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ON SIMILARITY SOLUTION OF A BOUNDARY LAYER PROBLEM FOR POWER-LAW FLUIDS

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Abstract. The boundary layer equations for the non-Newtonian power law fluid are examined under the classical conditions of uniform flow past a semi infinite flat plate. We investigate the behavior of the similarity solution and employing the Crocco-like transformation we establish the power series representation of the solution near the plate.

Keywords: similarity solution, boundary layer problem, power series solution

MSC 2010: 35Q35, 34B40

1. Mathematical formulation

Consider a steady two-dimensional laminar flow of a power-law fluid with constant speed V_∞ over a semi-infinite flat plate at zero incidence. In the absence of the body force and external pressure gradients, the laminar boundary layer equations expressing conservation of mass and momentum are governed by ([2], [14]):

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.1}$$

$$v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{1}{\varrho}\frac{\partial \tau}{\partial z}, \tag{1.2}$$

where the y and z axes are taken along and perpendicular to the plate, v and w are the velocity components of the fluid parallel and normal to the plate, $\tau = \kappa\left|\partial v/\partial z\right|^{n-1} \times \partial v/\partial z$ is the shear stress, and $\nu = \gamma|\partial v/\partial z|^{n-1}$ $(\gamma = \kappa/\varrho)$ is the kinematic viscosity. The case $n = 1$ corresponds to a Newtonian fluid, $0 < n < 1$ is referred to as the pseudo-plastic non-Newtonian fluid and $n > 1$ describes the dilatant fluid. The appropriate boundary conditions are

$$v(y,0)=0,\quad w(y,0)=0, \tag{1.3}$$

$$v(y,z)\to V_\infty \quad \text{as } z\to\infty, \tag{1.4}$$

where $V_\infty>0$ represents the mainstream velocity.

The continuity equation (1.1) is satisfied by introducing a stream function $\psi(y,z)$ such that $v=\partial\psi/\partial z$, $w=-\partial\psi/\partial y$. Then the momentum equation can be transformed into an ordinary differential equation by the transformation $\eta=(\mathrm{Re}/(y/L))^{1/(n+1)}(z/L)$, $\psi(y,z)=LV_\infty(\mathrm{Re}/(y/L))^{-1/(n+1)}f(\eta)$, where η is the similarity variable, $f(\eta)$ is the dimensionless stream function, L is the characteristic length and Re is the generalized Reynolds number defined as $\mathrm{Re}=\varrho V_\infty^{2-n}L^n/\kappa$. The partial differential equation (1.2) is transformed into an autonomous third order non-linear differential equation

$$(|f''|^{n-1}f'')'+\frac{1}{n+1}ff''=0, \tag{1.5}$$

where primes denote differentiation with respect to η. The transformed boundary conditions are

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=\lim_{\eta\to\infty}f'(\eta)=1. \tag{1.6}$$

The nondimensional velocity components can be expressed by $f(\eta)$ as $v(y,z)=V_\infty f'(\eta)$, $w(y,z)=V_\infty(n+1)^{-1}\mathrm{Re}_y^{-1/(n+1)}(\eta f'(\eta)-f(\eta))$, where $\mathrm{Re}_y=V_\infty^{2-n}y^n\times(\kappa/\varrho)^{-1}$. We note that when $n=1$ (Newtonian fluid), the present problem is reduced to the classical Blasius problem [3]. Equation (1.5) is referred to as the generalized Blasius equation. We shall use the shooting method, and we replace the condition at ∞ by one at $\eta=0$ $f''(0)=\gamma$. The real number γ has a physical meaning: it provides the wall shear stress $\eta_{\mathrm{wall}}=\gamma V_\infty^n\mathrm{Re}_y^{n/(n+1)}y^{-n}|\gamma|^{n-1}\gamma$ and the non-dimensional drag coefficient $C_D=(n+1)^{1/(n+1)}\mathrm{Re}^{-n/(n+1)}|\gamma|^{n-1}\gamma$ ([2], [13]). For the Newtonian case ($n=1$), it was found in [13] that $\gamma_0=0.33205$. Highly accurate numerical results for γ have been provided in [1], [6], [9]. In 1908, Blasius [3] obtained a numerical solution to (1.5)–(1.6) for $n=1$ in the form of a power series for small values of η:

$$f(\eta)=\eta^2\sum_{k=0}^{\infty}\Big(-\frac{1}{2}\Big)^k\frac{A_k\gamma_0^{k+1}}{(3k+2)!}\eta^{3k}, \tag{1.7}$$

where $A_k=\sum\limits_{j=0}^{k-1}\binom{3k-1}{3r}A_rA_{k-r-1}$ if $k\geqslant 2$, $A_0=A_1=1$.

The main goal of this paper is to examine the properties of the solution to the boundary value problem (1.5)–(1.6) and to give a power series expansion of the solution for small value of η.

2. Initial value problem of the generalized Blasius equation

Instead of (1.5)–(1.6) consider the initial value problem (1.5) with

$$f(0)=0, \quad f'(0)=0, \quad f''(0)=\gamma, \tag{2.1}$$

where $\gamma \in \mathbb{R}$ is the shooting parameter. For an appropriate γ the solution to (1.5)–(2.1) satisfies (1.6). The boundary condition at infinity indicates that $\gamma > 0$. In the case $n=1$, K. Töpfer [15] realized that there is a second group invariance such that if h denotes the solution to (1.5) with initial conditions $h(0)=0$, $h'(0)=0$ and $h''(0)=1$, then the solution f with initial solutions $f(0)=0$, $f'(0)=0$ and $f''(0)=\gamma$ can be obtained by $f(\eta)=\gamma^{1/3}h(\gamma^{1/3}\eta)$. It therefore suffices to compute h and then rescale it so that the rescaled function has the desired asymptotic behavior for large η, namely $f'(\infty)=1$. The true value of the second derivative at the origin is then $\gamma = [\lim\limits_{\eta\to\infty} h'(\eta)]^{-3/2}$. Analogously to Töpfer we get

Theorem 2.1. *Assume that f is the solution of* (1.5)–(1.6) *such that* $f''(0)=\gamma$ *and h is the solution of*

$$(|h''|^{n-1}h'')' + \frac{1}{h+1}hh'' = 0, \quad n>0, \quad n\neq 2, \tag{2.2}$$

$$h(0)=0, \quad hx(0)=0, \quad h''(0)=1. \tag{2.3}$$

Then

$$f(\eta)=\gamma^{(2n-1)/3}h(\gamma^{(2-n)/3}\eta), \quad \gamma = [\lim_{\eta^*\to\infty} h'(\eta^*)]^{-3/(n+1)}. \tag{2.4}$$

Proof. Let us introduce the scaling transformation $h=\lambda^\kappa f$, $\eta^*=\lambda^\mu\eta$ for (1.5), where κ and μ are real, non-zero parameters. We determine κ and μ such that the boundary conditions are substituted by suitable conditions. After simple calculations we get that when $\kappa=(1-2n)/(2-n)\mu$ the governing differential equation (1.5) is left invariant by the new variables h and η^* and the primes for h denotes the derivative with respect to η^* in (2.2). The initial conditions for f correspond to (2.3) with the choice of $\lambda=\gamma$ when $h''(0)=\gamma^{\kappa-2\mu}f''(0)=\gamma^{\kappa-2\mu+1}$. Hence with $\kappa=(1-2n)/3$, $\mu=(2-n)/3$, we have for the power of γ that $\kappa-2\mu+1=0$ i.e., $h=\gamma^{(1-2n)/3}f$,

$\eta^* = \gamma^{(2-n)/3}\eta$ and we get $h''(0) = 1$. Then (2.4) holds and γ is determined by the boundary condition at ∞ as

$$1 = \lim_{\eta\to\infty} f'(\eta) = \lim_{\eta\to\infty} r^{\kappa-\mu} h'(\eta^*) = \lim_{\eta\to\infty} \gamma^{(n+1)/3} h'(\eta^*).$$

□

R e m a r k. Applying (2.4) it is possible to determine the value γ numerically by solving the initial value problem (2.2)–(2.3) for different values of n.

We note that the case $n = 2$ was treated in the paper [10]. For the existence and uniqueness of solutions of (1.5)–(1.6) we refer to [5]:

Theorem 2.2. *Let $n > 0$, then there exists a unique solution of problem* (1.5)–(1.6). *Furthermore, if $0 < n \leqslant 1$ then $f'' > 0$ for all $\eta > 0$, and if $n > 1$ there exists $\eta_0 > 0$ such that $f'' > 0$ on $[0, \eta_0)$.*

For positive γ it was deduced that f, f' and f'' are positive on $(0, \eta_0)$, and the solution f exists on $(0, \infty)$ [5]. We assume that f is the solution to (1.5)–(1.6). Then there exist $r > 0$ and $\eta_r > 0$ such that $f \equiv f_r$ on $(0, \infty)$, $0 \leqslant f_r' \leqslant 1$ for $[0, \eta_r]$ and f satisfies

$$(|f''|^{n-1} f'')' + \frac{1}{n+1} f f'' = 0,$$
$$f(0) = 0, \quad f'(0) = 0, \quad f'(\eta_r) = 1,$$

where η_r is unknown. We employ the following Crocco-like transformation for (1.5)–(1.6): $s = f'$ and $G = f''$, and we arrive at the problem [12]

$$G^n G'' + (n-1) G^{n-1} G'^2 + \frac{s}{n(n+1)} = 0, \tag{2.5}$$
$$G'(0) = 0, \quad G(1) = 0. \tag{2.6}$$

We note that $G(0) = f''(0)$. Here we are interested in the positive solution of (2.5)–(2.6) in $[0, 1)$. It was shown that there exists a unique r such that the initial value problem

$$G^n G'' + (n-1) G^{n-1} G'^2 + \frac{s}{n(n+1)} = 0, \tag{2.7}$$
$$G(0) = r, \quad G'(0) = 0 \tag{2.8}$$

has a continuous, unique, positive solution which vanishes for 1, and $r > 0$ is the shooting parameter [5]. Our task is to determine $r = f''(0) = \gamma$ such that G is positive on $[0, 1)$ and $G(1) = 0$.

Let g be the inverse function of f_r': $g(f_r'(\eta)) = \eta$, then $g(0) = 0$ and $g(1) = \eta_r$. Moreover for any $s \in (0,1)$ $G(s) = 1/g'(s)$, $\eta = g(s)$.

Our aim is to give an approximate power series solution to problem (2.7)–(2.8) and also to problem (1.5)–(1.6) for small values of η and $n > 0$; moreover, to present a method for the determination of the coefficients.

3. Power series representation of the local solution

The object of this section is to determine the local solution G of (2.7)–(2.8) near the origin. We will consider (2.7)–(2.8) as a system of certain differential equations, namely, the special Briot-Bouquet differential equations. For this type of differential equations we refer to the book by E. Hille [8] and E. L. Ince [11]. In order to establish the existence of a power series representation of $G(s)$ about $s = 0$ we refer to the following theorem [4]:

Theorem 3.1. *Consider the system of equations*

$$\begin{cases} \xi \dfrac{\mathrm{d}z_1}{\mathrm{d}\xi} = u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi \dfrac{\mathrm{d}z_2}{\mathrm{d}\xi} = u_2(\xi, z_1(\xi), z_2(\xi)), \end{cases} \tag{3.1}$$

where functions u_1 and u_2 are holomorphic functions of ξ, $z_1(\xi)$, and $z_2(\xi)$ near the origin, and moreover $u_1(0,0,0) = u_2(0,0,0) = 0$. Then a holomorphic solution of (3.1) satisfying the initial conditions $z_1(0) = 0$, $z_2(0) = 0$ exists if none of the eigenvalues of the matrix

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial z_1}\Big|_{(0,0,0)} & \dfrac{\partial u_1}{\partial z_2}\Big|_{(0,0,0)} \\ \dfrac{\partial u_2}{\partial z_1}\Big|_{(0,0,0)} & \dfrac{\partial u_2}{\partial z_2}\Big|_{(0,0,0)} \end{bmatrix} \tag{3.2}$$

is a positive integer.

This theorem ensures the existence of formal solutions $z_1 = \sum\limits_{k=0}^{\infty} a_k \xi^k$ and $z_2 = \sum\limits_{k=0}^{\infty} b_k \xi^k$ for system (3.1), and also the convergence of formal solutions.

Theorem 3.2. *Let $n > 0$. The initial value problem (2.7)–(2.8) has a unique analytic solution of the form $G(s) = Q(s^3)$ in the neighborhood of $s = 0$, where Q is a holomorphic solution to*

$$Q'' = -\frac{2}{3}\frac{Q'}{s} - (n-1)\frac{Q'^2}{Q^n} - \frac{1}{n(n+1)\delta^2}\frac{1}{Q^n s} \tag{3.3}$$

near zero satisfying $Q(0) = r$, $Q'(0) = -1/[6n(n+1)r^n]$.

P r o o f. Let us formulate (2.7) as a system of Briot-Bouquet type differential equations (3.1) and take the solution in the form $G(s) = Q(s^\delta)$, $s \in (0,1)$, where $Q \in C^2(0,1)$. Substituting $G(s) = Q(s^\delta)$ into (2.7) we get that Q satisfies

$$Q''(s^\delta) = -\frac{\delta-1}{\delta}s^{-\delta}Q' - (n-1)\frac{Q'^2}{Q^n} - \frac{1}{n(n+1)\delta^2}s^{3-2\delta}\frac{1}{Q^n}.$$

Introducing a variable ξ by $\xi = s^\delta$ we have

$$Q''(\xi) = -\frac{\delta-1}{\delta}\frac{Q'}{\xi} - (n-1)\frac{Q'^2}{Q^n} - \frac{1}{n(n+1)\delta^2}\xi^{3/\delta-2}\frac{1}{Q^n}. \tag{3.4}$$

Take the function Q in the form $Q(\xi) = r + q\xi + z(\xi)$, for some constant q, and $z \in C^2(0,a)$, $z(0) = 0$, $z'(0) = 0$. Therefore Q fulfils the conditions $Q(0) = r$, $Q'(0) = q$, $Q'(\xi) = q + z'(\xi)$, $Q''(\xi) = z''(\xi)$. The initial condition $G(0) = r$ is satisfied. We restate (3.4) as a system of equations

$$\left.\begin{aligned} z_1(\xi) &= z(\xi) \\ z_2(\xi) &= z'(\xi) \end{aligned}\right\} \quad \text{with} \quad \left.\begin{aligned} z_1(0) &= 0 \\ z_2(0) &= 0 \end{aligned}\right\}.$$

Due to (3.4) we get that

$$\begin{aligned} z''(\xi) = &- \frac{\delta-1}{\delta}\frac{q+z_2(\xi)}{\xi} - (n-1)\frac{(q+z_2(\xi))^2}{(r+q\xi+z_1(\xi))^n} \\ &- \frac{1}{n(n+1)\delta^2}\xi^{\frac{3}{\delta}-2}\frac{1}{(r+q\xi+z_1(\xi))^n}. \end{aligned}$$

We arrange the system of equations (3.1) as follows

$$\begin{aligned} u_1(\xi, z_1(\xi), z_2(\xi)) &= \xi z_2 \\ u_2(\xi, z_1(\xi), z_2(\xi)) &= -\frac{\delta-1}{\delta}(q+z_2(\xi)) - (n-1)\frac{(q+z_2(\xi))^2}{(r+q\xi+z_1(\xi))^n}\xi \\ &\quad - \frac{1}{n(n+1)\delta^2}\xi^{\frac{3}{\delta}-1}\frac{1}{(r+q\xi+z_1(\xi))^n}. \end{aligned}$$

In order to satisfy conditions $u_1(0,0,0)=0$ and $u_2(0,0,0)=0$ we must get zero for the power of ξ on the right-hand side of the second equation. Therefore $3/\delta-1=0$, i.e., $\delta=3$; moreover, we have that $-\frac{2}{3}q-\frac{1}{9}(n(n+1)r^n)^{-1}=0$, i.e.,

$$q=-\frac{1}{6n(n+1)}\frac{1}{r^n}. \tag{3.5}$$

The initial conditions (2.8) are satisfied. For u_1 and u_2 we find that

$$\frac{\partial u_1}{\partial z_1}\Big|_{(0,0,0)}=0,\ \frac{\partial u_1}{\partial z_2}\Big|_{(0,0,0)}=0,\ \frac{\partial u_2}{\partial z_1}\Big|_{(0,0,0)}=\frac{1}{9(n+1)}\frac{1}{r^{n+1}},\ \frac{\partial u_2}{\partial z_2}\Big|_{(0,0,0)}=-\frac{2}{3}.$$

Therefore the eigenvalues of matrix (3.2) at $(0,0,0)$ are 0 and $-\frac{2}{3}$. Since both the eigenvalues are non-positive, applying Theorem 3.1 we get the existence of unique analytic solutions z_1 and z_2 at zero. Thus we get the analytic solution $Q(\xi)=r+q\xi+z(\xi)$ satisfying (3.4) with $Q(0)=r$, $Q'(0)=q$, where q is determined by (3.5). □

Remark. It follows that the solution $G(s)$ to (2.7)–(2.8) has an expansion near zero of the form $G(s)=\sum\limits_{k=0}^{\infty}a_k s^{3k}$.

4. Determination of the local solution

In this section we give a method for the determination of the coefficients of the power series solution. We seek a solution of the form

$$G(s)=a_0+a_1s^3+a_2s^6+\dots,\quad s>0, \tag{4.1}$$

with coefficients $a_k\in\mathbb{R}$, $k=0,1,\dots$. From Section 3 we get that $a_0=r$ and $a_1=q=-1/\left[6n(n+1)r^n\right]$. Near zero we have $G(s)>0$ and $G'(s)<0$ and

$$G'(s)=\sum_{k=0}^{\infty}3(k+1)a_{k+1}s^{3k},\quad G''(s)=\sum_{k=0}^{\infty}3(k+1)(3k+2)a_{k+1}s^{3k+1}. \tag{4.2}$$

Hence, for G^n, G^{n-1} and G'^2 we get

$$G^n(s)=\sum_{k=0}^{\infty}A_ks^{3k},\quad G^{n-1}(s)=\sum_{k=0}^{\infty}B_ks^{3k},\quad G'^2(s)=s^4\sum_{k=0}^{\infty}C_ks^{3k} \tag{4.3}$$

where the coefficients A_k, B_k and C_k can be expressed in terms of a_k. One can apply the J. C. P. Miller formula (see [7]) for the determination of A_k, B_k and C_k:

$$A_0 = a_0^n, \quad B_0 = a_0^{n-1}, \quad C_0 = 9a_1^2 \quad \text{and for } k = 1, 2, \dots$$

$$A_k = \frac{1}{ka_0} \sum_{j=0}^{k-1} [(k-j)n - j] A_j a_{k-j}, \tag{4.4}$$

$$B_k = \frac{1}{ka_0} \sum_{j=0}^{k-1} [(k-j)(n-1) - j] B_j a_{k-j}, \tag{4.5}$$

$$C_k = \frac{1}{ka_0} \sum_{j=0}^{k-1} [2k - 3j] C_j c_{k-j}, \quad c_k = 3(k+1)a_{k+1}. \tag{4.6}$$

Substituting (4.4)–(4.6) into the equation (2.5) we compare the like powers of s and we get

$$a_1 = -\frac{1}{6n(n+1)} r^{-n}, \quad a_2 = -\frac{5n-3}{360n^2(n+1)^2} r^{-2n-1}, \tag{4.7}$$
$$a_3 = -\frac{10n^2 + 17n - 15}{25920n^3(n+1)^3} r^{-2-3n}, \dots.$$

Then we get

$$G(s) = r\Big(1 - \frac{1}{6n(n+1)} r^{-(n+1)} s^3 - \frac{5n-3}{360n^2(n+1)^2} r^{-2(n+1)} s^6 - \dots\Big). \tag{4.8}$$

Since $G(s) = 1/g'(s)$ and $g(0) = 0$ we have $g(s) = \int_0^s G^{-1}(t)\,\mathrm{d}t$ and

$$g(s) = \frac{1}{r}\Big[s + \frac{1}{24n(n+1)} r^{-(n+1)} s^4 + \frac{5n+7}{2520n^2(n+1)^2} r^{-2(n+1)} s^7 + \frac{10n^2 + 137n + 33}{259200n^3(n+1)^3} r^{-3(n+1)} s^{10}\Big] + O(s^{13}),$$

hence

$$r\eta = f_r'(\eta) + \frac{1}{24n(n+1)} r^{-(n+1)} f_r'(\eta)^4 + \frac{5n+7}{2520n^2(n+1)^2} r^{-2(n+1)} f_r'(\eta)^7 + \frac{10n^2 + 137n + 33}{259200n^3(n+1)^3} r^{-3(n+1)} f_r'(\eta)^{10} + O(s^{13}).$$

This implies a similar down stream velocity

$$f_r'(\eta) = r\eta - \frac{1}{4!n(n+1)} r^{3-n} \eta^4 - \frac{10n-21}{7!n^2(n+1)^2} r^{5-2n} \eta^7 - \frac{560n^2 - 2054n + 1869}{10!n^3(n+1)^3} \gamma_0^{7-3n} \eta^{10} + O(\eta^{10}),$$

and consequently, the stream function is obtained as

$$f_r(\eta) = \frac{r}{2}\eta^2 - \frac{1}{5!n(n+1)}r^{3-n}\eta^5 - \frac{10n-21}{8!n^2(n+1)^2}r^{5-2n}\eta^8$$
$$- \frac{560n^2-2054n+1869}{11!n^3(n+1)^3}r^{7-3n}\eta^{11} + O(\eta^{11}).$$

These numerical results indicate that a power series solution similar to (1.7) for $n = 1$ can be obtained for $n > 0$ in the form

$$f(\eta) = \eta^2 \sum_{j=0}^{k-1} \frac{b_k(n)\gamma^{k(2-n)+1}}{(3k+2)!n^k(n+1)^k}\eta^{3k},$$

where $b_k = b_k(n)$ are polynomials of order $(k-1)$.

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