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MAXIMAL REGULARITY, THE LOCAL INVERSE FUNCTION THEOREM, AND LOCAL WELL-POSEDNESS FOR THE CURVE SHORTENING FLOW

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Abstract. Local well-posedness of the curve shortening flow, that is, local existence, uniqueness and smooth dependence of solutions on initial data, is proved by applying the Local Inverse Function Theorem and L^2-maximal regularity results for linear parabolic equations. The application of the Local Inverse Function Theorem leads to a particularly short proof which gives in addition the space-time regularity of the solutions. The method may be applied to general nonlinear evolution equations, but is presented in the special situation only.

Keywords: curve shortening flow, maximal regularity, local inverse function theorem

MSC 2010: 35K93, 35B65, 35B30, 35K90, 46T20

1. Introduction

Optimal or maximal regularity results for linear evolution equations on Banach spaces are now widely used in order to prove local existence, uniqueness, regularity of solutions of abstract nonlinear parabolic evolution equations of the form

$$\dot{u} + F(u) = f \quad \text{on } (0,T), \quad u(0) = u_0; \tag{1.1}$$

among the first articles, we cite for example Da Prato & Grisvard [14], Amann [5], [6], Angenent [7], [8], Clément & Li [13], but we mention also the monograph by Lunardi [27] and more recent works by Escher, Prüss & Simonett [20], Prüss [31]

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and Amann [3], [4]. In these articles, but also in [10], [22], [23], [32], [33] (the list is not exhaustive), the authors applied the contraction mapping principle in order to prove local existence and uniqueness of solutions. The contraction mapping principle is of course a standard tool in nonlinear analysis, although finding an appropriate contraction is sometimes quite technical. We want to show that, besides another advantage, an application of the Inverse Function Theorem avoids this problem; in some sense, the problem is hidden in the proof of the Inverse Function Theorem.

In [7], [8], Angenent remarked that optimal regularity of underlying linear evolution equations not only gives local existence and uniqueness of solutions of the nonlinear equation (1.1), but also time regularity of solutions and continuous/smooth dependence of solutions on data (see also the monograph by Lunardi [27]). The solutions are as regular and the dependence is as smooth as F is, that is, equation (1.1) behaves very much like an ordinary differential equation for which analoguous results are classical. In order to achieve his goal, Angenent applied—besides the contraction mapping principle—the Implicit Function Theorem in a most elegant way (see also [20] where the so-called parameter trick was further developed).

In this article, we show that the Inverse Function Theorem may be an efficient alternative to the contraction mapping principle and the Implicit Function Theorem: by applying the Inverse Function Theorem, the proof of local existence of solutions is simpler *and* gives continuous/smooth dependence on data at the same time, at least in the context of L^p-maximal regularity. An application of the Inverse Function Theorem is moreover natural since optimal/maximal regularity just translates the fact that a certain linear operator is an isomorphism between appropriate Banach spaces. In order to avoid much abstract notation, we illustrate the approach only in the particular case of the curve shortening flow equation

$$u_t - \kappa(u) = 0, \tag{1.2}$$

but the interested reader will certainly understand how to apply the Inverse Function Theorem in different, more abstract situations. For the curve shortening flow we show in addition that the approach via the Inverse Function Theorem yields—with very little additional effort—smooth solutions. Smoothness is here obtained without the use of the Implicit Function Theorem but follows from smooth dependence on data.

We have chosen the example of the curve shortening flow because it is one of the simplest examples of geometric flows. Analytic properties of the flow and its applications in physics or image analysis have been widely studied in literature. Moreover, local existence and uniqueness of various types of solutions for appropriate initial data is well known; at least this question is not an issue at all even in specialized monographs (see, for example, [12], [19], [35]). Among the possible approaches to obtain short time existence of solutions, we mention the geometric measure theory (Brakke [11]), the theory of quasilinear parabolic equations (here one frequently refers to Ladyzhenskaya et al. [25] in combination with a reparametrization argument by DeTurck [18], but this approach comprises also the above mentioned use of optimal/maximal regularity and the contraction mapping principle/the Implicit Function Theorem, see Huisken & Polden [24]), the level set approach in combination with the theory of viscosity solutions (Giga [21] and references therein), or variational approaches (Almgren et al. [1], Deckelnick [16], Luckhaus & Sturzenhecker [26]).

2. The curve shortening flow equation—functional setting

The *curve shortening flow equation* for parametrizations of closed curves is the partial differential equation

$$\begin{cases} u_t - \kappa(u) = 0 & \text{in } [0,T]\times\mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for } (t,x)\in[0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x\in\mathbb{R}. \end{cases} \tag{2.1}$$

Here, each $u(t,\cdot)\colon\ \mathbb{R}\to\mathbb{R}^d$ is the parametrization of a closed curve Γ_t in $\mathbb{R}^d$. By definition, a *parametrization of a closed curve* $\Gamma\subseteq\mathbb{R}^d$ is a 2π-periodic, continuously differentiable function $u\colon\ \mathbb{R}\to\mathbb{R}^d$ such that $\Gamma=\{u(x)\colon\ x\in\mathbb{R}\}$ and $\inf_{x\in\mathbb{R}}|u_x(x)|>0$ (the latter assumption on the derivative guarantees in particular that u is locally injective and therefore an immersion).

In the following, we consider the Sobolev spaces

$$H^k_{\mathrm{per}} := \{u\in H^k_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)\colon\ u(x)=u(x+2\pi)\},$$

and we denote analogously by C^k_{per} the space of all 2π-periodic, k times continuously differentiable functions. We assume that the initial value u_0 in (2.1) is a parametrization in the Sobolev space H^2_{per}. This space has the simple advantage of being a Hilbert space. Moreover, this Sobolev space is continuously embedded into C^1_{per} so that the set of all parametrizations in H^2_{per} is open. Finally, given a parametrization $u\in H^2_{\mathrm{per}}$, we can define the associated *curvature vector field* $\kappa(u)$ by

$$\kappa(u) := \frac{1}{|u_x|}\Big(\frac{u_x}{|u_x|}\Big)_x = \frac{u_{xx}}{|u_x|^2} - \frac{u_x}{|u_x|}\Big\langle\frac{u_{xx}}{|u_x|^2}\,\frac{u_x}{|u_x|}\Big\rangle = P^\perp\frac{u_{xx}}{|u_x|^2}.$$

Here,

$$P^\perp v := v - \frac{u_x}{|u_x|}\Big\langle v\,\frac{u_x}{|u_x|}\Big\rangle$$

is the orthogonal projection along the tangent space $\langle u_x \rangle$ onto the normal space along u.

3. Reduction of the curve shortening flow equation

Following an idea of DeTurck in [18], one may introduce reparametrizations of a solution u of the curve shortening flow equation in such a way that the reparametrizations satisfy a strictly parabolic equation (see, for example, Zhu [35]). This strictly parabolic equation can be obtained by "projecting" the time derivative of the function u into the space which is normal along u_0 (the equation thus obtained therefore depends on u_0). Here, we somehow proceed in the opposite way (see also Deckelnick [16], K. Deckelnick, G. Dziuk & C. M. Elliott [17] and K. Mikula & D. Ševčovič [29]): instead of projecting the curve shortening flow equation into normal direction we rather leave out the normal projection $P^{\perp}$ which appears in the definition of the curvature vector $\kappa(u)$. That is, instead of the curve shortening flow equation (2.1) we consider the problem

$$\begin{cases} v_t - \dfrac{v_{xx}}{|v_x|^2} = 0 & \text{in } [0,T] \times \mathbb{R}, \\ v(t,x) = v(t,x+2\pi) & \text{for } (t,x) \in [0,T] \times \mathbb{R}, \\ v(0,x) = u_0(x) & \text{for } x \in \mathbb{R}; \end{cases} \tag{3.1}$$

(the same initial value u_0 as in (2.1)!). In this problem, $(v(t,\cdot))$ is again a family of parametrizations of closed curves in $\mathbb{R}^d$. We show in this section that a smooth solution v of the problem (3.1) and a smooth solution u of the curve shortening flow equation (2.1) parametrize the same family of curves. As a consequence, if one is only interested in the evolution of the associated curves, it suffices to solve the reduced problem (3.1). For simplicity, we work only with C^∞ solutions here and we do not try to find the weakest possible regularity on v which ensures existence and uniqueness of sufficiently regular reparametrizations θ (see the following lemma).

Lemma 1. *Let $v \in C^\infty([0,T]; C^\infty_{\text{per}})$ be a solution of the problem* (3.1) *(in particular, all functions $v(t,\cdot)$ are parametrizations of closed curves, that is, $\inf_{(t,x)} |v_x(t,x)| > 0$). Then there exists a unique function*

$$\theta \in C^\infty([0,T] \times \mathbb{R}), \quad \theta = \theta(t,x),$$

(the same existence time as for v!) satisfying

$$\begin{cases} \theta_t + \dfrac{1}{|v_x(t,\theta)|} \left\langle \dfrac{v_{xx}(t,\theta)}{|v_x(t,\theta)|^2} \, \dfrac{v_x(t,\theta)}{|v_x(t,\theta)|} \right\rangle = 0 & \textit{in } [0,T] \times \mathbb{R}, \\ \theta(0,x) = x & \textit{for } x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Proof. For every fixed $x \in \mathbb{R}$ the equation (3.2) is an ordinary differential equation for the function $\theta(\cdot, x)$. For this ordinary differential equation, the classical results for local/global existence and uniqueness of solutions and smooth dependence on initial data apply and yield the claim. □

Let v and θ be as in Lemma 1 and define

$$u(t,x) := v(t, \theta(t,x)) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}.$$

Note carefully that u is 2π-periodic in the second variable. Moreover, by the chain rule and since v and θ are solutions of (3.1) and (3.2), respectively, we have

$$\begin{aligned} u_t(t,x) &= v_t(t,\theta(t,x)) + v_x(t,\theta(t,x))\,\theta_t(t,x) \\ &= \frac{v_{xx}(t,\theta(t,x))}{|v_x(t,\theta(t,x))|^2} - \frac{v_x(t,\theta(t,x))}{|v_x(t,\theta(t,x))|}\Big\langle \frac{v_{xx}(t,\theta(t,x))}{|v_x(t,\theta(t,x))|^2} \frac{v_x(t,\theta(t,x))}{|v_x(t,\theta(t,x))|} \Big\rangle \\ &= \kappa(v(t,\theta(t,x))) \\ &= \kappa(u(t,x)). \end{aligned}$$

In the last equality we have used the equality $\kappa(v(t,\theta(t,x))) = \kappa(u(t,x))$, that is, the curvature vector at the point $v(t,\theta(t,x)) = u(t,x)$ does not depend on the particular parametrization. Since we have also that

$$u(0,x) = v(0,\theta(0,x)) = v(0,x) = u_0(x),$$

the function u defined above is indeed a solution of the curve shortening flow equation.

4. Existence and regularity for the reduced problem by the local inverse function theorem

In this section we solve the reduced problem (3.1). More precisely, we prove existence and uniqueness of solutions which belong to the *maximal regularity space*

$$MR := H^1(0,T; H^1_{\text{per}}) \cap L^2(0,T; H^3_{\text{per}}).$$

This space is equipped with the natural norm, so that it becomes a Banach (or: Hilbert) space. One has two continuous embeddings

$$MR \subseteq C([0,T]; H^2_{\text{per}}) \subseteq C([0,T]; C^1_{\text{per}}).$$

The latter embedding follows from the Sobolev embedding $H^2_{\rm per} \subseteq C^1_{\rm per}$, while the former embedding follows from the interpolation theory [28, Corollary 1.14] and the fact that $H^2_{\rm per}$ is the trace space (or: interpolation space) between $H^1_{\rm per}$ and $H^3_{\rm per}$ associated with the maximal regularity space MR (see, for example, [28, Example 1.26, p. 37] in the case of Sobolev spaces on domains in $\mathbb{R}^d$; the case of the periodic Sobolev spaces is similar). The subset

$$U := \Big\{u \in MR\colon \inf_{(t,x)} |u_x(t,x)| > 0\Big\}$$

is, by the above embeddings, an open subset of the maximal regularity space. For every parametrization $u_0 \in H^2_{\rm per}$ there exists an element $u \in U$ such that $u(0) = u_0$. In fact, since $H^2_{\rm per}$ is the trace space associated with MR, there exists an element $\tilde{u} \in MR$ such that $\tilde{u}(0) = u_0$. Then, by a simple continuity and compactness argument, there exists $T' \in (0,T]$ such that $\inf\limits_{(t,x)\in[0,T']\times\mathbb{R}} |\tilde{u}_x(t,x)| > 0$. Now, the function $u(t,x) = \tilde{u}(\frac{T'}{T}t, x)$ belongs to U and satisfies $u(0) = u_0$.

Theorem 2 [Local existence and smooth dependence of local solutions on data]**.** *For every parametrization $u_0 \in H^2_{\rm per}$ and every $f \in L^2(0,T;H^1_{\rm per})$ there exists a local existence time $T' \in (0,T]$ and a constant $r > 0$ such that for every $v_0 \in H^2_{\rm per}$ and every $g \in L^2(0,T';H^1_{\rm per})$ with $\|v_0 - u_0\|_{H^2_{\rm per}} < r$ and $\|g - f\|_{L^2(0,T';H^1_{\rm per})} < r$ the problem*

$$\begin{cases} v_t - \dfrac{v_{xx}}{|v_x|^2} = g & \text{in } [0,T']\times\mathbb{R}, \\ v(t,x) = v(t,x+2\pi) & \text{for } (t,x)\in[0,T']\times\mathbb{R}, \\ v(0,x) = v_0(x) & \text{for } x\in\mathbb{R}, \end{cases} \tag{4.1}$$

admits a unique solution

$$v \in H^1(0,T';H^1_{\rm per}) \cap L^2(0,T';H^3_{\rm per}).$$

Moreover, the mapping which maps every pair $(g, v_0) \in B(f,r)\times B(u_0,r)$ (the open balls in $L^2(0,T';H^1_{\rm per})$ and $H^2_{\rm per}$, respectively) to the unique solution $v \in H^1(0,T';H^1_{\rm per})\cap L^2(0,T';H^3_{\rm per})$ is analytic (in the sense of [34, Definition 8.8, p. 362]*).*

In particular, for every parametrization $u_0 \in H^2_{\rm per}$ the problem (3.1) *admits a unique local solution $v \in H^1(0,T';H^1_{\rm per}) \cap L^2(0,T';H^3_{\rm per})$.*

Proof. *Existence.* Consider the function

$$\begin{aligned} G\colon\ U &\to L^2(0,T;H^1_{\rm per}) \times H^2_{\rm per} \\ v &\mapsto \Big(v_t - \frac{v_{xx}}{|v_x|^2}, v(0)\Big). \end{aligned}$$

It is analytic in the sense of [34]. We show that G is a local diffeomorphism. Denote by G' the Fréchet derivative of G. For every $\overline{v} \in U$ and every $w \in MR$,

$$G'(\overline{v})w = \Big(w_t - \frac{w_{xx}}{|\overline{v}_x|^2} + 2\, \frac{\overline{v}_{xx}}{|\overline{v}_x|^4} \langle \overline{v}_x, w_x \rangle, w(0) \Big).$$

Saying that $G'(\overline{v})$ is a linear isomorphism from MR onto $L^2(0,T;H^1_{\text{per}}) \times H^2_{\text{per}}$ is then clearly equivalent to saying that for every right-hand side $h \in L^2(0,T;H^1_{\text{per}})$ and every initial value $w_0 \in H^2_{\text{per}}$ the problem

$$\text{(4.2)} \qquad \begin{cases} w_t - \dfrac{w_{xx}}{|\overline{v}_x|^2} + 2\, \dfrac{\overline{v}_{xx}}{|\overline{v}_x|^4} \langle \overline{v}_x, w_x \rangle = h & \text{in } [0,T] \times \mathbb{R}, \\ w(t,x) = w(t, x+2\pi) & \text{for } (t,x) \in [0,T] \times \mathbb{R}, \\ w(0,x) = w_0 & \text{for } x \in \mathbb{R}, \end{cases}$$

admits a unique solution $w \in MR$. We take this fact for granted, or we refer to Section 5 below, where we briefly sketch why this linear, nonautonomous problem has L^2-maximal regularity in H^1_{per}.

Now the problem (4.1) can be solved in the following way. Given a parametrization $u_0 \in H^2_{\text{per}}$ and a function $f \in L^2(0,T;H^1_{\text{per}})$, there exists an element $\overline{v} \in U$ such that $\overline{v}(0) = u_0$. Since $G'(\overline{v})$ is linear and continuously invertible (by the above granted assumption), and by the Local Inverse Function Theorem [34, Theorem 4.F, p. 172], there exists a neighbourhood $\overline{V} \subseteq U$ of $\overline{v}$ and a neighbourhood $\overline{W} \subseteq L^2(0,T;H^1_{\text{per}}) \times H^2_{\text{per}}$ of $G(\overline{v}) =: (\overline{f}, u_0)$ such that G is a diffeomorphism between $\overline{V}$ and $\overline{W}$. More precisely, G and its local inverse G^{-1} are analytic [34, Corollary 4.37, p. 172].

Now, choose first $r > 0$ so small that $B(\overline{f}, 2r) \times B(u_0, r) \subseteq \overline{W}$, and choose then $T' \in (0,T]$ so small such that

$$\|f - \overline{f}\|_{L^2(0,T';H^1_{\text{per}})} < r;$$

here it is crucial that we work with L^2-maximal regularity, since this ensures that such a time T' exists. Let $v_0 \in H^2_{\text{per}}$ and $g \in L^2(0,T';H^1_{\text{per}})$ be such that $\|v_0 - u_0\|_{H^2_{\text{per}}} < r$ and $\|g - f\|_{L^2(0,T';H^1_{\text{per}})} < r$. We extend g by $\overline{f}$ on $(T',T]$ and we denote this extension by Eg. Then $Eg \in L^2(0,T;H^1_{\text{per}})$ and

$$\begin{aligned} \|Eg - \overline{f}\|_{L^2(0,T;H^1_{\text{per}})} &= \|g - \overline{f}\|_{L^2(0,T';H^1_{\text{per}})} \\ &\leqslant \|g - f\|_{L^2(0,T';H^1_{\text{per}})} + \|f - \overline{f}\|_{L^2(0,T';H^1_{\text{per}})} \\ &< 2r. \end{aligned}$$

In particular, $(Eg, v_0) \in B(\overline{f}, 2r) \times B(u_0, r) \subseteq \overline{W}$. Since $G\colon \overline{V} \to \overline{W}$ is invertible, there exists $v \in \overline{V} \subseteq MR$ such that $G(v) = (Eg, v_0)$. By definition of G and Eg, this implies that the restriction of v to $[0,T'] \times \mathbb{R}$ is a local solution of (4.1).

The mapping which maps every

$$(g, v_0) \in B(f, r) \times B(u_0, r) \subseteq L^2(0, T'; H^1_{\mathrm{per}}) \times H^2_{\mathrm{per}}$$

to the local solution $v \in H^1(0, T'; H^1_{\mathrm{per}}) \cap L^2(0, T'; H^3_{\mathrm{per}})$ is the composition of the affine extension operator E (the sum of a linear operator and a constant), the inverse G^{-1} and a linear restriction operator, and it is thus analytic.

Since the parametrization $u_0 \in H^2_{\mathrm{per}}$ and the right-hand side $f \in L^2(0, T; H^1_{\mathrm{per}})$ are arbitrary (so that we may take $f = 0$), the above arguments yield in particular the existence of a local solution v of (3.1).

Uniqueness. Let v_1, $v_2 \in H^1(0, T'; H^1_{\mathrm{per}}) \cap L^2(0, T'; H^3_{\mathrm{per}})$ be two solutions of (4.1). Then the difference $z := v_1 - v_2$ is a solution of the problem

$$\begin{cases} z_t - \dfrac{z_{xx}}{|v_{1x}|^2} + \dfrac{z_x \cdot (v_1 + v_2)_x}{|v_{1x}|^2\,|v_{2x}|^2} v_{2xx} = 0 & \text{in } [0, T'] \times \mathbb{R}, \\ z(t, x) = z(t, x + 2\pi) & \text{for } (t, x) \in [0, T'] \times \mathbb{R}, \\ z(0, x) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

Multiplying the first line by $z_t\,|v_{1x}|^2$ and integrating over 2π, we obtain for almost every $t \in (0, T')$

$$\begin{aligned} \int_0^{2\pi} |z_t|^2\,|v_{1x}|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \int_0^{2\pi} |z_x|^2 &= -\int_0^{2\pi} \frac{z_x \cdot (v_1 + v_2)_x}{|v_{2x}|^2}\, v_{2xx} \cdot z_t \\ &\leqslant \frac{1}{2} \int_0^{2\pi} |z_t|^2\,|v_{1x}|^2 + \frac{1}{2}\left\| \frac{|(v_1 + v_2)_x|}{|v_{1x}|\,|v_{2x}|^2}\,|v_{2xx}| \right\|_\infty^2 \int_0^{2\pi} |z_x|^2, \end{aligned}$$

and as a consequence

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_0^{2\pi} |z_x|^2 \leqslant c(t) \int_0^{2\pi} |z_x|^2$$

with a function $c \in L^1(0, T')$. By integrating this differential inequality and using the initial condition $z(0, x) = 0$, one finds first $z_x = 0$. Then, by inserting this into the partial differential equation for z, one obtains $z_t = 0$ and hence $z = 0$, that is, $v_1 = v_2$. □

The smooth dependence on the data implies higher space regularity if the data are more regular, too.

Corollary 3. *Let* $u_0 \in H^2_{\text{per}}$*,* $f \in L^2(0,T;H^1_{\text{per}})$ *and* $T' \in (0,T]$ *be as in Theorem 2, and let* $v \in H^1(0,T';H^1_{\text{per}}) \cap L^2(0,T';H^3_{\text{per}})$ *be the local solution of* (4.1) *with* $g = f$ *and* $v_0 = u_0$*. If* $u_0 \in H^{2+k}_{\text{per}}$ *and* $f \in L^2(0,T;H^{1+k}_{\text{per}})$ *for an integer* $k \geqslant 0$*, then* $v \in H^1(0,T';H^{1+k}_{\text{per}}) \cap L^2(0,T';H^{3+k}_{\text{per}}) \subseteq C([0,T'];H^{2+k}_{\text{per}})$*.*

P r o o f. Note that $u_0 \in H^{2+k}_{\text{per}}$ if and only if the mapping $h \mapsto u_0(\cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into H^2_{per}. Similarly, $f \in L^2(0,T;H^{1+k}_{\text{per}})$ if and only if the mapping $h \mapsto f(\cdot, \cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into $L^2(0,T;H^1_{\text{per}})$. Since $v(\cdot, \cdot + h)$ is the (unique) solution of (4.1) with the initial value u_0 replaced by $u_0(\cdot + h)$ and the right-hand side f replaced by $f(\cdot, \cdot + h)$, the smooth dependence of solutions on initial data (Theorem 2) implies that the mapping $h \mapsto v(\cdot, \cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into $H^1(0,T';H^1_{\text{per}}) \cap L^2(0,T';H^3_{\text{per}}) \cap C([0,T'];H^2_{\text{per}})$. This gives the desired regularity. □

With little additional effort, we now show that the unique local solution v found in Theorem 2 is smooth for $t > 0$. Note that the following corollary may also be proved by applying the beautiful argument of Angenent (see [7], [20]); there, the Implicit Function Theorem first gives the time regularity while the space regularity can for example be obtained by using the equation (3.1). Here, the smooth dependence on data implies first the space regularity, and the time regularity is obtained in the second place.

Corollary 4. *Let* $v \in H^1(0,T';H^1_{\text{per}}) \cap L^2(0,T';H^3_{\text{per}})$ *be a solution of the homogeneous problem* (3.1) *(the existence of a local solution is guaranteed by Theorem* 2*). Then*

$$v \in C^\infty((0,T'];C^\infty_{\text{per}}).$$

P r o o f. We start by showing space regularity. Note that if

$$v \in H^1_{\text{loc}}((0,T'];H^{1+k}_{\text{per}}) \cap L^2_{\text{loc}}((0,T'];H^{3+k}_{\text{per}}) \quad \text{for some } k \geqslant 0,$$

then, for almost every $t \in (0,T')$, $v(t) \in H^{3+k}_{\text{per}}$. By Corollary 3, this implies that $v \in H^1([t,T'];H^{2+k}_{\text{per}}) \cap L^2([t,T'];H^{4+k}_{\text{per}})$ for almost every $t \in (0,T')$, and therefore

$$v \in H^1_{\text{loc}}((0,T'];H^{2+k}_{\text{per}}) \cap L^2_{\text{loc}}((0,T'];H^{4+k}_{\text{per}}) \subseteq C((0,T'];H^{3+k}_{\text{per}}).$$

Hence, an induction on $k \geqslant 0$ shows that $v \in C((0,T'];C^\infty_{\text{per}})$. This regularity and the equality

$$v_t = \frac{v_{xx}}{|v_x|^2}$$

imply first that $v \in C^1((0,T'];C^\infty_{\text{per}})$, and then, by iterating this argument, that $v \in C^\infty((0,T'];C^\infty_{\text{per}})$. □

5. Maximal regularity for the linear, nonautonomous problem

In this section, we briefly sketch the idea why the linear problem (4.2) has L^2-maximal regularity in $H^1_{\rm per}$. Let us first note that the problem (4.2) is a special case of the linear, nonautonomous problem

$$(5.1)\qquad \begin{cases} w_t - m(t,x)w_{xx} + b(t,x)w_x = h & \text{in } [0,T]\times\mathbb{R},\\ w(t,x) = w(t,x+2\pi) & \text{for } (t,x)\in[0,T]\times\mathbb{R},\\ w(0,x) = w_0(x) & \text{for } x\in\mathbb{R}, \end{cases}$$

where $m \in C([0,T]; H^1_{\rm per}(\mathbb{R}))$ and $b \in L^2(0,T; H^1_{\rm per}(\mathbb{R};\mathbb{R}^{d\times d}))$ are two given functions such that, for some fixed $\varepsilon > 0$, $m(t,x) \in [\varepsilon, 1/\varepsilon]$ for every $(t,x)\in[0,T]\times\mathbb{R}$.

In order to prove L^2-maximal regularity of the problem (5.1) in $H^1_{\rm per}$, it is convenient to proceed in several steps and to consider the following three cases:

(1) the case when $m(t,\cdot) = m(\cdot) \in H^1_{\rm per}$ does not depend on time and b vanishes identically (autonomous case);
(2) the case when m is arbitrary (but satisfies the conditions above) and b vanishes identically;
(3) the general case (m and b satisfy the conditions above).

The first case is of course the simplest one. In order to prove L^2-maximal regularity in $H^1_{\rm per}$, it suffices to know that the operator $-mw_{xx}$ with domain $H^3_{\rm per}$ generates an analytic C_0-semigroup on the Hilbert space $H^1_{\rm per}$ and to refer to [15]. Alternatively, one can show by variational methods that the operator $-mw_{xx}$ with domain $H^1_{\rm per}$ has L^2-maximal regularity on $H^{-1}_{\rm per}$ and then to use a similarity argument.

Once the first case is settled, the cases 2 and 3 follow by perturbation arguments (using the Neumann series, for example). For the second case, due to the continuity of m, one may apply either [30, Theorem 2.5], [2, Theorem 7.1], or [9, Theorem 2.7] in combination with the first case. The third, general case follows similarly [2, Theorem 7.1].

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