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# Harmonic and Minimal Unit Vector Fields on the Symmetric Spaces $G_2$ and $G_2/SO(4)$ *

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**Abstract**

The exceptional compact symmetric spaces $G_2$ and $G_2/SO(4)$ admit cohomogeneity one isometric actions with two totally geodesic singular orbits. These singular orbits are not reflective submanifolds of the ambient spaces. We prove that the radial unit vector fields associated to these isometric actions are harmonic and minimal.

**Key words:** harmonic unit vector field, minimal unit vector field, Lie group, Riemannian symmetric space, isometric action

**2000 Mathematics Subject Classification:** 53C35, 53C40, 57S15

## 1 Introduction

Let $N$ be a compact orientable Riemannian manifold which admits smooth unit vector fields. Take a smooth unit vector field $\xi$ on $N$ and consider $\xi$ as an imbedding of $N$ into its unit tangent sphere bundle $T_1N$ equipped with the Sasaki metric. Then we can define the energy $\mathrm{E}(\xi)$ of $\xi$ as the energy of the corresponding map and the volume $\mathrm{Vol}(\xi)$ of $\xi$ as the volume of the compact orientable submanifold $\xi(N)$ in $T_1N$. Therefore we get two functionals on the space of the smooth unit vector fields on $N$. $\xi$ is called a harmonic unit vector field if it is a critical point of the energy functional E, and $\xi$ is said to be minimal if it is a critical point of the volume functional Vol.

The relevant critical point criteria have been derived in the papers [10] and [4] by using differential one-forms. Hence, the concepts of harmonic and minimal unit vector fields can be extended to the case when $N$ is not compact or not

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orientable. A lot of examples of harmonic or minimal unit vector fields have been constructed on two-point homogeneous spaces and Einstein manifolds (see [2] and [3]).

On Riemannian symmetric spaces, harmonic and minimal unit vector fields can be obtained by using special cohomogeneity one isometric actions as follows. Take a cohomogeneity one isometric action $\lambda\colon L \times N \to N$ of a compact connected Lie group $L$ on a Riemannian symmetric space $N$. Assume that the action has a totally geodesic singular orbit $F$. Then the principal orbits of $L$ coincide with the tubes around $F$, and the unit speed geodesics emanating perpendicularly from $F$ intersect orthogonally each principal orbit of $L$. The union of all principal orbits is an open and dense domain in $N$ and the tangent vectors of the geodesics mentioned above yield a radial unit vector field $\xi$ on it.

A connected submanifold $F$ of $N$ is said to be reflective if $F$ is a connected component of the fixed point set of an involutive global isometry of $N$. Hence, a reflective submanifold is necessarily totally geodesic. It has been proved in the paper [1] that *if the isometric action $\lambda\colon L \times N \to N$ has a reflective singular orbit $F$, then the radial unit vector field $\xi$ associated to $\lambda$ is harmonic and minimal.*

The cohomogeneity one isometric actions of the Lie groups $SU(3) \times SU(3)$ and $SU(3)$ on $G_2$ and $G_2/SO(4)$, respectively, have been discussed in detail in [9]. Both the isometric actions have got exactly two totally geodesic singular orbits, but they are not reflective submanifolds. The purpose of this paper is to show that the radial unit vector fields associated to these actions are also harmonic and minimal.

## 2 The concepts of harmonic and minimal unit vector fields

Let $N$ be a $d$-dimensional connected Riemannian manifold with the metric $\langle\ ,\ \rangle$ and with the Levi–Civita connection $\nabla$. Let $\mathfrak{X}_1(N)$ denote the set of all smooth unit vector fields on $N$, and assume that $\mathfrak{X}_1(N)$ is non-empty. Take a smooth unit vector field $\xi$ on $N$ and the unit tangent sphere bundle $T_1N$ of $N$ with the Sasaki metric. Clearly, $\xi$ can be regarded as an immersion $\xi\colon N \to T_1N$.

Consider the tensor fields $A_\xi$, $L_\xi$ of type (1,1) defined by the relations

$$A_\xi(X) = -\nabla_X \xi, \qquad L_\xi(X) = X + A_\xi^t(A_\xi(X)),$$

where $X$ is a vector field on $N$ and $A_\xi^t$ denotes the transpose of $A_\xi$. Assume now that $N$ is a compact orientable Riemannian manifold. Then the energy $\mathrm{E}(\xi)$ of the map $\xi\colon N \to T_1N$ and the volume $\mathrm{Vol}(\xi)$ of the submanifold $\xi(N)$ can be expressed by the formulae

$$\mathrm{E}(\xi) = \frac{1}{2}\int_N \mathrm{Tr}\ L_\xi\, dv = \frac{d}{2}\ \mathrm{Vol}(N) + \frac{1}{2}\int_N |\nabla\xi|^2\, dv, \quad \mathrm{Vol}(\xi) = \int_N \sqrt{det\, L_\xi}\, dv,$$

where $dv$ is the volume form of $N$. Therefore we get two functionals E and Vol on $\mathfrak{X}_1(N)$. The unit vector field $\xi$ is called harmonic if $\xi$ is a critical point of the energy functional E. Furthermore, $\xi$ is said to be a minimal unit vector field if $\xi$ is a critical point of the volume functional Vol. Concerning the latter concept, we remark that if $\xi$ is a minimal unit vector field, then $\xi(N)$ is a minimal submanifold in $T_1N$ (for proof see [4]).

Let $\xi^{\perp}$ denote the $(d-1)$-dimensional distribution on $N$ which is perpendicular to $\xi$. The critical point conditions for the functionals E and Vol have been determined in the papers [10] and [4] by using differential forms. Accordingly, $\xi$ is a harmonic unit vector field if and only if the differential one-form $\nu_\xi$ defined by the relation

$$\nu_\xi(X) = \mathrm{Tr}(Y \mapsto (\nabla_Y A_\xi^t)(X)),$$

where $X$ and $Y$ are vector fields on $N$, vanishes on $\xi^{\perp}$.

In order to give a criterion for the minimal unit vector fields on $N$ we need the tensor field $K_\xi$ of type (1,1) defined by $K_\xi = -\sqrt{det\, L_\xi} \cdot L_\xi^{-1} \circ A_\xi^t$. Then $\xi$ is a minimal unit vector field if and only if the one-form $\omega_\xi$ defined by

$$\omega_\xi(X) = \mathrm{Tr}(Y \mapsto (\nabla_Y K_\xi)(X))$$

vanishes on the distribution $\xi^{\perp}$.

It is clear that the two critical point conditions above also make sense in the case when the manifold $N$ is not compact or not orientable. Then $\xi$ is said to be harmonic, respectively minimal, if the one-form $\nu_\xi$, respectively $\omega_\xi$, vanishes on $\xi^{\perp}$.

Hereafter, we assume that $\xi$ is a geodesic vector field, which means that $\nabla_\xi \xi = 0$ is valid, and the distribution $\xi^{\perp}$ is integrable. Then the critical point conditions can be reformulated as follows (for details and proof see [2]).

The distribution $\xi^{\perp}$ yields a foliation of $N$ the leaves of which are integral submanifolds of $\xi^{\perp}$. Let us take at each point $p$ of $N$ the mean curvature of the integral submanifold of $\xi^{\perp}$ through $p$. Hence, we obtain the mean curvature function $h\colon N \to \mathbb{R}$ associated to the foliation. Let $R$ and $Ric$ denote the curvature tensor and the Ricci tensor of $N$, respectively. Then $\xi$ is a harmonic unit vector field if and only if

$$dh(X) = Ric(\xi, X) \tag{1}$$

holds for each vector field $X$ perpendicular to $\xi$.

Observe that the tensor $A_{\xi(p)}$ at a point $p \in N$ corresponds to the shape operator of the one-codimensional integral submanifold of $\xi^{\perp}$ through $p$. Since the shape operators are self-adjoint endomorphisms, we can take local orthonormal vector fields $E_1, \dots, E_{d-1}$ tangent to $\xi^{\perp}$ and local smooth functions $\lambda_1, \dots, \lambda_{d-1}$ such that $A_\xi(E_i) = \lambda_i E_i$ $(i = 1, \dots, d-1)$ hold. Then $\xi$ is a minimal unit vector field if and only if the relation

$$\sum\nolimits_{i=1}^{d-1} \frac{1}{1+{\lambda_i}^2} \left(d\lambda_i(E_j) - (1-\lambda_i\lambda_j)\, \langle R(\xi, E_i)E_i, E_j\rangle\right) = 0 \tag{2}$$

is valid for each index $j$ $(j = 1, \dots, d-1)$.

## 3 Matrix representation of the Lie algebra of $G_2$

As is well-known, the exceptional compact Lie group $G_2$ is isomorphic to the group of automorphisms of the algebra of Cayley numbers. This implies that $G_2$ is a closed subgroup of the special orthogonal Lie group $SO(7)$. The Lie algebra of $SO(7)$ can be identified with the Lie algebra $\mathfrak{so}(7)$ of the real skew-symmetric $7 \times 7$ matrices. Let $M_{i,j}$ $(i,\ j = 1, \dots, 7)$ denote the $7 \times 7$ matrix, where the entry in $i$th row and in $j$th column is equal to 1 and all the other entries vanish. Clearly, we need the skew-symmetric matrices $A_{i,j} = M_{i,j} - M_{j,i}$ $(i \neq j)$. Consider now the elements

$$\begin{aligned}
P_0 &= A_{3,2} + A_{6,7},\\
Q_0 &= A_{4,5} + A_{6,7},\\
D_1 &= 2A_{2,1} + A_{5,6} + A_{7,4},\\
D_2 &= 2A_{5,1} + A_{6,2} + A_{3,7},\\
D_3 &= 2A_{6,1} + A_{2,5} + A_{4,3},\\
F_1 &= 2A_{3,1} + A_{6,4} + A_{7,5},\\
F_2 &= 2A_{4,1} + A_{3,6} + A_{2,7},\\
F_3 &= 2A_{1,7} + A_{2,4} + A_{3,5},\\
\hat{D}_1 &= A_{6,5} + A_{7,4},\\
\hat{D}_2 &= A_{2,6} + A_{3,7},\\
\hat{D}_3 &= A_{5,2} + A_{4,3},\\
\hat{F}_1 &= A_{6,4} + A_{5,7},\\
\hat{F}_2 &= A_{3,6} + A_{7,2},\\
\hat{F}_3 &= A_{4,2} + A_{3,5}
\end{aligned}$$

of $\mathfrak{so}(7)$. It can be verified that the 14 matrices listed above form a basis of the Lie algebra $\mathfrak{g}_2$ of the compact Lie group $G_2$ (for details see Lecture 14 in [8]). This means that using this subalgebra $\mathfrak{g}_2$ of $\mathfrak{so}(7)$ and the exponential map $\exp\colon \mathfrak{so}(7) \to SO(7)$, we obtain the exceptional compact Lie group $G_2 = \exp(\mathfrak{g}_2)$.

Considering an element $X$ of $\mathfrak{g}_2$, the endomorphism $\operatorname{ad} X\colon \mathfrak{g}_2 \to \mathfrak{g}_2$ is defined by $\operatorname{ad} X(Y) = [X, Y]$ for $Y \in \mathfrak{g}_2$, where $[\ ,\ ]$ denotes the bracket operation in $\mathfrak{g}_2$. It is clear that $\mathfrak{u}_0 = \mathbb{R}P_0 + \mathbb{R}Q_0$ presents a 2-dimensional Abelian subspace of $\mathfrak{g}_2$. Let $\alpha$ be a linear form on the linear space $\mathfrak{u}_0$. Take the subspace

$$\mathfrak{u}_\alpha = \{Y \in \mathfrak{g}_2 \mid (\operatorname{ad} X)^2(Y) = -\alpha(X)^2\, Y \text{ for all } X \in \mathfrak{u}_0\}.$$

The linear form $\alpha$ $(\alpha \neq 0)$ is called a root of $\mathfrak{g}_2$ if $\mathfrak{u}_\alpha \neq \{0\}$ holds, and $\mathfrak{u}_\alpha$ is said to be the root subspace corresponding to $\alpha$.

Let $\varepsilon_1$, $\varepsilon_2$ be the dual basis of $P_0$, $Q_0$ in the space $\mathfrak{u}_0^*$ of linear forms. We need the linear forms $\alpha_1 = \varepsilon_1$, $\alpha_2 = \varepsilon_2$, $\alpha_3 = \varepsilon_1 + \varepsilon_2$ and $\beta_1 = \varepsilon_1 + 2\varepsilon_2$, $\beta_2 = 2\varepsilon_1 + \varepsilon_2$, $\beta_3 = \varepsilon_1 - \varepsilon_2$, which are the roots of $\mathfrak{g}_2$ up to the sign. Concerning the root subspaces, it can be verified that

$$\mathfrak{u}_{\alpha_i} = \mathbb{R}D_i + \mathbb{R}F_i, \qquad \mathfrak{u}_{\beta_i} = \mathbb{R}\hat{D}_i + \mathbb{R}\hat{F}_i \qquad (i = 1, 2, 3)$$

are true. Hence, we obtain the root space decomposition

$$\mathfrak{g}_2 = \mathfrak{u}_0 + \sum\nolimits_{i=1}^{3} \mathfrak{u}_{\alpha_i} + \sum\nolimits_{i=1}^{3} \mathfrak{u}_{\beta_i},$$

where the components are orthogonal with respect to the Killing form $B$ of $\mathfrak{g}_2$.

It can be seen that the subspace $\mathfrak{l} = \mathfrak{u}_0 + \sum_{i=1}^{3} \mathfrak{u}_{\beta_i}$ of $\mathfrak{g}_2$ is a Lie algebra and $\mathfrak{l}$ is isomorphic to $\mathfrak{su}(3)$. By the exponential map we obtain the closed subgroup $L = \exp(\mathfrak{l})$ of $G_2$, which is isomorphic to the Lie group $SU(3)$. Mention must be made that $L$ is not a symmetric subgroup of $G_2$.

Let us take now the complementary subspaces

$$\mathfrak{k} = \mathfrak{u}_0 + \mathfrak{u}_{\alpha_3} + \mathfrak{u}_{\beta_3}, \qquad \mathfrak{p} = \mathfrak{u}_{\alpha_1} + \mathfrak{u}_{\alpha_2} + \mathfrak{u}_{\beta_1} + \mathfrak{u}_{\beta_2} \tag{3}$$

in $\mathfrak{g}_2$. Then $\mathfrak{k}$ is also a subalgebra of $\mathfrak{g}_2$ and $K = exp(\mathfrak{k})$ presents a symmetric subgroup of $G_2$, which is isomorphic to the Lie group $SO(4)$.

Notice that $H_i = \frac{1}{2}[D_i, F_i]$, $\hat{H}_i = \frac{1}{2}[\hat{D}_i, \hat{F}_i]$ $(i = 1, 2, 3)$ are elements of $\mathfrak{u}_0$. Using these vectors, we obtain the subalgebras $\mathfrak{v}_i = \mathbb{R}D_i + \mathbb{R}F_i + \mathbb{R}H_i$ and $\hat{\mathfrak{v}}_i = \mathbb{R}\hat{D}_i + \mathbb{R}\hat{F}_i + \mathbb{R}\hat{H}_i$ of $\mathfrak{g}_2$, which are isomorphic to $\mathfrak{su}(2)$.

## 4 Harmonic and minimal unit vector field on $G_2$

In what follows we use the notation introduced in Section 3. For the general theory about Riemannian symmetric spaces, we refer to [5].

Let $e$ denote the identity element of $G_2$. The Killing form $B$ of the Lie algebra $\mathfrak{g}_2$ is negative definite. Therefore we can take the inner product $\langle\ ,\ \rangle_e$ on the tangent space $T_eG_2 = \mathfrak{g}_2$ at $e$ defined by $\langle v_1, v_2\rangle_e = -B(v_1, v_2)$ for $v_1, v_2 \in \mathfrak{g}_2$. Consider the left smooth action $\lambda\colon (G_2 \times G_2) \times G_2 \to G_2$ defined by $\lambda((g_1, g_2), h) = g_1 h (g_2)^{-1}$ for $g_1, g_2, h \in G_2$ and endow $G_2$ with the invariant Riemannian metric $\langle\ ,\ \rangle$ derived from the inner product $\langle\ ,\ \rangle_e$. Then $G_2$ turns into a compact symmetric space of rank two. As is well-known, the curvature tensor at $e$ can be expressed by

$$R(v_1, v_2)v_3 = -\tfrac{1}{4}\left[\,[v_1, v_2], v_3\right] \tag{4}$$

for $v_1, v_2, v_3 \in \mathfrak{g}_2$. Using the relation (4), it can be seen that the maximal sectional curvature of the symmetric space $G_2$ is equal to $\chi = \frac{1}{16}$.

*Consider now the natural isometric action $\lambda\colon (L \times L) \times G_2 \to G_2$, where the compact Lie group $L$ is isomorphic to $SU(3)$.* For brevity, the product Lie group $L \times L$ will be denoted by $\hat{L}$. It is evident that $\hat{L}(e) = L$ holds and the normal complementary subspace of $T_eL$ in $T_eG_2$ coincides with $\mathfrak{n} = \sum_{i=1}^{3} \mathfrak{u}_{\alpha_i}$. In what follows we summarize the basic facts concerning the action $\lambda$ (for details and proof see [9]).

Select a unit vector $u$ in $\mathfrak{n}$. Take the closed geodesic $\gamma\colon \mathbb{R} \to G_2$ defined by $\gamma(t) = \exp(t\,u)$ and the numbers $\kappa = \frac{\chi}{12}$, $r = \frac{\pi}{4\sqrt{\kappa}}$. Then $\gamma$ intersects orthogonally all the orbits of $\hat{L}$, and therefore $\lambda$ is a cohomogeneity one action. The compact Lie group $\hat{L}$ has exactly two singular orbits $\hat{L}(e) = L$ and $\hat{L}(\gamma(r))$, which are non-reflective totally geodesic submanifolds. Both of them are isometric to the compact symmetric space $SU(3)$. The other orbits $\hat{L}(\gamma(t))$, $0 < t < r$, are principal and can be thought of as tubes around $L$.

Let $U$ denote the union of all principal orbits of $\hat{L}$. Obviously, $U$ is a connected open domain of $G_2$ and $U = G_2 \setminus (L \cup \hat{L}(\gamma(r)))$ holds. The tangent vectors of geodesics starting perpendicularly from $L$ and parametrized by arc length yield a smooth unit vector field $\xi$ on the open domain $U$. This vector field $\xi$ is perpendicular to the principal orbits and invariant under the action $\lambda$. Hence, we call $\xi$ the radial unit vector field associated to $\lambda$.

In what follows we use the notation

$$v_1 = \hat{D}_2/4,\; v_2 = \hat{F}_2/4,\;\; v_3 = \hat{D}_3/4,\;\; v_4 = \hat{F}_3/4,\;\; v_5 = H_1/(4\sqrt{3}),$$
$$v_6 = \hat{D}_1/4,\;\; v_7 = \hat{F}_1/4,\;\; v_8 = \hat{H}_1/4,\;\; w_1 = D_2/(4\sqrt{3}),\;\; w_2 = F_2/(4\sqrt{3}),$$
$$w_3 = D_3/(4\sqrt{3}),\;\; w_4 = F_3/(4\sqrt{3}),\;\; w_5 = F_1/(4\sqrt{3}),\;\; w_6 = D_1/(4\sqrt{3}).$$

Notice that the vectors $v_1, \ldots, v_8$ present an orthonormal basis of $T_eL$ and the vectors $w_1, \ldots, w_6$ present an orthonormal basis of $\mathfrak{n}$.

Select now the unit vector $u = w_6$ in $\mathfrak{n}$ and take the closed geodesic $\gamma$ defined by $\gamma(t) = \exp(tu)$ for $t \in \mathbb{R}$. Recall that $\dot{\gamma}(t) = \xi(\gamma(t))$ holds for all $t \in (0, r)$. Then the tensor $A_{\xi(\gamma(t))}$ at $\gamma(t)$ presents the shape operator of the principal orbit $\hat{L}(\gamma(t))$. To describe the operator $A_{\xi(\gamma(t))}$ in explicit form we need the orthonormal vectors $z_j = \frac{1}{2}(\sqrt{3}\,v_j + w_j)$ and $\hat{z}_j = \frac{1}{2}(v_j - \sqrt{3}\,w_j)$, $j = 1, 2, 3, 4$. Take the Jacobi operator $R_u\colon T_eG_2 \to T_eG_2$ with respect to $u$, which is defined by $R_u(v) = R(v, u)u$ for all $v \in T_eG_2$. Then, by means of (4) we obtain that

$$\begin{gathered} R_u(z_j) = \kappa\, z_j, \quad R_u(\hat{z}_j) = 9\kappa\, \hat{z}_j, \quad R_u(v_5) = 4\kappa\, v_5, \\ R_u(w_5) = 4\kappa\, w_5, \quad R_u(v_s) = 0 \end{gathered} \tag{5}$$

hold for $j \in \{1, 2, 3, 4\}$ and $s \in \{6, 7, 8\}$. Concerning the shape operator $A_{\xi(\gamma(t))}$, $0 < t < r$, we can state the following proposition (for proof see Section 3 in [9]).

**Proposition 1** *Let $Z_j$, $\hat{Z}_j$, $V_s$ and $W_5$ be the parallel vector fields along $\gamma$ defined by $Z_j(0) = z_j$, $\hat{Z}_j(0) = \hat{z}_j$, $V_s(0) = v_s$ and $W_5(0) = w_5$ for $j \in \{1, 2, 3, 4\}$, $s \in \{5, 6, 7, 8\}$. Then the relations*

$$A_{\dot{\gamma}(t)}(Z_j(t)) = -\sqrt{\kappa}\, \cot(4\sqrt{\kappa}\, t)\, Z_j(t) + \tfrac{\sqrt{3}\,\kappa}{\sin(4\sqrt{\kappa}\, t)}\, \hat{Z}_j(t), \tag{6}$$

$$A_{\dot{\gamma}(t)}(\hat{Z}_j(t)) = \tfrac{\sqrt{3}\,\kappa}{\sin(4\sqrt{\kappa}\, t)}\, Z_j(t) - 3\sqrt{\kappa}\, \cot(4\sqrt{\kappa}\, t)\, \hat{Z}_j(t) \tag{7}$$

*are valid and the numbers*

$$\mu_1(t) = \sqrt{\kappa}\,\big(-2\,\cot(4\sqrt{\kappa}\, t) + \sqrt{4\cot^2(4\sqrt{\kappa}\, t) + 3}\big),$$

$$\mu_2(t) = \sqrt{\kappa}\,\big(-2\,\cot(4\sqrt{\kappa}\, t) - \sqrt{4\cot^2(4\sqrt{\kappa}\, t) + 3}\big)$$

*are principal curvatures of $\hat{L}(\gamma(t))$ with multiplicity 4. Furthermore, the equalities*

$$\begin{aligned}
A_{\dot\gamma(t)}(V_5(t)) &= 2\sqrt{\kappa}\,\tan(2\sqrt{\kappa}\,t)\,V_5(t),\\
A_{\dot\gamma(t)}(W_5(t)) &= -2\sqrt{\kappa}\,\cot(2\sqrt{\kappa}\,t)\,W_5(t),\\
A_{\dot\gamma(t)}(V_s(t)) &= 0 \qquad (s = 6,7,8)
\end{aligned}$$

*hold.*

We are ready now to prove the following theorem.

**Theorem 1** *The radial unit vector field $\xi$ associated to the isometric action $\lambda\colon (L\times L)\times G_2 \to G_2$ is harmonic and minimal.*

**Proof** Using the matrix representation of $\mathfrak{g}_2$ discussed in Section 3 and (4), it can be seen that the relations

$$\begin{gathered}
R(u,z_j)z_j = \kappa\,u - (-1)^j\sqrt{3}\kappa\,v_6, \quad R(u,\hat z_j)\hat z_j = 9\kappa\,u + (-1)^j 3\sqrt{3}\kappa\,v_6,\\
R(u,z_j)\hat z_j = 0, \quad R(u,\hat z_j)z_j = 0, \quad R(u,v_5)v_5 = 4\kappa\,u,\\
R(u,w_5)w_5 = 4\kappa\,u, \quad R(u,v_s)v_s = 0
\end{gathered}$$

hold for $j\in\{1,2,3,4\}$ and $s\in\{6,7,8\}$. Let us take the orthonormal vectors $e_j = a\,z_j + b\,\hat z_j$ and $e_{4+j} = -b\,z_j + a\,\hat z_j$ with two real numbers $a$ and $b$ for which $a^2+b^2=1$ holds. Then, it follows from the equalities above that the equations

$$R(u,e_j)e_j = (a^2+9b^2)\kappa\,u + (-1)^j\sqrt{3}(3b^2-a^2)\kappa\,v_6, \tag{8}$$

$$R(u,e_{4+j})e_{4+j} = (9a^2+b^2)\kappa\,u + (-1)^j\sqrt{3}(3a^2-b^2)\kappa\,v_6 \tag{9}$$

are valid.

Consider now the shape operator $A_{\dot\gamma(t)}$ of the principal orbit $\hat L(\gamma(t))$ for some $t\in(0,r)$. The relations (6) and (7) in Proposition 1 imply that we can take two numbers $a(t)$ and $b(t)$ such that $E_j(t) = a(t)\,Z_j(t) + b(t)\,\hat Z_j(t)$ and $E_{4+j}(t) = -b(t)\,Z_j(t) + a(t)\,\hat Z_j(t)$ are orthonormal eigenvectors of $A_{\dot\gamma(t)}$ with the eigenvalues $\mu_1(t)$ and $\mu_2(t)$, respectively. In addition, we need the further eigenvectors $E_9(t) = V_5(t)$, $E_{10}(t) = W_5(t)$, $E_{11}(t) = V_6(t)$, $E_{12}(t) = V_7(t)$ and $E_{13}(t) = V_8(t)$ of $A_{\dot\gamma(t)}$. In what follows we denote the corresponding eigenvalues of $A_{\xi(\gamma(t))}$ by $\lambda_i(t)$, $i = 1,\dots,13$.

Since $G_2$ is a Riemannian symmetric space, the parallel transport along geodesics preserves curvature. Hence, the relations above imply that

$$\langle R(\xi(\gamma(t)),E_i(t))E_i(t),E_k(t)\rangle = 0 \tag{10}$$

is valid for $i\in\{1,\dots,13\}$ and $k\in\{1,\dots,10,12,13\}$.

Observe that the integral submanifolds of the distribution $\xi^\perp$ coincide with the principal orbits of $\hat L$. It is obvious that the eigenvalues of $A_\xi$ and the mean curvature function $h$ are constant on a fixed principal orbit $\hat L(\gamma(t))$. Then we can verify that $Ric(\xi(\gamma(t)), E_i(t)) = 0$ holds for each index $i \in \{1, \ldots, 13\}$. Therefore the critical point condition (1) is satisfied, and this means that $\xi$ is a harmonic unit vector field.

It remains to prove the minimal property of the unit vector field $\xi$. Observe that $\lambda_{11} = 0$ holds by Proposition 1. Moreover, by means of (8) and (9) we obtain that

$$\begin{aligned}
&\textstyle\sum_{i=1}^{13} \frac{1}{1+\lambda_i(t)^2} \langle R(\xi(\gamma(t)), E_i(t))E_i(t), E_{11}(t)\rangle \\
&\quad = \textstyle\sum_{j=1}^{4} \big( \frac{1}{1+\mu_1(t)^2} \langle R(\dot\gamma(t), E_j(t))E_j(t), E_{11}(t)\rangle \\
&\qquad + \frac{1}{1+\mu_2(t)^2} \langle R(\dot\gamma(t), E_{4+j}(t))E_{4+j}(t), E_{11}(t)\rangle \big) = 0
\end{aligned}$$

is valid. Then the relation above and (10) imply that the critical point condition (2) is also satisfied, and therefore $\xi$ is a minimal unit vector field. $\square$

**Remark 1** In the paper [1], Proposition 2 is crucial for the proof of the fact that the radial unit vector field $\xi$ is harmonic and minimal. Among others, Proposition 2 implies that the tubes around the reflective totally geodesic submanifold $F$ are curvature-adapted submanifolds of the ambient space $N$.

In the present case, however, the relations (5), (6), (7) show that the Jacobi operator $R_{\dot\gamma(t)}$ and the shape operator $A_{\dot\gamma(t)}$ of the principal orbit $\hat{L}(\gamma(t))$ at $\gamma(t)$ do not commute. This means that the principal orbits of $\hat{L}$ are not curvature-adapted submanifolds of $G_2$. Hence, Proposition 2 in [1] is not valid for the Jacobi operator $R_u \colon T_eG_2 \to T_eG_2$.

## 5 Harmonic and minimal unit vector field on $G_2/SO(4)$

Consider now the coset space $G_2/K$, where the symmetric subgroup $K$ is isomorphic to $SO(4)$. In what follows we also denote this coset space $G_2/K$ by $N$. As is usual, we identify the tangent space $T_oN$ of $N$ at the point $o = eK$ with the subspace $\mathfrak{p}$ presented by (3).

Take the smooth action $\lambda \colon G_2 \times N \to N$ defined by $\lambda(g, hK) = ghK$ for $g, h \in G_2$. Endow $N$ with the Riemannian metric $\langle\ ,\ \rangle$ for which the action $\lambda$ is isometric and $\langle v_1, v_2\rangle_o = -B(v_1, v_2)$ holds for all $v_1, v_2 \in \mathfrak{p}$. As is well-known, then $N = G_2/K$ turns into a compact symmetric space of rank two.

*Hereafter, we discuss the natural isometric action $\lambda \colon L \times N \to N$, where the Lie group $L$ is isomorphic to $SU(3)$.* We list some basic facts about this isometric action (for more details see [9]).

$L(o)$ is a totally geodesic singular orbit which is isometric to the 4-dimensional complex projective space $\mathbb{C}P^2$. Let Exp denote the exponential map of $N$. Then $L(o) = \mathrm{Exp}(\mathfrak{p} \cap \mathfrak{l})$ holds and $\mathfrak{p} \cap \mathfrak{n}$ presents the normal complementary subspace of $T_oL(o) = \mathfrak{p} \cap \mathfrak{l}$ in $T_oN$. Since $\mathfrak{p} \cap \mathfrak{n}$ is not a Lie triple system, $L(o)$ is a non-reflective totally geodesic submanifold of $N$. The closed geodesics starting perpendicularly from $L(o)$ intersect orthogonally all the orbits of $L$, and therefore $\lambda$ is a cohomogeneity one action.

Denote by $\chi$ the maximal sectional curvature of $N$ and take the numbers $\kappa = \frac{\chi}{12}$, $r = \frac{\pi}{4\sqrt{\kappa}}$. Select a unit vector $u$ of the subspace $\mathfrak{p} \cap \mathfrak{n}$ and take the geodesic $\gamma$ defined by $\gamma(t) = \mathrm{Exp}(tu)$ for $t \in \mathbb{R}$. Then $L(\gamma(r))$ presents the other singular orbit of $\lambda$. It can be seen that $L(\gamma(r))$ is also a non-reflective totally geodesic submanifold of $N$ and $L(\gamma(r))$ is isometric to the Riemannian symmetric space $SU(3)/SO(3)$. The principal orbits $L(\gamma(t))$, $0 < t < r$, coincide with the tubular hypersurfaces around $L(o)$.

Consider now the union $U$ of all principal orbits of $L$. The tangent vectors of the unit speed geodesics emanating perpendicularly from $L(o)$ yield a unit vector field $\xi$ on this connected open domain $U$. Then, applying the argument developed in Section 4, we can prove the following statement.

**Theorem 2** *The radial unit vector field $\xi$ associated to the isometric action $\lambda\colon L \times G_2/K \to G_2/K$ is harmonic and minimal.*

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