingale. According to an obvious extension of Itô's martingale representation theorem, there exists a unique $\mathcal{G}_t$-progressively measurable process Z_t with values in $\mathbb{R}^{k\times d}$ such that

$$(8) \qquad E\int_0^\infty |Z_t|^2 \,\mathrm{d}t < \infty,$$

$$M_t = E^{\mathcal{G}_0}\left[\xi + \int_0^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_0^\infty g(s, y_s)\,\mathrm{d}B_s\right] + \int_0^t Z_s\,\mathrm{d}W_S, \quad 0 \leqslant t \leqslant \infty.$$

Let

$$(9) \qquad Y_t \doteq E^{\mathcal{G}_t}\left[\xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s\right], \quad 0 \leqslant t \leqslant \infty.$$

Then it is not difficult to check that (8) and (9) are equivalent to

$$(10) \quad Y_t = \xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s - \int_t^\infty Z_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant \infty.$$

We show that $\{Y_t\}$ and $\{Z_t\}$ are in fact $\mathcal{F}_t$-measurable. For Y_t, this is obvious since for each t,

$$Y_t = E^{\mathcal{G}_t}\left[\xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s\right] = E(\Theta/\mathcal{F}_t \vee \mathcal{F}^B_{0,t}),$$

where $\Theta = \xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s$ is indeed $\mathcal{F}^W_{0,\infty} \vee \mathcal{F}^B_{t,\infty}$-measurable. Hence, $\mathcal{F}^B_{0,t}$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = E(\Theta/\mathcal{F}_t).$$

Now

$$\int_t^\infty Z_s\,\mathrm{d}W_s = \xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s - Y_t$$

and the right-hand side is $\mathcal{F}^W_{0,\infty} \vee \mathcal{F}^B_{t,\infty}$-measurable. Thus, from Itô's martingale representation theorem, $\{Z_s,\ s > t\}$ is $\mathcal{F}^W_{0,s} \vee \mathcal{F}^B_{t,\infty}$-adapted. Consequently, Z_s is $\mathcal{F}^W_{0,s} \vee \mathcal{F}^B_{t,\infty}$-measurable for any $t < s$, and, thus, Z_t is $\mathcal{F}_t$-measurable. So $(Y, Z) \in B^2$. Therefore, equation (10) yields a mapping from B^2 to B^2, and we denote it by φ, that is,

$$\varphi\colon\ (y, z) \to (Y, Z).$$

If φ is a contractive mapping with respect to the norm $\|\cdot\|_B$, then by the fixed-point theorem there exists a unique $(y, z) \in B^2$ satisfying (10), which is just the unique solution to BDSDE (2).

Now we are in the position to prove that φ is a contractive mapping. Suppose $(y^i, z^i) \in B^2$, let (Y^i, Z^i) be the map φ of (y^i, z^i) $(i = 1, 2)$, that is

$$\varphi(y^i, z^i) = (Y^i, Z^i), \quad i = 1, 2.$$

We denote

$$\hat{Y} = Y^1 - Y^2, \quad \hat{Z} = Z^1 - Z^2, \quad \hat{y} = y^1 - y^2, \quad \hat{z} = z^1 - z^2,$$
$$\hat{f}_t = f(t, y^1, z^1) - f(t, y^2, z^2), \quad \hat{g}_t = g(t, y^1) - g(t, y^2).$$

By Lemma 2, we have

$$\|\varphi(y^1, z^1) - \varphi(y^2, z^2)\|_B^2 = \|(\hat{Y}, \hat{Z})\|_B^2 \leqslant C l_{[0,\infty]} \|(\hat{y}, \hat{z})\|_B^2.$$

Due to $l_{[0,\infty]} \leqslant 1/2C$, it follows that φ is a contractive mapping from B^2 to B^2.

Step 2. Since $\int_0^\infty v(t)\,\mathrm{d}t < \infty$; $\int_0^\infty u^2(t)\,\mathrm{d}t < \infty$, there exists a sufficiently large constant T such that

$$\left(\int_T^\infty v(s)\,\mathrm{d}s\right)^2 + \int_T^\infty u^2(s)\,\mathrm{d}s \leqslant \frac{1}{2C}.$$

Let

$$f_1(t, y, z) \doteq I_{[T,\infty]}(t) f(t, y, z), g_1(t, y) \doteq I_{[T,\infty]}(t) g(t, y),$$

then (H1)–(H3) hold for f_1 and g_1, whose Lipschitz coefficients are $\overline{v}(t) = I_{[T,\infty]} v(t)$ and $\bar{u}(t) = I_{[T,\infty]} u(t)$. Obviously,

$$\left(\int_0^\infty \overline{v}(s)\,\mathrm{d}s\right)^2 + \int_0^\infty \bar{u}^2(s)\,\mathrm{d}s \leqslant \frac{1}{2C}.$$

By Step 1, there exists a unique $(\tilde{y}, \tilde{z}) \in B^2$ satisfying

$$\tilde{y}_t = \xi + \int_t^\infty f_1(s, \tilde{y}_s, \tilde{z}_s)\,\mathrm{d}s + \int_t^\infty g_1(s, \tilde{y}_s)\,\mathrm{d}B_s - \int_t^\infty \tilde{z}_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant \infty.$$

For $(\tilde{y}_t, \tilde{z}_t)$ given as above, let us consider the infinite BDSDE

$$\begin{cases} \overline{y}_t = \int_t^T f(s, \overline{y}_s + \tilde{y}_s, \overline{z}_s + \tilde{z}_s)\,\mathrm{d}s \\ \qquad + \int_t^T g(s, \overline{y}_s + \tilde{y}_s)\,\mathrm{d}B_s - \int_t^T \overline{z}_s\,\mathrm{d}W_s, \ \ 0 \leqslant t \leqslant T; \\ \overline{y}_t \equiv 0, \ \ \overline{z}_t \equiv 0, \ \ t > T. \end{cases}$$

According to the results of Pardoux and Peng [10], the above BDSDE has a unique solution $(\overline{y}, \overline{z})$ in $[0, T]$, thus the above BDSDE has a unique solution such that $(\overline{y}, \overline{z}) \equiv (0, 0)$ for every $t > T$. Let

$$y \doteq \overline{y} + \tilde{y}, \quad z \doteq \overline{z} + \tilde{z}.$$

It is easy to check that (y_t, z_t) is the unique solution of (2). □

4. Continuous dependence theorem

In this section we will discuss the convergence of solutions of infinite horizon BDSDEs. First we give the following continuous dependence theorem.

Theorem 3. *Suppose* $\xi_i \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ $(i = 1, 2)$, *and* (H1)–(H3). *Let* (y^i, z^i) *be the solutions of BDSDE* (2) *corresponding to the terminal data* $\xi = \xi_1$, $\xi = \xi_2$, *respectively. Then there exists a constant* $\overline{C} > 0$ *such that*

$$\|(y^1 - y^2, z^1 - z^2)\|_B^2 \leqslant \overline{C} E|\xi_1 - \xi_2|^2.$$

Proof. Set $\hat{y} := y^1 - y^2$, $\hat{z} := z^1 - z^2$. Since $(\int_0^\infty v(s)\,\mathrm{d}s)^2 + \int_0^\infty u^2(s)\,\mathrm{d}s < \infty$, we can choose a strictly increasing sequence $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \infty$ such that

$$l_{[t_i, t_{i+1}]} = \left(\int_{t_i}^{t_{i+1}} v(s)\,\mathrm{d}s\right)^2 + \int_{t_i}^{t_{i+1}} u^2(s)\,\mathrm{d}s \leqslant \frac{1}{2C}, \quad i = 0, 1, \ldots, n.$$

Applying Lemma 2, we have

$$\begin{aligned}\|(\hat{y}, \hat{z}) I_{[t_i, t_{i+1}]}\|_B^2 &\leqslant C E|\hat{y}_{t_{i+1}}|^2 + C l_{[t_i, t_{i+1}]} \|(\hat{y}, \hat{z}) I_{[t_i, t_{i+1}]}\|_B^2 \\ &\leqslant C E|\hat{y}_{t_{i+1}}|^2 + \frac{1}{2}\|(\hat{y}, \hat{z}) I_{[t_i, t_{i+1}]}\|_B^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|(\hat{y}, \hat{z}) I_{[t_i, t_{i+1}]}\|_B^2 &\leqslant 2C E|\hat{y}_{t_{i+1}}|^2 \\ &\leqslant 2C E\left(\left(\sup_{t_{i+1} \leqslant s \leqslant t_{i+2}} |\hat{y}_s|\right)^2 + \int_{t_{i+1}}^{t_{i+2}} \|\hat{z}_s\|^2\,\mathrm{d}s\right) \\ &= 2C\|(\hat{y}, \hat{z}) I_{[t_{i+1}, t_{i+2}]}\|_B^2, \quad i = 0, 1, \ldots, n-1.\end{aligned} \tag{11}$$

In particular, we have

$$\|(\hat{y}, \hat{z}) I_{[t_n, \infty]}\|_B^2 \leqslant 2C E|\xi_1 - \xi_2|^2. \tag{12}$$

From (11) and (12), it follows that

$$\begin{aligned}\|(\hat{y}, \hat{z})\|_B^2 &\leqslant \sum_{i=0}^{n} \|(\hat{y}, \hat{z}) I_{[t_i, t_{i+1}]}\|_B^2 \\ &\leqslant (2C + (2C)^2 + \ldots + (2C)^{n+1}) E|\xi_1 - \xi_2|^2 \\ &= \frac{2C((2C)^{n+1} - 1)}{2C - 1} E|\xi_1 - \xi_2|^2 = \overline{C} E|\xi_1 - \xi_2|^2.\end{aligned}$$

Thus the desired result is obtained. □

Now we can assert the following convergence theorem for infinite horizon BDSDEs.

Theorem 4. *Suppose* $\xi, \xi_i \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ $(i = 1, 2, \ldots)$, *let* (H1)–(H3) *hold for* f *and* g. *Let* (y^i, z^i) *be the solutions of the following BDSDE:*

$$(13)\quad y_t^i = \xi_i + \int_t^\infty f(s, y_s^i, z_s^i)\,\mathrm{d}s + \int_t^\infty g(s, y_s^i)\,\mathrm{d}B_s - \int_t^\infty z_s^i\,\mathrm{d}W_s, \quad 0 \leqslant t < \infty.$$

If $E|\xi_i - \xi|^2 \to 0$ *as* $i \to \infty$, *then there exists a pair* $(y, z) \in B^2$ *such that* $\|(y^i - y, z^i - z)\|_B \to 0$ *as* $i \to \infty$. *Furthermore,* (y, z) *is the solution of the following BDSDE:*

$$(14)\quad y_t = \xi + \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s + \int_t^\infty g(s, y_s)\,\mathrm{d}B_s - \int_t^\infty z_s\,\mathrm{d}W_s, \quad 0 \leqslant t < \infty.$$

Proof. For any $n, m \geqslant 1$, let (y^n, z^n) and (y^m, z^m) be the solutions of (13) corresponding to ξ_n and ξ_m, respectively. Due to Theorem 3, there exists a constant $\overline{C} > 0$ such that

$$\begin{aligned}\|(y^n - y^m, z^n - z^m)\|_B^2 &\leqslant \overline{C} E|\xi_n - \xi_m|^2 \\ &\leqslant 2\overline{C}(E|\xi_n - \xi|^2 + E|\xi_m - \xi|^2) \to 0 \quad \text{as } n, m \to \infty,\end{aligned}$$

which means that $\{(y^i, z^i), i = 1, 2, \ldots\}$ is a Cauchy sequence in B^2. Thus, there exists a pair $(y, z) \in B^2$ such that $\|(y^i - y, z^i - z)\|_B \to 0$ as $i \to \infty$. Hence,

$$\begin{aligned}E\biggl|\int_t^\infty &(f(s, y_s^i, z_s^i) - f(s, y_s, z_s))\,\mathrm{d}s\biggr|^2 \\ &\leqslant E\biggl(\int_0^\infty (v(s)|y_s^i - y_s| + u(s)\|z_s^i - z_s\|)\,\mathrm{d}s\biggr)^2 \\ &\leqslant 2\biggl[\biggl(\int_0^\infty v(s)\,\mathrm{d}s\biggr)^2 + \int_0^\infty u^2(s)\,\mathrm{d}s\biggr] \cdot \|(y^i - y, z^i - z)\|_B^2 \to 0 \quad \text{as } i \to \infty,\end{aligned}$$

and

$$\begin{aligned}E\biggl|\int_t^\infty &(g(s, y_s^i) - g(s, y_s))\,\mathrm{d}B_s\biggr|^2 \\ &\leqslant E\int_0^\infty u^2(s)|y_s^i - y_s|^2\,\mathrm{d}s \\ &\leqslant \biggl(\int_0^\infty u^2(s)\,\mathrm{d}s\biggr) \cdot \|(y^i - y)\|_S^2 \to 0, \quad \text{as } i \to \infty.\end{aligned}$$

Thus, for any $t \in \mathbb{R}^+$, $\int_t^\infty f(s, y_s^i, z_s^i)\,\mathrm{d}s \to \int_t^\infty f(s, y_s, z_s)\,\mathrm{d}s$ and $\int_t^\infty g(s, y_s^i)\,\mathrm{d}B_s \to \int_t^\infty g(s, y_s)\,\mathrm{d}B_s$ in $L^2(\Omega, \mathcal{F}, P)$. Taking the limit on both sides of (13), we deduce that (y, z) is the solution to BDSDE (14). The desired result is obtained. □

The following corollary shows the relation between the solution of infinite horizon BDSDE (2) and of the finite time BDSDE

$$y_t = E^{\mathcal{F}_T}[\xi] + \int_t^T f(s, y_s, z_s)\,\mathrm{d}s + \int_t^T g(s, y_s)\,\mathrm{d}B_s$$

$$(15) \qquad - \int_t^T z_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty.$$

Corollary 5. *Assume* $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$, *let* (H1)–(H3) *hold for* f *and* g. *Let* (y, z) *be the solution of BDSDE* (2). *For any* $T > 0$, *let* (y^T, z^T) *be the solutions of the finite time interval BDSDE* (15). *Then* $(y^T, z^T) \to (y, z)$ *in* B^2 *as* $T \to \infty$.

Proof. Note that $E^{\mathcal{F}_T}[\xi] \to \xi$ in $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ as $T \to \infty$. The proof is straightforward from Theorem 4. □

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