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UNIQUENESS OF LIMIT CYCLES BOUNDED BY TWO INVARIANT PARABOLAS*

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Abstract. In this paper we consider a class of cubic polynomial systems with two invariant parabolas and prove in the parameter space the existence of neighborhoods such that in one the system has a unique limit cycle and in the other the system has at most three limit cycles, bounded by the invariant parabolas.

Keywords: stability, limit cycles, center, bifurcation

MSC 2010: 92D25, 34C05, 58F14, 58F21

1. Introduction

A polynomial system is a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities

$$(1)\quad \dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij}x^i y^j, \quad \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij}x^i y^j \quad \text{with } a_{ij}, b_{ij} \in \mathbb{R}.$$

The problem of analyzing periodic solutions has been widely studied and, consequently, there is extensive literature on them. This activity reflects the breadth of interest in Hilbert's 16th problem and the fact that such systems are often used in mathematical models. Suppose that the origin of (1) is a critical point of center-focus type. We are concerned with the number of limit cycles (that is, isolated periodic solutions) which bifurcate from a critical point or from a center.

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Let us assume that the origin is a critical point of (1) and transform the system to canonical form

$$\dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y), \tag{2}$$

where p and q are polynomials without linear terms.

Let $n = \max(\partial P, \partial Q)$, where the symbol ∂ denotes 'degree of'. A function h is said to be *invariant* with respect to (2) if there is a polynomial $k(x, y)$, called the cofactor, with $\partial k < n$ such that $\dot{h} = hk$. Here $\dot{h} = h_x P + h_y Q$ is the rate of change of h along orbits.

It is interesting to note that the existence of algebraic trajectories has been known to strongly influence the behavior of polynomial systems. For instance, quadratic systems ($n = 2$) with an invariant ellipse, hyperbola, or a pair of straight lines can have no limit cycles other than possibly the ellipse itself. Moreover, if there is an invariant line, there cannot be more than one limit cycle (see [3]). The case of a parabola was considered in [5].

For the cubic, there exist different classes of systems in which there may coexist an invariant curve or straight lines with limit cycles (see [2], [6], [7], [10]).

2. The system as a perturbation of Hamiltonians

We can write system (2) as the perturbation with a small real parameter ε of a Hamiltonian system

$$X_\mu\colon \begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y} + \varepsilon f(x, y, \varepsilon), \\[2mm] \dot{y} = \dfrac{\partial H}{\partial x} + \varepsilon g(x, y, \varepsilon), \end{cases}$$

where $f(x, y, \varepsilon), g(x, y, \varepsilon) \in \mathbb{R}[x, y, \varepsilon]$.

We say that this system with $\varepsilon = 0$ is the unperturbed system and we will impose for it the existence of a center at the origin. In this way, by the Poincaré-Andronov Theorem [4], we shall study the bifurcation of limit cycles from the unperturbed Hamiltonian center.

If the first-order Melnikov function is not identical zero, then the integral on the orbits of the Hamiltonian system is given by

$$M_0(h) = \int_0^{T(h)} \Bigl(f \frac{\partial H}{\partial x} + g \frac{\partial H}{\partial y}\Bigr)\, \mathrm{d}t,$$

where $T(h)$ is the period of the periodic orbit $H(x, y) = h$ of the unperturbed system ($\varepsilon = 0$). Then, the form of the function $M_0(h)$ and the result on the relation between the number of zeros of $M_0(h)$ and the number of limit cycles of the system follows from [4, Theorem 3.1].

3. Preliminary results

Let us consider the parabolas

$$c(x,y) = y - x^2 + 1 \quad \text{and} \quad c_b(x,y) = y + x^2 - bx - 1$$

and the cubic system

$$(3) \qquad X_\mu\colon \begin{cases} \dot{x} = L(16+b^2)x - (16+b^2)y - b(b-8L-b^2L)x^2 \\ \qquad + (4b-16L-3b^2L)x^3 - b(6+b^2+2bL)xy \\ \qquad + (8+3b^2+8bL)x^2y + b(b+2L)y^2 - 4(b+2L)xy^2, \\ \dot{y} = (16+b^2)x + L(16+b^2)y + 4b(2-bL)x^2 \\ \qquad - (16+3b^2-16bL-2b^3L)x^3 + b(5b+16L)xy \\ \qquad - (4b+2b^3+48L+11b^2L)x^2y - b(8-bL)y^2 \\ \qquad + 4(4+2b^2+5bL)xy^2 - 8(b+2L)y^3, \end{cases}$$

where $\mu = (L,b) \in \mathbb{R}^2$.

It is easy to verify that for all $\mu = (L,b) \in \mathbb{R}^2$

$$\dot{c}(x,y) = \frac{\partial c(x,y)}{\partial x}\,\dot{x} + \frac{\partial c(x,y)}{\partial y}\,\dot{y} = c(x,y)k(x,y)$$

and

$$\dot{c_b}(x,y) = \frac{\partial c_b(x,y)}{\partial x}\,\dot{x} + \frac{\partial c_b(x,y)}{\partial y}\,\dot{y} = c_b(x,y)k_b(x,y),$$

where the cofactors are

$$\begin{aligned} k(x,y) = {} & 16x + b^2x + 8bx^2 - 32Lx^2 - 6b^2Lx^2 + 16Ly + b^2Ly \\ & + 16xy + 6b^2xy + 16bLxy - 8by^2 - 16Ly^2, \\ k_b(x,y) = {} & -16x - b^2x + 16bLx + b^3Lx + 8bx^2 - 32Lx^2 \\ & - 6b^2Lx^2 - 16by - b^3y - 16Ly - b^2Ly + 16xy + 6b^2xy + 16bLxy \\ & - 8by^2 - 16Ly^2. \end{aligned}$$

Then the parabolas are invariant curves of the system (3).

Rescaling the parameters $L \to L\varepsilon$, $b \to b\varepsilon$, we can write the system (3) as the perturbation, with a small real parameter ε, of a Hamiltonian system

$$X_{\mu,\varepsilon}\colon \begin{cases} \dot{x} = 8(-2+x^2)y + \varepsilon f(x,y,\varepsilon), \\ \dot{y} = -16x(-1+x^2-y^2) + \varepsilon g(x,y,\varepsilon), \end{cases}$$

where the perturbation is given by

$$\begin{aligned} f(x,y,\varepsilon) &= 16Lx + b^2\varepsilon^2 Lx - b^2\varepsilon x^2 + 8b\varepsilon Lx^2 + b^3\varepsilon^3 Lx^2 + 4bx^3 - 16Lx^3 \\ &\quad - 3b^2\varepsilon^2 Lx^3 - b^2\varepsilon y - 6bxy - b^3\varepsilon^2 xy - 2b^2\varepsilon^2 Lxy + 3b^2\varepsilon x^2 y \\ &\quad + 8b\varepsilon Lx^2 y + b^2\varepsilon y^2 + 2b\varepsilon Ly^2 - 4bxy^2 - 8Lxy^2, \\ g(x,y,\varepsilon) &= b^2\varepsilon x + 8bx^2 - 4b^2\varepsilon^2 Lx^2 - 3b^2\varepsilon x^3 + 16b\varepsilon Lx^3 + 2b^3\varepsilon^3 Lx^3 \\ &\quad + 16Ly + b^2\varepsilon^2 Ly + 5b^2\varepsilon xy + 16b\varepsilon Lxy - 4bx^2 y - 2b^3\varepsilon^2 x^2 y \\ &\quad - 48Lx^2 y - 11b^2\varepsilon^2 Lx^2 y - 8by^2 + b^2\varepsilon^2 Ly^2 + 8b^2\varepsilon xy^2 \\ &\quad + 20b\varepsilon Lxy^2 - 8by^3 - 16Ly^3. \end{aligned}$$

Let us denote by Ω_μ the region enclosed by the two invariant parabolas. Then in Ω_0 we find an integral factor of the system $X_{0,0}$ as

$$\varphi(x) = \frac{1}{(2-x^2)^3} > 0, \quad x \in \left(-\sqrt{2}, \sqrt{2}\right).$$

So, the Hamiltonian function H in Ω_0 is given by

$$H(x,y) = h = \frac{4(3-2x^2+y^2)}{(2-x^2)^2}, \quad |x| < 1, \tag{4}$$

satisfying the symmetry $H(-x,-y) = H(x,y)$ and $H(0,0) = 3$, $H(1,0) = 4$. Consequently, for $h \in (3,4)$, $H^{-1}(h)$ are periodic orbits that extend from the origin to the invariant parabolas $c^{-1}(0)$ and $c_0^{-1}(0)$ (see Fig. 1).

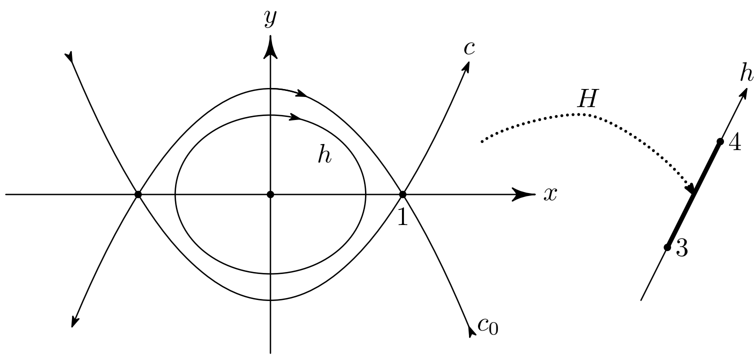

Figure 1. Levels of H.

Then for the system $X_{\mu,\varepsilon}$ in Ω_μ, we change the time as $t \to t\varphi(x)$ and for $(x, y) \in \Omega_\mu$ we have a C^∞-equivalent system

$$(5) \qquad Y_{\mu,\varepsilon}: \begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y} + \varepsilon f(x, y, \varepsilon)\varphi(x), \\ \dot{y} = \dfrac{\partial H}{\partial x} + \varepsilon g(x, y, \varepsilon)\varphi(x). \end{cases}$$

Theorem 1 (see [9]). *If $L = 0$, system (3) at the singularity $(0, 0)$ has an attracting weak focus of order one if $b > 0$ and a repelling weak focus of order one if $b < 0$.*

Theorem 2 (see [9]). *In the parameter space $\mathbb{R}^2$ there exists an open set $\mathcal{N}$ such that for all $(L, b) \in \mathcal{N}$ and $Lb \neq 0$, system (3) has at least one small-amplitude limit cycle enclosed by two invariant parabolas.*

4. Main results

We show that for system (3), there exist neighborhoods in the parameter space such that the system has a unique limit cycle and an upper bound of at most three limit cycles which are bounded by the two invariant parabolas.

The main result of this paper and its proof require the discussion of the existence of the real roots of the cubic equation.

It is known (see [1]) that the cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ has only one real root if $\Delta > 0$ and at most three real roots if $\Delta < 0$, where $\Delta = G^2 + 4W^3$, $W = a_0a_2 - a_1^2$, and $G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$.

In particular we consider a cubic equation whose coefficients depend of the parameters L, b, and h

$$\begin{cases} a_0 = h(b + 2L), \\ a_1 = \dfrac{2}{3}(28b - 9bh + 8L - 18hL), \\ a_2 = -\dfrac{4}{3}(19b - 15bh + 158L - 30hL), \\ a_3 = 4(42b - 14bh + 132L - 28hL). \end{cases}$$

Then $\Delta(L,b,h) = -\frac{1}{27}256h^2(b+2L)\delta(L,b,h)$, where

$$(6)\quad \begin{aligned}\delta(L,b,h) = &-390236b^4 + 397571b^4h - 114444b^4h^2 + 6480b^4h^3\\ &-626752b^3L + 396256b^3hL - 91296b^3h^2L + 20736b^3h^3L\\ &+4216416b^2L^2 - 3882696b^2hL^2 + 892512b^2h^2L^2 - 31104b^2h^3L^2\\ &+2526464bL^3 - 3457856bhL^3 + 1003392bh^2L^3 - 165888bh^3L^3\\ &+365632L^4 + 5423984hL^4 - 462528h^2L^4 - 145152h^3L^4.\end{aligned}$$

Theorem 3. *There exist $h \in (0,\infty)$ and an open set $\mathcal{N}$ such that*

i) *For all $(L,b) \in \mathcal{N} \cap \delta^{-1}_{L,b,h}(-\infty,0)$ and $Lb \neq 0$, system* (3) *has one and only one limit cycle enclosed by two invariant parabolas.*

ii) *For all $(L,b) \in \mathcal{N} \cap \delta^{-1}_{L,b,h}(0,\infty)$ and $Lb \neq 0$, system* (3) *has at most three limit cycles.*

Proof. The Melnikov integral along the orbits of the Hamiltonian system is reduced to

$$M_0(h) = \int_0^{T(h)} \Big[f(x,y,\varepsilon)\varphi(x)\frac{\partial H}{\partial x} + g(x,y,\varepsilon)\varphi(x)\frac{\partial H}{\partial y}\Big]\,\mathrm{d}t,$$

where $h \in (3,4)$ and $T(h)$ is the time return of the orbit $H^{-1}(h)$.

Equivalently, we can write

$$M_0(h) = \int_0^{T(h)} f(x,y,\varepsilon)\varphi(x)dy - g(x,y,\varepsilon)\varphi(x)\,\mathrm{d}x.$$

Then, in the region $R(h) \subseteq \Omega_\mu$ enclosed by the orbits $H^{-1}(h)$, we can use Green's Theorem and we have

$$M_0(h) = \iint_{R(h)} \Big[\frac{\partial}{\partial x}(f(x,y,\varepsilon)\varphi(x)) + \frac{\partial}{\partial y}(g(x,y,\varepsilon)\varphi(x))\Big]\,\mathrm{d}x\,\mathrm{d}y,$$

and by Fubini's Theorem,

$$M_0(h) = \int_{-x(h)}^{x(h)}\int_{-y(x,h)}^{y(x,h)} (A_0(x) + A_1(x)y + A_2(x)y^2)\,\mathrm{d}y\,\mathrm{d}x$$

where

$$
\begin{aligned}
y(x,h) &= \tfrac{1}{2}\sqrt{-12+8x^2+h(-2+x^2)^2},\\
A_0(x) &= (64L+4b^2\varepsilon^2L+6b^2\varepsilon x+64b\varepsilon Lx+4b^3\varepsilon^3Lx\\
&\quad +16bx^2-4b^3\varepsilon^2x^2-128Lx^2-36b^2\varepsilon^2Lx^2-9b^2\varepsilon x^3\\
&\quad +16b\varepsilon Lx^3+4b^3\varepsilon^3Lx^3+16bx^4+2b^3\varepsilon^2x^4+2b^2\varepsilon^2Lx^4)/(-2+x^2)^4,\\
A_1(x) &= b(-44-2b^2\varepsilon^2+38b\varepsilon x+112\varepsilon Lx\\
&\quad -14x^2-5b^2\varepsilon^2x^2-12b\varepsilon^2Lx^2-4b\varepsilon x^3-8\varepsilon Lx^3)/(-2+x^2)^4,\\
A_2(x) &= 2(b+2L)(-28+3b\varepsilon x+2x^2)/(-2+x^2)^4.
\end{aligned}
$$

It is easy to see that the integral of the linear term y is zero. Moreover, due to the symmetry of $H^{-1}(h)$ and that of the interval of integration, and considering only even powers of y, the above integral can be written as

$$M_0(h) = 4\int_0^{x(h)}\int_0^{y(x,h)} (\hat{A}_0(x)+\hat{A}_2(x)y^2)\,\mathrm{d}y\,\mathrm{d}x,$$

where

$$
\begin{aligned}
\hat{A}_0(x) &= (64L+4b^2\varepsilon^2L+16bx^2-4b^3\varepsilon^2x^2-128Lx^2-36b^2\varepsilon^2Lx^2\\
&\quad +16bx^4+2b^3\varepsilon^2x^4+2b^2\varepsilon^2Lx^4)/(-2+x^2)^4,\\
\hat{A}_2(x) &= 2(b+2L)(-28+2x^2)/(-2+x^2)^4,
\end{aligned}
$$

and integrating with respect to y leads to

$$M_0(h) = \frac{4}{3}\int_0^{x(h)} (3\hat{A}_0(x)+\hat{A}_2(x)y^2(x,h))y(x,h)\,\mathrm{d}x.$$

As the argument of the above integral is an even function in the variable x, by the Second Mean Value Theorem for Integrals, there is $\xi(h)\in(0,x(h))$ such that

$$M_0(h) = \frac{4}{3}[C_0+C_2\xi^2(h)+C_4\xi^4(h)+C_6\xi^6(h)]\int_0^{x(h)} \frac{y(x,h)}{(-2+x^2)^4}\,\mathrm{d}x, \tag{7}$$

where $x(h)$ is the positive solution of the equation

$$-12+8x^2+h(-2+x^2)^2=0,\quad |x|<1,\ h\in(3,4),$$

and C_i, $i=0,1,2,3$ are polynomials in the parameters h, L, ε, b given by

$$
\begin{aligned}
C_0 &= 4(42b-14bh+132L+3b^2\varepsilon^2L-28hL),\\
C_2 &= -4(19b+3b^3\varepsilon^2-15bh+158L+27b^2\varepsilon^2L-30hL),\\
C_4 &= 2(28b+3b^3\varepsilon^2-9bh+8L+3b^2\varepsilon^2L-18hL),\\
C_6 &= h(b+2L).
\end{aligned}
$$

Let $p(h)_{L,b,\varepsilon} = C_0 + C_2\xi^2(h) + C_4\xi^4(h) + C_6\xi^6(h)$, be a cubic polynomial in $\xi^2(h)$.

In order to study the positive roots of the equation $p(h)_{L,b,\varepsilon} = 0$, we consider in terms of the parameters the conditions for the existence of roots (see [1]), namely, for the equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0, \quad \text{where}$$
$$a_3 = C_0, \quad a_2 = \frac{C_2}{3}, \quad a_1 = \frac{C_4}{3}, \quad \text{and} \quad a_0 = C_6.$$

To simplify the calculations for sufficiently small ε, we assume that the functions f, g in (5) are not dependent on ε; then $\Delta(L,b,h) = \frac{1}{27}(-256h^2(b+2L)^2\delta(L,b,h))$, where $\delta(L,b,h)$ is the polynomial given in (6).

(i) If $(L,b) \in \mathcal{N} \cap \delta^{-1}_{Lb\varepsilon}(-\infty,0)$, $Lb \neq 0$, by Theorem 1 and Theorem 2, system (3) has a limit cycle. As $\delta < 0$, we have $\Delta(L,b,h) > 0$ and by [1], for ε small enough, the polynomial $p(h)_{L,b,\varepsilon}$ has only one real root $\xi(h) \in (0, x(h))$.

When we apply the Mean Value Theorem to (7), the value of $\xi(h)$ does not necessarily coincide with the real root of the polynomial $p(h)_{L,b,\varepsilon} = 0$, for this reason we have proved that there is at most one limit cycle bifurcating from the Hamiltonian.

(ii) If $(L,b) \in \mathcal{N} \cap \delta^{-1}_{Lb\varepsilon}(0,\infty)$, $Lb \neq 0$, by Theorem 1 and Theorem 2, system (3) has a limit cycle. As $\delta > 0$, we have $\Delta(L,b,h) < 0$ and by [1], for ε small enough, the polynomial $p(h)_{L,b,\varepsilon}$ has at most three real roots. Therefore, three is an upper bound for the number of limit cycles of system (3). □

In order to illustrate the result stated in Theorem 3, Part i), Fig. 2 shows a numerical simulation of the system (3) with $L = 0.02$ and $b = 0.4$, which corresponds to the case of an attracting limit cycle. The simulation was obtained using the Pplane7 Software with MATLAB [8]

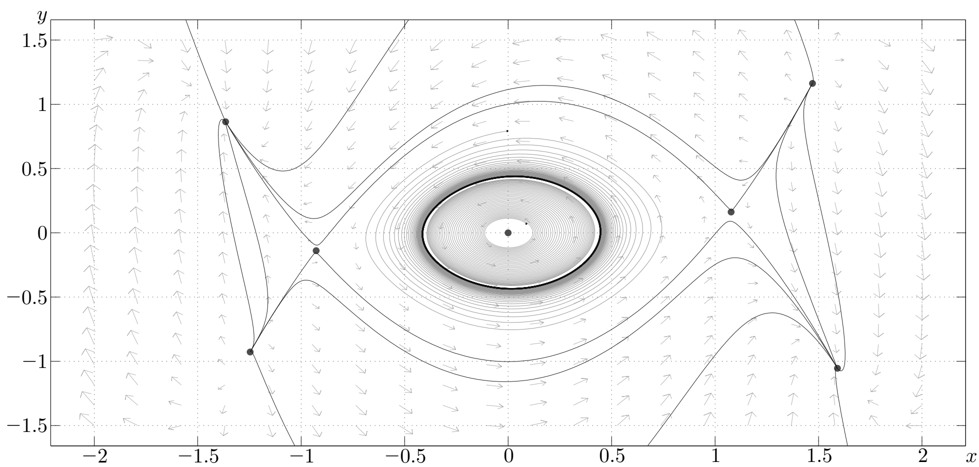

Figure 2. Numerical simulation of an attracting limit cycle.

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