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ASYMPTOTIC PROPERTIES OF ONE DIFFERENTIAL EQUATION WITH UNBOUNDED DELAY

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Abstract. We study the asymptotic behavior of the solutions of a differential equation with unbounded delay. The results presented are based on the first Lyapunov method, which is often used to construct solutions of ordinary differential equations in the form of power series. This technique cannot be applied to delayed equations and hence we express the solution as an asymptotic expansion. The existence of a solution is proved by the retract method.

Keywords: asymptotic expansion, retract method

MSC 2010: 34E05

1. Introduction

The first method of Lyapunov is a well known technique used to study the asymptotic behavior of ordinary differential equations in the form of a linear system with perturbation. This method uses the solution in the form of a convergent power series, for details see [1]. The results for equations in the implicit form [2] or for integro-differential equations [8] were derived by modifying the first method of Lyapunov. The existence of solutions with a certain asymptotic form were proved in the results cited using Ważewski's topological method. For analogous representations of solutions for a retarded differential equation, see [6], [7]. The perturbation has a polynomial form in both cases. In this paper, we study an equation in the form

$$\dot{y}(t) = -a(t)y(t) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \prod_{j=1}^{n} \big(y(\xi_j(t))\big)^{i_j} \tag{1.1}$$

where $\mathbf{i} = (i_1, \dots, i_n)$ is a multiindex, $i_j \geqslant 0$ are integers and $|\mathbf{i}| = \sum_{j=1}^{n} i_j$. The continuous functions $\xi_j(t)$ satisfy $t > \xi_j(t) \geqslant r_0$ for all $t \in [t_0, \infty)$ and the function $\xi(t)$, which is defined as $\xi(t) = \min_{1 \leqslant i \leqslant n} \xi_i(t)$, is nondecreasing for $t \geqslant t_0$. Therefore, all asymptotic relations such as the Landau symbols o, O and the asymptotic equivalence $\sim$ will be considered for $t \to \infty$. This fact will not be pointed out in the sequel.

The function $a(t)$ satisfies the following conditions:

(C1) $a(t)$ is continuous and positive on the interval $[t_0, \infty)$ and $1/a(t) = O(1)$,

(C2) $(t - \xi(t))\widetilde{a}(t) = o\,(A(t))$ where the functions $A(t)$, $\widetilde{a}(t)$ are defined as $A(t) = \int_{t_0}^{t} a(u)\,\mathrm{d}u$, $\widetilde{a}(t) = \max_{u \leqslant t}(a(u))$.

Further conditions for continuous functions $c_{\mathbf{i}}(t)\colon [t_0, \infty) \to \mathbb{R}$ will be given later. In order to apply the first method of Lyapunov to the equation (1.1) we assume the solution in the form of a formal series

$$y(t, C) = \sum_{n=1}^{\infty} f_n(t)\varphi^n(t, C) \tag{1.2}$$

where $\varphi(t, C)$ is the solution of the homogeneous equation $\dot{y}(t) = -a(t)y(t)$ given by the formula $\varphi(t, C) = C\exp(-A(t))$, the function $f_1(t) \equiv 1$, and the functions $f_k(t)$ for $k = 2, \dots, n$ are particular solutions of a certain system of auxiliary differential equations. Using Ważewski's topological method in the form as used in [3] and [4] for differential equations with unbounded delay and finite memory, we prove the existence of a solution $y_n(t, C) \sim Y_n(t, C) = \sum_{k=1}^{n} f_k(t)\varphi^k(t, C)$.

2. Preliminaries

Lemma 2.1. *Let a function $a(t)$ satisfy conditions* (C1), (C2). *Then*

$$A(t) \sim A(\xi^i(t)) \text{ as } t \to \infty \text{ for any integer } i \in \mathbb{N} \tag{2.1}$$

where $\xi^1(t) = \xi(t)$, and for $i > 1$, the functions $\xi^i(t)$ are defined by

$$\xi^{i+1}(t) = \xi(\xi^i(t)).$$

P r o o f. First, we see that, by virtue of condition (C2), the assertion is true for $i = 1$:

$$\int_{\xi(t)}^{t} a(u)\,\mathrm{d}u \leqslant (t - \xi(t))\widetilde{a}(t) = o(A(t)) \text{ and } \lim_{t\to\infty} \frac{A(\xi(t))}{A(t)}$$
$$= 1 - \lim_{t\to\infty} \frac{\int_{\xi(t)}^{t} a(u)\,\mathrm{d}u}{A(t)} = 1.$$

The assumption $\xi(t) \not\to \infty$ for $t \to \infty$ implies that there exists a constant $\xi(\infty)$ and condition (C2) is not satisfied. If $\xi(t) \to \infty$ for $t \to \infty$, then $\xi^i(t) \to \infty$ for $t \to \infty$, too. Now we use the assertion for $i = 1$ substituting $\xi^i(t)$ for t and the proof follows by induction. □

Remark 2.1. Note that condition (C1) implies the divergence of the integral $\int_{t_0}^{\infty} a(u)\,\mathrm{d}u$, which has two consequences.

First, the function $\varphi(t, C)$ satisfies the relation $\varphi^k(t, C) = o\big(\varphi^l(t, C)\big)$ for $k > l$, which guarantees that the sequence $\{\varphi^n(t, C)\}_{n=1}^{\infty}$ is asymptotic.

Second, the divergence implies the relation $1/A(t) = o(1)$ which is suitable for asymptotic estimation.

In order to specify the asymptotic behavior of the solution of the auxiliary equations we consider the equation

$$\dot{y}(t) = na(t)y(t) + f(t) \tag{2.2}$$

where $n > 0$ is a constant and the properties of the function $f(t)$ are described by a function $k(t)$, a constant K, and the relations

(F1) $\lim\limits_{t\to\infty} f(t)\exp(\tau k(t)) = 0$ for all $\tau < K$,

(F2) $\lim\limits_{t\to\infty} |f(t)|\exp(\tau k(t)) = \infty$ for all $\tau > K$.

The asymptotic behavior of the solution of equation (2.2) depends on the relation between the functions $k(t)$ and $na(t)$.

Lemma 2.2. *Let either $k(s) - k(t) = o(\int_t^s na(u)\,\mathrm{d}u)$ or $k(s) - k(t) = O(\int_t^s na(u)\,\mathrm{d}u)$ and $K = 0$ where K is the constant used in assumptions* (F1), (F2). *Now if the function $f(t)$ satisfies assumption* (F1), *then there exists at least one solution $Y(t)$ of equation* (2.2) *satisfying also assumption* (F1). *If the function $f(t)$, moreover, satisfies assumption* (F2), *then the solution $Y(t)$ also satisfies assumption* (F2).

Proof. We may rewrite assumptions (F1), (F2) for the function $f(t)$ satisfying them so that, for sufficiently large t and constants $\tau_1, \tau_2 > 0$, the function $f(t)$ satisfies the inequality

$$\exp\left((K - \tau_2)k(t)\right) \leqslant |f(t)| \leqslant \exp\left((K + \tau_1)k(t)\right),$$

and also, for the desired solution $Y(t) = \int_t^{\infty} -f(s)\exp\int_t^s -na(u)\,\mathrm{d}u\,\mathrm{d}s$, we have estimates of the solution of equation (2.2)

$$\begin{aligned} \exp((K+\tau_1)k(t)) \int_t^{\infty} \exp\Big\{ - (K+\tau_1)(k(t) - k(s)) - \int_t^s na(u)\,\mathrm{d}u \Big\}\,\mathrm{d}s \geqslant |Y(t)| \\ \geqslant \exp((K-\tau_2)K(t)) \int_t^{\infty} \exp\Big\{ - (K-\tau_2)\tau(k(s) - k(t)) - \int_t^s na(u)\,\mathrm{d}u \Big\}\,\mathrm{d}s. \end{aligned}$$

Now utilizing the assumptions of this lemma, we see that the asymptotic behavior of exponents involved in both integrands are the same as the asymptotic behavior of the function $\int_t^s na(u)\,\mathrm{d}u$. As the function $(na(t))^{-1}$ is bounded, the integral $\int_t^s na(u)\,\mathrm{d}u$ is divergent for $s\to\infty$ and the integrals on both sides of the inequalities are convergent and there exist constants A_1, A_2 such that

$$A_1\exp\left((K-\tau_2)k(t)\right)\leqslant |Y(t)|\leqslant A_2\exp\left((K+\tau_1)k(t)\right).$$

Assumption (F1) implies the second inequality, which ensures the convergence and thus the existence of the integral defining $Y(t)$ which is the solution of the given equation. □

To make the specification of the coefficients of the power series which is the product of the power series raised to a power easier, we use the following notation: $\mathfrak{s}=(\mathfrak{s}_1,\dots,\mathfrak{s}_n)$ is an ordered n-tuple of sequences $\mathfrak{s}_j=\left\{\mathfrak{s}_j^k\right\}_{k=1}^{\infty}$ of nonnegative integers with a finite sum $|\mathfrak{s}_j|=\sum\limits_{k=1}^{\infty}\mathfrak{s}_j^k$, and we denote $\mathfrak{s}!=\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{\infty}\mathfrak{s}_j^k!$, $\mathbf{i}(\mathfrak{s})!=\prod\limits_{j=1}^{n}|\mathfrak{s}_i|!$, $V(\mathfrak{s})=\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{\infty}k\mathfrak{s}_j^k$, $\mathbf{i}(\mathfrak{s})=(|\mathfrak{s}_1|,\dots,|\mathfrak{s}_n|)$. For any ordered n-tuple of sequences (of numbers or functions) $\mathcal{C}=(\mathbf{c}_1,\dots,\mathbf{c}_n)$ where $\mathbf{c}_j=\{c_j^k\}_{k=1}^{\infty}$, we denote $\mathcal{C}^{\mathfrak{s}}=\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{\infty}\left(c_j^k\right)^{\mathfrak{s}_j^k}$ where $\left(c_j^k\right)^0=1$ for every c_j^k. Then it is possible to write

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{\infty}c_j^kx^k\right)^{i_j}=\sum_{k=|\mathbf{i}|}^{\infty}x^k\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{C}^{\mathfrak{s}}$$

where the symbol $\sum\limits_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}$ denotes the sum over all $\mathfrak{s}$ such that $V(\mathfrak{s})=k$, $\mathbf{i}(\mathfrak{s})=\mathbf{i}$ and, for empty set of $\mathfrak{s}$, this symbol equals 0.

3. Main results

We assume that the formal solution of equation (1.1) is expressed in the form (1.2) where $\varphi(t,C)$ is the general solution of the equation $\dot{y}(t)=-a(t)y(t)$. Consequently, $\varphi(t,C)=C\exp(-A(t))$ where $C\neq 0$ is a constant, $f_1(t)=1$ and $f_k(t)$, $k\geqslant 2$ for the time being are unknown functions. Substituting $y(t)$ in equation (1.1) and matching the coefficients at the same powers $\varphi^k(t,C)$, we obtain an auxiliary system of linear differential equations

$$\dot{f}_k(t)=(k-1)a(t)f_k(t)+\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{F}^{\mathfrak{s}}\tag{3.1}$$

where $\mathcal{F}(t)$ is the n-tuple of sequences $\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty}$ i.e. $\mathcal{F}(t)=\big(\ldots\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty},\ldots\big)$. The facts $V(\mathfrak{s})=k\geqslant 2$ and $|\mathbf{i}(\mathfrak{s})|\geqslant 2$ imply $\mathfrak{s}_i^l=0$ for $l\geqslant k$. Moreover, the auxiliary system (3.1) is recurrent.

Theorem 3.1. *For the functions* $c_{\mathbf{i}}(t)$, *let* $\lim\limits_{t\to\infty} c_{\mathbf{i}}(t)\exp(-\tau A(t))=0$ *for all positive* τ. *Then there exists a sequence* $\{f_k(t)\}_{k=1}^{\infty}$ *of solutions of the auxiliary system* (3.1)

$$f_k(t)=\int_t^{\infty}-a(s)\exp\Big\{-\int_t^s(k-1)a(u)\,\mathrm{d}u\Big\}\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{|\mathbf{i}(\mathfrak{s})|!}{\mathbf{i}(\mathfrak{s})!}\mathcal{F}^{\mathfrak{s}}\,\mathrm{d}s \tag{3.2}$$

such that $\lim\limits_{t\to\infty} f_k(t)\exp(-\tau A(t))=0$ *for all* τ.

P r o o f. Formula (3.2) can be obtained by integrating the system (3.1). When applying Lemma 2.2, we put $k(t)=A(t)$. Condition (C2) proves that for the function $y(t)$ satisfying assumption (F1) of Lemma 2.2, the function $y(\xi^j(t))$ satisfies this assumption, too. Therefore, the sum and the product of functions verifying assumption (F1) of Lemma 2.1 satisfy the assumptions of Lemma 2.2. Using Lemma 2.2, we can then easily show the convergence of (3.2) and the desired property. □

R e m a r k 3.1. An assertion analogous to the one of Theorem 3.1 with the property described by assumption (F2) of Lemma 2.2 cannot be proved as the sum of functions verifying the assumption (F2) need not satisfy this assumption.

Let $\|\cdot\|$ denote the maximum norm on $C^0[r^*,t_0]$. Moreover, we denote

$$y_k(t)=\sum_{l=1}^{k}f_l(t)\varphi^l(t,C),\qquad \overset{k}{\Sigma}(t)=\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\alpha)=\mathbf{i}\\ V(\alpha)=k}}\frac{\mathbf{i}(\alpha)!}{\alpha!}\mathcal{F}^{\alpha}.$$

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *hold and let*

$$\lim_{t\to\infty} f_{k+1}^{-1}(t)\exp(-\tau A(t))=0$$

where $\tau<1$ *is a constant. We denote* $r^*=\min\limits_{t\geqslant t_0}(\xi(t))$. *Then for every* $C\neq 0$ *and* $\psi\in C^0[r^*,t_0], \|\psi\|\leqslant 1, \psi(t_0)=0$, *there exists a solution* $y_C(t)$ *of equation* (1.1) *such that*

$$|y_C(t)-y_k(t)|\leqslant\sigma|f_{k+1}(t)\varphi^{k+1}(t,C)| \tag{3.3}$$

for $t\in[t_C,\infty)$ *where the functions* $f_k(t)$ *are solutions* (3.2) *of system* (3.1), $\sigma>1$ *is a constant.* t_C *is a function of the parameter* C *and of* σ,k.

P r o o f. The existence of the solution $y_C(t)$ is proved by Theorem 1 in [3], which is based on the retract method and the second method of Lyapunov. A sufficient condition for the existence of a solution of the equation with unbounded delay and finite memory is described there. The theory of this type of equations (referred to as p-type retarded functional differential equation) is given in [5]. In this case we put $p(t,\vartheta) = t + \vartheta(t - \psi(t))$ and the function on the right hand side of the equation $f(t, y_t)\colon \mathbb{R} \times C^0[-1,0] \to \mathbb{R}$ is defined by the formula:

$$f(t,\psi) = -a(t)\psi(p(t,0)) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \prod_{l=1}^{n} \psi^{i_l}(p(t,\vartheta_{i_l}(t)))$$

where $\vartheta_{i_l}(t) = -(t - \xi_{i_l}(t))/(t - \xi(t))$. The set ω used in Theorem 1 is defined as

$$\omega = \{(y,t)\colon\ y_k(t) - \sigma|f_{k+1}|(t)\varphi^{k+1} < y < y_k(t) + \sigma|f_{k+1}(t)|\varphi^{k+1},\ t > t_C\}.$$

Note that the numbers p, n used in Theorem 1 in [3] equal 1 and, consequently, the indices of functions δ, ϱ are omitted, i.e., $\delta = y_k(t) + \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$ and $\varrho = y_k(t) - \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$. We verify the inequalities

$$\delta'(t) > f(t,\pi) \text{ and } \varrho'(t) < f(t,\pi)$$

where $\pi \in C([p(t,-1),t],\mathbb{R})$ is such that $(\theta,\pi(\theta)) \in \omega$ for all $\theta \in [p(t,-1),t)$ and $\pi(t) = \delta(t)$ or $\pi(t) = \varrho(t)$, respectively, for a sufficiently large t. As the sequence $\{\varphi^k(t,C)\}_{k=1}^{\infty}$ is asymptotic, we can rearrange the terms in these inequalities with respect to the powers of the functions $\varphi^k(t,C)$. We verify the first inequality.

First, for sufficiently large t, $f_{k+1}\varphi^{k+1}(t,C) \neq 0$ and the derivative $\delta'(t)$ exists:

$$\begin{aligned}\delta'(t) = {} & \sum_{l=1}^{k} \left(f_l'(y) - la(t)f_l(t)\right)\varphi^l(t,C) \\ & + \sigma \operatorname{sign}(f_{k+1}(t)) \left(f_{k+1}'(t) - (k+1)a(t)f_{k+1}(t)\right)\varphi^{k+1}(t,C).\end{aligned}$$

Second, for $\pi(t) = \delta(t)$ there exist suitable positive constants such that

$$\begin{aligned}f(t,\pi_t) = {} & -a(t)\left(y_k(t) + \sigma|f_{k+1}(t)|\varphi^{k+1}(t,C)\right) \\ & + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \prod_{l=1}^{n} \left(y_k(t) + K_l\sigma|f_{k+1}(t)|\varphi^{k+1}(t,C)\right)^{i_l}.\end{aligned}$$

Since the system (3.1) is recurrent, the coefficients at $\varphi^l(t,C)$ after substituting $y(t,C)$ in the form (1.2) and $y(t) = y_k(t) \pm \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$ in the sum

$\sum\limits_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t)\big(\mathbf{y}(\xi(t))\big)^{\mathbf{i}}$ coincide for $l = 1, \dots, k+1$, i.e.

$$\begin{aligned} f(t, \pi_t) = & -a(t)\bigg(\sum_{l=1}^{k} f_l(t)\varphi^l(t, C) + \sigma|f_{k+1}|(t)\varphi^{k+1}(t, C)\bigg) \\ & + \sum_{j=1}^{k+1} \overset{j}{\Sigma}(t)\varphi(t, C)^j + \varphi(t, C)^{k+2}R(t) \end{aligned}$$

where $R(t)$ is a function satisfying $\lim\limits_{t\to\infty} R(t)\exp(-\tau\int_{t_0}^{t} \mathrm{d}u/g(u)) = 0$ for all positive τ.

Now we can evaluate the sign of the difference $\delta'(t) - f(t, \pi_t)$ (with $\pi(t) = \delta(t)$):

$$\begin{aligned} \delta'(t) - f(t, \pi_t) = & \sum_{l=1}^{k} \Big(f_l'(y) - \frac{(l-1)f_l(t)}{g(t)} - \overset{l}{\Sigma}(t)\Big)\varphi^l(t, C) \\ & + \bigg[\sigma \operatorname{sign}(f_{k+1}(t))\Big(f_{k+1}'(t) - \frac{kf_{k+1}(t)}{g(t)}\Big) - \overset{k+1}{\Sigma}(t)\bigg]\varphi^{k+1}(t, C) - \varphi(t, C)^{k+2}R(t). \end{aligned}$$

The functions $f_k(t)$ are solutions of (3.1) for $l = 1, \dots, k$. Therefore, the minimal power of $\varphi(t, C)$ in the difference $\delta'(t) - f(t, \pi_t)$ is $k+1$. Moreover, the term $\varphi(t, C)^{k+2}R(t)$ and higher powers are very small for sufficiently large t, the sign of this difference is given by the factor at the power $\varphi(t, C)^{k+1}$, i.e.

$$\begin{aligned} \operatorname{sign}(\delta'(t) - f(t, \pi_t)) &= \sigma \operatorname{sign}(f_{k+1}(t))\Big(f_{k+1}'(t) - \frac{kf_{k+1}(t)}{g(t)}\Big) - \overset{k+1}{\Sigma}(t) \\ &= \sigma \operatorname{sign}(f_{k+1}(t))\overset{k+1}{\Sigma}(t) - \overset{k+1}{\Sigma}(t) = \sigma \operatorname{sign}(f_{k+1}(t))\overset{k+1}{\Sigma}(t). \end{aligned}$$

Due to definition (3.2) of $f_{k+1}(t)$, we obtain $\operatorname{sign}(\delta'(t) - f(t, \pi_t)) = -1$ and the inequality $\delta'(t) > f(t, \pi_t)$ holds, too. A similar consideration for the difference $\varrho'(t) - f(t, \pi_t)$ (with $\pi(t) = \varrho(t)$) gives $\varrho'(t) < f(t, \pi_t)$. Now we may use Theorem 1 in [3] to obtain the existence of a solution satisfying the estimate (3.6). □

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied and let there exist a sequence* $\{K_k\}_{k=1}^{\infty}$, $K_0 = 1$ *such that the assumptions of Theorem* 3.2 *are satisfied for every* K_k, *i.e.,* $\lim\limits_{t\to\infty} f_{K_k}^{-1}(t)\exp(-\tau A(t)) = 0$. *Then there exists an asymptotic expansion of the solution* $y_C(t)$ *in the form*

$$y_C(t) \approx \sum_{k=1}^{\infty} F_k(t), \ \textit{where } F_k(t) = \sum_{l=K_{k-1}}^{K_k - 1} f_l(t)\varphi^l(t, C)$$

and $f_l(t)$ *are solutions of* (3.2).

P r o o f. Since the assumptions of Theorem 3.2 are fulfilled for every K_k, there exists a solution $y_C(t)$ satisfying the inequality in this theorem. Then the existence of an asymptotic expansion follows from the fact that the sequence $\{F_k\}^\infty$ is asymptotic, i.e., $\lim\limits_{t\to\infty} F_{k+1}(t)/F_k(t) = 0$ and the assertion is proved. $\square$

E x a m p l e 1. We study the asymptotic properties of the solutions of the equation

$$\dot{y}(t) = -y\cos(ty(\xi(t)) = -y(t) + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k}y(t)(y(\xi(t)))^{2k}}{(2k)!}$$

on the interval $[1,\infty)$ for two various delays $r_1(t) = r > 0$, i.e., $\xi_1(t) = t - r$, and $r_2(t) = \ln t$, i.e., $\xi_2(t) = t - \ln t$. In this case we have $a(t) = 1$, $A(t) = t - 1$, $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$, $c_{(1,2k)} = (-1)^{k+1}t^{2k}/(2k)!$ (for other multiindices $c_{\mathbf{i}} = 0$). If we denote $\mathcal{F} = (\{f_i(t)\}_{i=1}^\infty, \{f_i(\xi(t))\mathrm{e}^{i(t-\xi(t))}\}_{i=1}^\infty)$, the system of auxiliary differential equations of the form

$$\dot{f}_k(t) = (k-1)f_k(t) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^{2i}}{(2i)!}\sum_{\substack{h\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{F}^{\mathfrak{s}}$$

has a particular solution $f_{2k} = 0$. First, $f_2(t) = 0$ is due to $\dot{f}_2(t) = f_2(t)$. We will prove by induction that the equation for the function f_{2k} has the form $\dot{f}_{2k}(t) = f_{2k}(t)$, therefore, the odd ($|\mathbf{i}(\mathfrak{s})| = 1 + 2l$) sum of odd exponents (due to the induction hypothesis) is not even (2k) and every product on the right-hand side of the auxiliary equation contains zero multiplicands (f_{2i}). The asymptotic form of the solutions f_{2k+1} depends on the delay $r_i(t)$ but the property $f_{2k-1}(t) \sim f_{2k-1}(\xi(t))$ holds for both $r_i(t)$.

First, for $r_1(t)$ the solutions have the asymptotic form $f_{2k+1} = t^{2k}(c_{2k+1} + O(1/t))$, where $c_1 = 1$ and c_{2k+1} are given by the recurrent formula

$$c_{2k+1} = \frac{1}{2k}\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}\sum_{\substack{\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=2k+1}}\mathcal{C}^{\mathfrak{s}_1}\mathcal{C}_r^{\mathfrak{s}_2}, \quad \text{where } \mathcal{C} = \{c_i\}_{i=1}^\infty,\ \mathcal{C}_r = \{c_i\exp(ir)\}_{i=1}^\infty.$$

Second, we have the relation $\exp(k(A(t) - A(\xi(t)))) = \exp(k\ln t) = t^k$ for the delay $r_2(t)$ and the function f_3 satisfies the equation $\dot{f}_3(t) = 2f_3(t) + \frac{1}{2}t^4$ and we obtain the solution $f_3(t) = t^4(-\frac{1}{4} + O(1/t))$. Applying induction for the solutions f_{2k+1} in the form $f_{2k-1}(t) = t^{p(k)}(d(k) + O(1/t))$, we see that the main power of t in the sum on the right hand side of the equation for f_{2k-1} is at the product $t^2 f_1(t)f_1(\xi(t))tf_{2k-3}(\xi(t))t^{2k-3} = t^{2k+p(k-1)}(d(k-1) + O(1/t))$ and we obtain the equation $\dot{f}_{2k+1}(t) = 2kf_{2k+1}(t) + t^{2k+p(k-1)}(d(k-1) + O(1/t))$. The solution $f_{2k-1}(t)$

has the asymptotic form $f_{2k+1} = -t^{2k+p(k-1)}\left(d(k-1)/2k + O(1/t)\right)$. The constants $d(k)$ and $p(k)$ satisfy the recurrent formulas $d(k) = -d(k-1)/2k$, $p(k) = p(k-1)+2k$, otherwise $d(k) = (-1)^{k-1}2^{-k}/(k-1)!$ and $p(k) = (k+2)(k-1)$. By Theorem 3.3, we obtain the existence of a pair of asymptotic expansions $y_1(t)$, $y_2(t)$ of the solutions for two different delays $r_1(t)$, $r_2(t)$:

$$y_1(t) \approx \sum_{k=1}^{\infty} t^{2(k-1)} c_{2k-1} \mathrm{e}^{(2k-1)t} C^{2k-1},$$

$$y_2(t) \approx \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^{(k+2)(k-1)}}{2^k (k-1)!} \mathrm{e}^{(2k-1)t} C^{2k-1}.$$

R e m a r k 3.2. This example shows a fundamental dependence of the asymptotic properties of the expansion on the magnitude of the delay. For a small delay ($r_1(t) \to 0$), the expansion $y_1(t)$ converges to the expansion of the solution of an ordinary equation $\dot{y}(t) = -y\cos(ty(t))$. For a sufficiently large delay $r_2(t) = \ln(t)$, the expansion $y_2(t)$ is the same as for the equation $\dot{y}(t) = -y(t) + t^2 y(t) y^2(t - \ln t)/2$, i.e., the expansions for the perturbation with infinite sum and for the perturbation with only the first summand are the same.

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