Groupoids and the Associative Law IIIA. (Primitive Extensions of SH-Groupoids and their Semigroup Distances)

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Received 15. October 2008

Szász-Hájek groupoids (shortly SH-groupoids) are groupoids containing just one non-associative (ordered) triple of elements. These groupoids were studied by G. Szász in [10] and [11], P. Hájek in [2] and [3] and later in [6], [7], [8] and [9].The present paper is a continuation of [12]. SH-groupoids of type (a, a, a) having infinite countable underlying set and an arbitrary given finite semigroup distance are constructed.

1. Preliminaries

A groupoid $E(\cdot)$ is called *SH-groupoid* if the set $\{(a, b, c) \in E^{(3)} \mid a \cdot bc \neq ab \cdot c\}$ of non-associative triple contains just one element.

Let $H(\cdot)$ be a subgroupoid of an SH-groupoid $E(\cdot)$ having the non-associative triple (a, b, c). Then either $\{a, b, c\} \in H$ and $H(\cdot)$ is an SH-groupoid having the non-associative triple (a, b, c), or $H(\cdot)$ is a semigroup in the opposite case.

Let κ be a congruence on SH-groupoid $E(\cdot)$. If (a, b, c) is the corresponding non-associative triple then either $(a \cdot bc, ab \cdot c) \in \kappa$ and then $E/\kappa(\cdot)$ is a semigroup, or $(a \cdot bc, ab \cdot c) \notin \kappa$ and then $E/\kappa(\cdot)$ is an SH-groupoid.

An SH-groupoid $G(\cdot)$ is called *SH-groupoid of type* (a, a, a) if there exists an element $a \in G$ such that (a, a, a) is the corresponding non-associative triple of the groupoid $G(\cdot)$.

1.1 Szász's theorem. *Let $E(\cdot)$ be an SH-groupoid and let (a, b, c) be the only non-associative triple of $E(\cdot)$. If $x, y \in E$ are such that $x \cdot y \in \{a, b, c\}$ then $x \cdot y \in \{x, y\}$.*

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2000 Mathematics Subject Classification. 20N05

Key words and phrases. Groupoid, non-associative triple, semigroup distance.

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Let $G(\diamond)$ and $G(*)$ be groupoids having the same underlying set G. Then $\mathrm{dist}(G(\diamond), G(*))$ denotes $\mathrm{card}\{(x, y) \in G^2 \mid x \diamond y \neq x * y\}$.

Let $G(\cdot)$ be a groupoid. Let $\mathrm{sdist}(G(\cdot))$ be the minimum of cardinal numbers $\mathrm{dist}(G(\cdot), G(*))$, where $G(*)$ runs through the set of all semigroups having the underlying set G. The number $\mathrm{sdist}(G(\cdot))$ is called *semigroup distance* of the groupoid $G(\cdot)$.

1.2 Definition. Let $G(\cdot)$ be an SH-groupoid. A semigroup $G(*)$ having the same underlying set G is called *nearest semigroup of* $G(\cdot)$ if $\mathrm{dist}(G(*), G(\cdot)) = \mathrm{sdist}(G(\cdot))$.

1.3 Definition. A groupoid $G(\cdot)$ is called *primitive extension of its subgroupoid* $H(\cdot)$ if there exists an element $p \in G, p \notin H$ such that $G(\cdot)$ is generated by the set $H \cup \{p\}$.

1.4 Lemma. *Let $G(\cdot)$ be a primitive extension of a subgroupoid $H(\cdot)$ generated by the set $H \cup \{p\}$. Then $p \notin H$ and the groupoid $P(\cdot)$ generated by the one-element set $\{p\}$ is a semigroup.*

1.5 Lemma. *Let $G(\cdot)$ be an SH-groupoid of type (a, a, a). Then $G(\cdot)$ contains at least four different elements a, $b = aa$, $c = a \cdot aa$ and $d = aa \cdot a$. Furthermore, $G(\cdot)$ satisfies just one of the following two conditions:*

(i) $a(a.aa) = a(aa.a) = aa.aa = (aa.a)a = (a.aa)a,$

(ii) $a(a.aa) = aa.aa = (aa.a)a \neq a(aa.a) = (a.aa)a.$

1.6 Definition. Let $E(\cdot)$ is an SH-groupoid having the non-associative triple (a, a, a). $E(\cdot)$ will be called *SH-groupoid of the first kind* if it satisfies the condition (i). In the opposite case $E(\cdot)$ will be called *SH-groupoid of the second kind.*

1.7 Definition. A groupoid $G(\cdot)$ will be called *stratified groupoid* if there exists a mapping σ of the G to the set of natural numbers satisfying the condition

$$\sigma(x \cdot y) = \sigma(x) + \sigma(y)$$

for every $x, y \in G$.

In this case the mapping σ will be called *stratifying function* on $G(\cdot)$. Finally, for each natural number n consider the set $S_n = \{x \in G; \sigma(x) = n\}$. Each non-empty set S_n will be called *n-th stratification* of the set G.

1.8 Definition. Let $G(\cdot)$ be a stratified groupoid and let σ be the corresponding stratifying function. A congruence κ on $G(\cdot)$ will be called *stratified congruence* if for all $x, y \in G$ $(x, y) \in \kappa$ implies $\sigma(x) = \sigma(y)$.

2. Minimal SH-groupoids and their nearest semigroups

From now on, we will deal only with SH-groupoids of type (a, a, a). An SH-groupoid $G(\cdot)$ of type (a, a, a) is called *minimal* if it is generated by the one-element set $\{a\}$.

2.1 Construction. For each natural number $k \geq 5$ consider pair-wise different elements $a^5, \dots, a^k, a^{k+1}, \dots$. Consider another six different elements a, b, c, d, e, f and put $G = \{a, b, c, d, e, f, a^5, \dots, a^k, a^{k+1}, \dots\}$.

Further, denote by λ a mapping of the set G to the set of natural numbers such that $\lambda(a) = 1$, $\lambda(b) = 2$, $\lambda(c) = 3 = \lambda(d)$, $\lambda(e) = 4 = \lambda(f)$ and $\lambda(a^k) = k$ for every natural number $k \geq 5$.

Finally, define on G a binary operation $\cdot$ in the way that the condition $\lambda(xy) = = \lambda(x) + \lambda(y)$ for every $x, y \in G$ is satisfied.

Especially, put at first:

(i) $b = a \cdot a$;

(ii) $c = a \cdot b$ and $d = b \cdot a$,

(iii) $e = a \cdot c = b \cdot b = d \cdot a$ and $f = c \cdot a = a \cdot d$;

(iv) $a^5 = a \cdot e = a \cdot f = b \cdot c = b \cdot d = c \cdot b = d \cdot b = e \cdot a = f \cdot a$;

(v) $a^6 = b \cdot e = b \cdot f = c \cdot c = c \cdot d = d \cdot c = d \cdot d = e \cdot b = f \cdot b$;

(vi) $a^7 = c \cdot e = c \cdot f = d \cdot e = d \cdot f = e \cdot c = e \cdot d = f \cdot c = f \cdot d$;

(vii) $a^8 = e \cdot e = e \cdot f = f \cdot e = f \cdot f$.

Further, for each natural number $k \geq 5$ put:

(viii) $b \cdot a^k = a^k \cdot b = a^{k+2}$,

(ix) $c \cdot a^k = d \cdot a^k = a^k \cdot c = a^k \cdot d = a^{k+3}$,

(x) $e \cdot a^k = f \cdot a^k = a^k \cdot e = a^k \cdot f = a^{k+4}$.

Finally, for all natural numbers $k, m \geq 5$ put:

(xi) $a^k \cdot a^m = a^{k+m}$.

Then G becomes a groupoid which will be further denoted as $G(\cdot)$.

2.2 Lemma. *$G(\cdot)$ is a minimal free SH-groupoid of the second kind.*

Proof. It is obvious that $G(\cdot)$ is generated by one-element set $\{a\}$ and it holds $c = a \cdot b = a \cdot aa \neq aa \cdot a = b \cdot a = d$.

If $x, y, z \in G$ are such that $\lambda(x) + \lambda(y) + \lambda(z) = k \geq 5$ then $x \cdot yz = a^k = xy \cdot z$. There is only a finite number of ordered triples (x, y, z) having $\lambda(x) + \lambda(y) + \lambda(z) = 4$ and it is easy to check that each of such triples is associative. It is proved in [6] that $G(\cdot)$ is a minimal free SH-groupoid of type (a, a, a).

Further, $e = aa \cdot aa \neq a \cdot (aa \cdot a) = f$. Therefore $G(\cdot)$ is an SH-groupoid of the second kind and the condition $\lambda(xy) = \lambda(x) + \lambda(y)$ is satisfied for every $x, y \in G$. It means that $G(\cdot)$ is a stratified groupoid. Moreover, $\lambda(x)$ denotes just the length of the corresponding element $x \in G$.

2.3 Lemma. *Let $G(\cdot)$ be a minimal free SH-groupoid of the type (a, a, a). Then the set $\kappa = \{(x, x); x \in G\} \cup \{(e, f), (f, e)\}$ is a stratified congruence on $G(\cdot)$ and the corresponding groupoid $G/\kappa(\cdot)$ is a minimal free SH-groupoid of type (a, a, a) and it is the only infinite SH-groupoid of the first kind.*

Proof. It is easy to see, that κ is a congruence on $G(.)$ and, so, G/κ is an SH-groupoid having the only non-associative triple (a, a, a). The rest is obvious.

2.4 Remark. Let $G(\cdot)$ be the SH-groupoid from 2.1 and κ the congruence from 2.3. Put $a^4 = \{(e, f), (f, e)\}$ and denote by H the set $\{a, b, c, d, a^4, a^5, \ldots, a^k, a^{k+1}, \ldots\}$. Then H is the underlying set of $G/\kappa(\cdot)$ and the SH-groupoid $G/\kappa(\cdot)$ will be shortly denoted as $H(\cdot)$ in the sequel.

2.5 Lemma. sdist(H(·)) = 1.

Proof. Define on H new binary operations $\triangle$ and ∇ in the following way:

(i) $a\triangle b = d \neq a \cdot b$ and $x\triangle y = x \cdot y$ whenever $(x, y) \neq (a, b)$;

(ii) $b\nabla a = c \neq b \cdot a$ and $x\nabla y = x \cdot y$ whenever $(x, y) \neq (b, a)$.

It is obvious that $\lambda(x\triangle y) = \lambda(x) + \lambda(y) = \lambda(x\nabla y)$ for every $x, y \in H$. Further, $(a\triangle a)\triangle a = (a \cdot a)\triangle a = b\triangle a = b \cdot a = d = a\triangle b = a\triangle(a\triangle a)$ and $(a\nabla a)\nabla a = (a \cdot a)\nabla a = b\nabla a = c = a \cdot b = a\nabla b = a\nabla(a \cdot a) = a\nabla(a\nabla a)$. Therefore, $H(\triangle)$, $H(\nabla)$ are semigroups. Obviously, dist(H(·), H(△)) = 1 = dist(H(·), H(∇)) .

Furthermore, if $H(\diamond)$ is an arbitrary semigroup having dist(H(·), H(⋄)) = 1 then $a \diamond a = a \cdot a$. Indeed, in the opposite case we have $y = a \diamond a$ and $y \cdot b = y \diamond b = a \diamond a \diamond b = a \diamond c = ac = a^4$. It follows from this that $\lambda(y) = 2$. Therefore, we obtain $y = b$, a contradiction.

2.6 Lemma. *The SH-groupoid $H(\cdot)$ has only two nearest semigroups and they are $H(\triangle)$ and $H(\nabla)$.*

Proof. It follows immediately from 2.3 and 2.5.

2.7 Lemma. sdist(G(·)) = 2.

Proof. Define on G a binary operation $\triangleleft$ such that $a \triangleleft b = c$, $c \triangleleft a = e$ and $x \triangleleft y = x \cdot y$ in the remaining cases.

Then we have:

(i) $a \triangleleft (a \triangleleft a) = a \triangleleft b = a \cdot b = c = b \triangleleft a = (a \triangleleft a) \triangleleft a$,

(ii) $a \triangleleft (b \triangleleft a) = a \triangleleft c = a \triangleleft c = a \cdot c = e = c \triangleleft a = (a \cdot b) \triangleleft a = (a \triangleleft b) \triangleleft a$,

(iii) $a \triangleleft (a \triangleleft b) = a \triangleleft (a \cdot b) = a \triangleleft c = a \cdot c = e = b \cdot b = b \triangleleft b = (a \cdot a) \triangleleft b = (a \triangleleft a) \triangleleft b$,

(iv) $b \triangleleft (a \triangleleft a) = b \triangleleft (a \cdot a) = b \triangleleft b = e = c \triangleleft a = (b \triangleleft a) \triangleleft a$,

(v) $x \triangleleft (y \triangleleft z) = a^k = (x \triangleleft y) \triangleleft z$ whenever $\lambda(x) + \lambda(y) + \lambda(z) = k \geq 5$.

It means that $G(\triangleleft)$ is a semigroup and therefore sdist(G(·)) ≤ dist(G(·), G(◃)) = 2.

Suppose that sdist((G(·)) = 1. Then there is a semigroup $G(\Diamond)$ such that dist(G(·), G(◊)) = 1. Then just one of the conditions $a\Diamond a \neq a \cdot a$, $a\Diamond b \neq a \cdot b$, $b\Diamond a \neq b \cdot a$ has to be satisfied. Further, sdist(G(·)) is finite and therefore there exists natural number m such that $x\Diamond y = x \cdot y$ whenever $\lambda(x) + \lambda(y) \geq m$. For any natural number $k \geq m$ and each $x \in G$ it holds $(a^k)\Diamond x = a^{k+\lambda(x)} = x\Diamond(a^k)$.

Suppose first that $y = a\Diamond a \neq a \cdot a$. Then $a^{k+2} = (a^{k+1})\Diamond a = (a^k\Diamond a)\Diamond a = a^k\Diamond(a\Diamond a) = (a^k)\Diamond y = a^k \cdot y = a^{k+\lambda(y)}$. It follows from this that $\lambda(y) = 2$. But this takes place only if $y = a \cdot a$, a contradiction.

Suppose further that $y = a \triangleleft b \neq a \cdot b$. Then we have $a^{k+3} = a^{k+1} \triangleleft b = (a^k \triangleleft a) \triangleleft b = a^k \triangleleft (a \triangleleft b) = (a^k) \triangleleft y = a^k \cdot y = a^{k+\lambda(y)}$. It follows from this that $\lambda(y) = 3$. It means that $y = b \cdot a = d$ and $x \triangleleft y = x \cdot y$ holds for every $(x, y) \neq (a, b)$. Then we obtain

$f = a \cdot d = a \triangleleft d = a \triangleleft (b \cdot a) = a \triangleleft (b \triangleleft a) = (a \triangleleft b) \triangleleft a = d \triangleleft a = d \cdot a = e$, a contradiction again.

The remaining case $y = b \triangleleft a \neq ba$ is similar to the last one. It follows from this that $1 \neq$ sdist((G(·)) and the rest is clear.

2.8 Theorem. *There exist just only two infinite minimal SH-groupoids having non-associative triple (a, a, a). This is either the SH-groupoid $H(\cdot)$ of the first kind having* sdist(H(·)) $=$ 1, *or it is the SH-groupoid $G(\cdot)$ of the second kind having* sdist(G(·)) $=$ 2.

Proof. It follows immediately from 2.5 and 2.7.

3. Primitive extensions of minimal SH-groupoids

Suppose that $F(\cdot)$ is an arbitrary SH-groupoid of type (a, a, a) generated by a two element set $\{a, p\}$. Then $F(\cdot)$ contains proper subgoupoids $H(\cdot)$ and $P(\cdot)$. Denote by W the set$F \setminus (H \cup P)$

If $p \cdot x \neq a \neq x \cdot p$ then $p \cdot xy = px \cdot y$ for every $x, y \in F$.

The corresponding element is determined by the ordered triple (p, x, y) and it can be understood as the word pxy. Similarly, $xp \cdot y = x \cdot py$ and also $xy \cdot p = x \cdot yp$ for every $x, y \in F$. Therefore, the corresponding element can be described as the word xpy or the word xyp, respectively..

Suppose that $n \geq 3$ and $x_1, x_2, \ldots, x_n, x_{n+1} \in F$. If there exists at least one natural number $1 \leq k \leq n+1$ such that $x_k = p$ then we have also

$$x_1 \cdot x_2 x_3 \ldots x_n x_{n+1} = x_1 x_2 \cdot x_3 \ldots x_n x_{n+1} = \cdots = x_1 x_2 \ldots x_n \cdot x_{n+1} .$$

It means that the corresponding element is described by the word $x_1 x_2 x_n \dot{x}_{n+1}$ containing at least once the element p.

3.1 Construction. Consider the SH-groupoid $H(\cdot)$ of the first kind constructed in 2.1 and 2.3. Let $p \notin H$ and let $P(\cdot)$ be the free semigroup generated by one-element set $\{p\}$. Suppose that infinte countable sets $H = \{a, b, c, d, a^4, \ldots, a^k, a^{k+1}, \ldots\}$ and $P = \{p, p^2, \ldots, p^n, p^{n+1}, \ldots\}$ are disjoint.

Further, for every two natural numbers i, j consider all natural numbers k such that $1 \leq k \leq \frac{(i+j)!}{i! \cdot j!}$. Let $w_{i,j,k}$ be pair-wise different elements and for given natural numbers i, j denote by $W_{i,j}$ the set containing all these elements. Of course, each of these elements can be understood as the word containing just i-times the element a and j-times the element p. Consider the lexicographic order on the set $W_{i,j}$ and suppose that the number k denotes just the place of the word $w_{i,j,k}$ in this order.

For each natural number $n \geq 2$ put $W_n = W_{1,n-1} \cup W_{2,n-2} \cup \cdots \cup W_{n-1,1}$ and let $W = W_1 \cup W_2 \cup \cdots \cup W_n \cup W_{n+1} \cup \ldots$.

Finally, suppose that the sets H, P and W are pair-wise disjoint and put $E = H \cup P \cup W$.

Define a mapping λ of the set E to the set of all natural numbers in the following way: $\lambda(a) = 1 = \lambda(p), \lambda(b) = 2, \lambda(c) = 3 = \lambda(d)$ and $\lambda(a^k) = k$ for each natural number $k \geq 4$. Further, put $\lambda(p^m) = m$ for each natural number m. Finally, for each $\lambda(w_{i,j,k})$ put $\lambda(w_{i,j,k}) = i + j$ for every two natural numbers i, j.

Define on E a binary operation $\cdot$ in the way that the following two conditions are satisfied:

(i) both groupoids $H(\cdot)$ and $P(\cdot)$ have to be proper subgroupoids of the constructed groupoid $E(\cdot)$;

(ii) each product $x \cdot y$ of element $x, y \in E$ such that $x \in W$ or $y \in W$ has to be equal to the word $w_{i,j,k}$ which is constructed form the words x and y in this order. The corresponding number k is determined by the placement of the word xy with the respect to the lexicographic order of the set W_{i+j}.

Then $E(\cdot)$ becomes an SH-groupoid with non-associative triple (a, a, a) and it is generated by the set $\{a, p\}$.

3.2 Lemma. *The groupoid $E(\cdot)$ is a stratified SH-groupoid of type (a, a, a) generated by two-element set $\{a, p\}$ and it is the only free primitive extension of the SH-groupoid $H(\cdot)$.*

Proof. It follows from 3.1 that the condition $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ is satisfied for every $x, y \in E$. There is only finite number of ordered triples (x, y, z) having $\lambda(x) + \lambda(y) + \lambda(z) = 3$ and it holds:

$a \cdot (a \cdot a) = a \cdot b = c \neq d = b \cdot a = (a \cdot a) \cdot a$;
$a \cdot (a \cdot p) = w_{2,1,1} = (a \cdot a) \cdot p$;
$a \cdot (p \cdot a) = w_{2,1,2} = (a \cdot p) \cdot a$;
$p \cdot (a \cdot a) = w_{2,1,3} = (p \cdot a) \cdot a$;
$a \cdot (p \cdot p) = w_{1,2,1} = (a \cdot p) \cdot p$;
$p \cdot (a \cdot p) = w_{1,2,2} = (p \cdot a) \cdot p$;
$p \cdot (p \cdot a) = w_{1,2,3} = (p \cdot p) \cdot a$;
$p \cdot (p \cdot p) = p^3 = (p \cdot p) \cdot p$.

Further, it is easy to check that each triple (x, y, z) having $\lambda(x) + \lambda(y) + \lambda(z) \geq 4$ is associative. It follows immediately from the definition because both products $x \cdot (y \cdot z)$ and $(x \cdot y) \cdot z$ represent the same element $w_{i,j,k}$ which is described by the word xyz whenever at least one of x, y, z is in W.

3.3 Lemma. *sidst*$(E(\cdot)) = 1 =$ sdist(H($\cdot$)).

Proof. Define on E a binary operation $*$ such that $c = b * a \neq b \cdot a$, $e = c * a \neq c \cdot a$ and $x * y = x \cdot y$ whenever $x, y \in E$ are such that $(b, a) \neq (x, y) \neq (c, a)$.

It is easy to see that $E(*)$ is a semigroup and the condition $\lambda(x * y) = \lambda(x) + \lambda(y)$ is satisfied for every $x, y \in E$. The rest follows immediately from the construction and from sdist(H($\cdot$)) = 1.

It is obvious that there are congruences on $E(\cdot)$ satisfying the condition $a \cdot a = p^n$ for an arbitrary given natural number n. If κ is one of such congruences then $(x \cdot b, x \cdot p^n) \in \kappa$ and $(b \cdot y, p^n \cdot y) \in \kappa$ for every $x, y \in E$.

Suppose, furthermore, that κ is a stratified congruence on $E(\cdot)$ and let σ be the corresponding stratifying function on E. Then $\sigma(a \cdot b) = \sigma(a \cdot p^n)$ and $\sigma(b \cdot a) = = \sigma(p^n \cdot a)$. It follows from this that $2 \times \sigma(a) = n \times \sigma(p)$.

3.4 Construction. Let $n \geq 2$ be an arbitrary given natural number and consider the SH-groupoid $E(\cdot)$ from 3.1. Define, at first, a stratifying function σ on the set E.

Put, for the simplicity, $\sigma(p) = 2$ and $\sigma(a) = n$. Then $\sigma(b) = 2n$, $\sigma(c) = 3n = \sigma(d)$ and $\sigma(a^m) = m \times n$ for each natural number $m \geq 4$, $\sigma(p^k) = 2k$ for each natural number k, $\sigma(w_{i,j,m}) = i \times n + 2j$ for each $w_{i,j,m} \in W$.

Define, further, a binary relation κ on the set E in the way that $(x, x) \in \kappa$ for each $x \in E$ and $(x, y) \in \kappa$ only if $\sigma(x) = \sigma(y)$. It follows from this that $(x, y) \in \kappa$ if and only if $(y, x) \in \kappa$ for every $x, y \in E$, $(b, p^n) \in \kappa$ and $(a, x) \notin \kappa$ for each $x \in E$.

Especially, put at first $(x, y) \in \kappa$ if and only if $x = y$ for each $x \in E$ such that $\sigma(x) < 2n$. Further, put $(x, y) \in \kappa$ for each natural number $k \geq 3n + 1$ and every $x, y \in E$ such that $\sigma(x) = k = \sigma(y)$.

It follows from this that there is only a finite number of the remaining ordered pairs $(x, y) \in E^{(2)}$ having $\sigma(x) = \sigma(y) \leq 3n$.

Define, finally, the relation κ step by step for the remaining $x, y \in E$. There is $E = H \cup P \cup W$ and the set W contains disjoint subsets $W_{i,j}$. Suppose that $x \neq y$, $\sigma(x) = \sigma(y)$ and let $2n \leq \sigma(x) \leq 3n$.

(i) If $\sigma(x) = 2n$ and $x \in H \cup P$ then either $x = b$, or $x = p^n$ and we have $(b, b) \in \kappa$, $(b, p^n) \in \kappa$, $(p^n, b) \in \kappa$ and $(p^n, p^n) \in \kappa$. If $n = 2m + 1$ then $x \notin W$. If $n = 2m$ then $x \in \{ap^m, p^m a\}$ and we put $(x, y) \in \kappa$ in that case if and only if $x = y$.

(ii) If $\sigma(x) = 2n + k$ and $1 \leq k < n$, then $x \notin H$. If $x \in P$ then $bp^k = p^{2n+k} = = p \cdot p^{2n+k-1} = pbp^{k-1} = p^2 \cdot p^{2n+k-2} = p^2 bp^{k-2} = \cdots = p^k b$. Therefore, we put $(x, y) \in \kappa$ if $x, y \in \{bp^k, pbp^{k-1}, \ldots, p^{k-1}bp, p^k b, p^{2n+k}\}$ and for the remaining $x, y \in E$ having $\sigma(x) = 2n + k = \sigma(y)$ we put $(x, y) \in \kappa$ if and only if $x = y$.

(iii) If $\sigma(x) = 3n$ and $n = 2m + 1$ then $x \notin P$. We have either $x \in \{c, d\}$ or $x \in W$. For $x = c$ we have $a \cdot b = a \cdot p^n$. If $c \neq y$ then $(c, y) \in \kappa$ if and only if $y = ap^n$. Similarly, if $x = d$ and $d \neq y$ then $(d, y) \in \kappa$ if and only if $y = p^n a$. Of course, $(c, d) \notin \kappa$. Finally, if $x, y \in W$ then $x, y \in W_{1,n}$ and we put $(x, y) \in \kappa$ if and only if $x = y$.

Let $\sigma(x) = 3n$ and $n = 2m$. If $x \in H$ then we put again $(c, d) \notin \kappa$, $(c, y) \in \kappa$ if $y \in \{c, ap^n\}$ and $(d, y) \in \kappa$ if $y \in \{d, p^n a\}$. Further, if $x \in P$ then we put $(x, y) \in \kappa$ if $x, y \in \{bp^m, pbp^{m-1}, \ldots, p^{m-1}bp, p^m b, p^{2n+m}\}$ similarly as in (ii). Finally, in the remaining cases if $x, y \in W$ and $\sigma(x) = 2n + 2m = \sigma(y)$ we put $(x, y) \in \kappa$ if and only if $x = y$.

3.5 Lemma. *The binary relation κ is a stratified congruence of the groupoid $E(\cdot)$.*

Proof. Suppose, at first, that $x, y, z \in E$ are pair-wise different elements such that $(x, y) \in \kappa$ and $(y, z) \in \kappa$.

It is obvious that $(x, z) \in \kappa$ whenever $\sigma(x) \geq 3n + 1$. It follows from $x \neq y$ that $2n \leq \sigma(x) \leq 3n$. Therefore, $\sigma(x) = \sigma(z)$ and it follows from the contruction of κ that $(x, z) \in \kappa$.

Further, let $r, s, t, u \in E$ and be such that $(r, s), (t, u) \in \kappa$. If $\sigma(r) + \sigma(t) \geq 3n + 1$ then obviously $(rt, su) \in \kappa$. Suppose that $r \neq s$ and $\sigma(r) + \sigma(t) \leq 3n$. Then $\sigma(r) \geq 2n$. Thus $\sigma(t) < n$ and $t = u$. It follows from the construction of κ that $(rt, su) \in \kappa$ in this case. In the remaining cases we have $r = s$, $t = u$ and the rest follows from the construction 3.4.

3.6 Lemma. *$E/\kappa(\cdot)$ is a groupoid containing (up to isomorphism) the SH-groupoid $H(\cdot)$ as a proper subgroupoid and it is generated by two element set $\{a, p\}$.*

Proof. It follows immediatelly from 3.4 and 3.5.

3.7 Construction. Let $n \geq 3$ be an arbitrary given natural number. Consider the groupoids $E(\cdot)$ from 3.1 and $E/\kappa(\cdot)$ from 3.6. For every two natural numbers i, j such that $1 \leq i + j \leq 2n$ consider pair-wise different elements $u_{i,j}$ of the set W such that $u_{i,j} = p^i a p^j$. Denote by U_{3n} the set of all such elements $u_{i,j}$. For every three natural numbers i, j, k such that $i + j + k \leq n$ and $1 \leq j$ consider pair-wise different elements $v_{i,j,k}$ of the set W such that $v_{i,j,k} = p^i a p^j a p^k$. Denote by V_{3n} the set of all such elements $v_{i,j,k}$.

Further, if $n = 2m$ then put $E_{2m} = \{a\} \cup P \cup U_{3n} \cup V_{3n}$ and denote by $E_{2m}(\cdot)$ the corresponding isomorphic image of the groupoid $E/\kappa(\cdot)$.

Finally, for each natural number $n = 2m + 1$ consider pair-wice different elements q_k and denote as Q the set $\{q_{3n+1}, q_{3n+2}, \dots\}$. Put $E_{2m+1} = \{a\} \cup P_{3n} \cup U_{3n} \cup V_{3n} \cup Q$. Denote $E_{2m+1}(\cdot)$ the corresponding isomorphic image of the constructed groupoid $E/\kappa(\cdot)$.

3.8 Lemma. $\mathrm{sdist}(\mathrm{E_n}, (\cdot)) \leq n$.

Proof. Define on E_n a new binary operation $*$ in the following way:

(i) $ap^n = p^n * a = p * (p^{n-1}a) = p^2 * (p^{n-2}a) = \cdots = p^{n-1} * (pa) \neq p^n a$;

(ii) $x * y = x \cdot y$ whenever $(x, y) \neq (a, a), p^n, a), (p^{n-1}, pa), \dots, (p, p^{n-1}a)$.

It is obvious that $\sigma(x) * y) = \sigma(x) \cdot y)$ for every $x, y \in E_n$. Therefore, $x * (y * z) = (x * y) * z$ whenever $\sigma(x) + \sigma(y) + \sigma(z) \geq 3n + 1$.

We have $a * (a * a) = a * (p^n) = ap^n = p^n * a = (a * a) * a$. Further, it is possible to check that if $(x, y, z) \in \{(p^{n-1}, p, a), (p^{n-2}, p, pa), \dots, (p, p, p^{n-2}a)$ then also $x * (y * z) = (x * y) * z$.

There is a finite number of remaining triples (x, y, z) having $\sigma(x) + \sigma(y) + \sigma(z) \leq 3n + 1$. In these cases, either $xy \neq p^k$ and $z \neq p^{n-k}a$, or $x \neq p^k$ and $yz \neq p^{n-k}a$. Therefore, $x * (y * z) = (x * y) * z$ again.

It was proved above that $E_n(*)$ is a semigroup having $\mathrm{dist}(\mathrm{E_n}(*), \mathrm{E_n}(\cdot)) = n$ and the rest is clear.

3.9 Lemma. $\mathrm{sdist}(\mathrm{E_{2m+1}}, (\cdot)) = 2m + 1$.

Proof. Let $E_{2m+1}(\star)$ be an arbitrary semigroup such that $\mathrm{dist}(\mathrm{E_{2m+1}}(\star), \mathrm{E_{2m+1}}(\cdot)) = \mathrm{sdist}(\mathrm{E_{2m+1}}(\cdot)$. Of course, at least one of the conditions $a \star a \neq p^{2m+1}$, $a \star p^{2m+1} \neq ap^{2m+1}$, $p^n \star a \neq p^{2m+1}a$ has to be satisfied.

It follows from 3.8 that there exists a natural number k such that $x \star y = x \cdot y$ whenever $\sigma(x) + \sigma(y) \geq k$. If $t = a \star a$ then $t \cdot p^k = t \star p^k = a \star (a \star p^k) = a \cdot ap^k = = aa \cdot p^k$. It follows from this that $\sigma(t) = 4m + 2$. There is only one element t having $\sigma(t) = 4m + 2$ and this is just the element $p^{2m+1} = a \cdot a$.

Therefore, either $a \star p^{2m+1} \neq a \cdot p^{2m+1}$, or $p^{2m+1} \star a \neq p2m + 1 \cdot a$. In both cases $\mathrm{dist}(\mathrm{E_{2m+1}}(\star), \mathrm{E_{2m+1}}(\cdot)) \geq 2\mathrm{m} + 1$. Thus $\mathrm{dist}(\mathrm{E_{2m+1}}(\star), \mathrm{E2m} + 1(\cdot)) \geq 2m + 1$ and the rest follows immediately from 3.8.

3.10 Proposition. $\mathrm{sdist}(\mathrm{E_{2m}}(\cdot)) = 2m.$

Proof. Let $E_{2m}(\star)$ be a semigroup having $\mathrm{dist}(\mathrm{E_{2m}}(\star), \mathrm{E_{2m}}(\cdot)) = \mathrm{sdist}(\mathrm{E_{2m}}(\cdot)$. It is obvious that at least one of the conditions $a \star a \neq p^{2m}$, $a \star p^{2m} \neq ap^{2m}$, $p^n \star a \neq \neq p^{2m}a$ has to be satisfied.

Suppose, at first, that $t = a \star a \neq a \cdot a = p^{2m}$. It follows from 3.8 that there is a natural number k such that $x \star y = x \cdot y$ whenever $\sigma(x) + \sigma(y) \geq k$. Especially, $t \cdot p^k = t \star p^k = (a \star a) \star p^k = a \star (a \star p^k) = a \star (ap^k) = a \cdot (ap^k) = (aa) \cdot p^k$. Therefore, $\sigma(t) = 4m$ and hence $t \in \{ap^m, pap^{m-1}, \ldots, p^{m-1}ap, p^m a\}$. It means that $t = u_{i,m-i}$ for suitable natural number $i \leq m$ in this case.

Further, $(a \star a) \star a = a \star (a \star a)$, and so $a \star u_{i,m-i} = u_{i,m-i} \star a$. But $a \cdot u_{i,m-i} = = v_{0,i,m-i} \neq v_{i,m-i,0} = u_{i,m-i} \cdot a$. It follows from this that either $a \star u_{i,m-i} \neq a \cdot u_{i,m-i}$, or $u_{i,m-i} \star a \neq u_{i,m-i} \cdot a$.

Let, for example, $a \star u_{i,m-i} \neg a \cdot u_{i,m-i}$ and suppose that

(i) $p^j \star p^k = p^j \cdot p^k$ for every $j + k \leq 3m$,

(ii) $a * p^j = a \cdot p^j$ for each $j \leq 2m$,

(iii) $a \star (ap^k) = a \cdot (ap^k)$ for each $k \leq m$.

It is easy to check that then $p^{2m+i+j} = a \cdot a \cdot p^{i+j} = a \star a \star p^{i+j} = u_{i,m-i} \star p^{i+j} = = u_{i,m-i} \cdot p^{i+j} = u_{i,m+j}$. This is a contradiction with the construction 3.4. It follows from this that at least one of the conditions (i),(ii), (iii) cannot be valid and therefore, there are at least m ordered pairs (x, y) such that $x \star y \neq x \cdot y$.

The similar assertion could be proved also for products $p^j \star u_{i,m-i}$. But then we obtain $\mathrm{dist}(\mathrm{E_{2m}}(\star), \mathrm{E_{2m}}(\cdot)) \geq 2m + 1$, a contradiction. Therefore, $a \star a = p^{2m} = a \cdot a$ and either $a \star p^{2m} \neq a \cdot p^{2m}$, or $p^{2m} \star a \neq p^{2m} \cdot a$.

Suppose, for example, that $s = p^{2m} \star a \neq p^{2m}a$. Then $\sigma(s) = 6m$, and hence $s = ap^{2m}$, or $s = u_{j,4m-j}$ and $j \geq 1$, or $s = v_{i,j,2m-i-j}$ and $j \geq 1$. It is tedious but possible to check that $\mathrm{dist}(\mathrm{E_{2m}}(\star), \mathrm{E_{2m}}(\cdot)) \geq 6m$ in each of these cases.

3.11 Theorem. *For each natural number n there exists at least one SH-groupoid $E_n(\cdot)$ of type (a, a, a) such that:*

(i) *E_n is an infinite countable set;*

(ii) *$E_n(\cdot)$ is generated by a two-element set $\{a, p\}$;*

(iii) $\mathrm{sdist}(\mathrm{E_n}(\cdot)) = n.$

Proof. It follows immediatelly from 3.4, 3.5, 3.6 and 3.7.

3.12 Corollary. *For each natural number n there is a finite SH-groupoid* $F_n(.)$ *such that* $\mathrm{sdist}(\mathrm{F_n}(\cdot)) = n$.

Proof. The construction 3.4 can by modified by the following conditions:
(i) if $(x, y) \in \kappa$ and $\sigma(x) \geq 3n$ then $\sigma(x) = \sigma(y)$;
(ii) $(x, y) \in \kappa$ whenever $\sigma(x) \geq 3n + 1$ and $\sigma(y) \geq 3n + 1$.
It follows from the condition (ii) that the set E/κ has to be finite in this case.

4. Comments and open problems

4.1 The groupoids $E_n(\cdot)$ of type (a, a, a) are SH-groupoids of the first kind. Is it possible to construct also SH-groupoids of type (a, a, a) satisfying the condition $a(aa \cdot a) \neq aa \cdot aa$ and having $sidst(E_n(\cdot)) = n$?

4.2 The condition $\sigma(x \cdot y) = \sigma(x) + \sigma(y)$ is important for proofs. Are there also primitive extensions of minimal SH-groupoids $E_n(\cdot)$ such that this condition is not satisfied?

References

[1] Drápal, A. and Kepka, T.: *Sets of associative triples*, Europ. J. Combinatorics **6** (1985), 227–261.

[2] Hájek, P.: *Die Szászschen Gruppoiden*, Matem.-fyz. časopis SAV **15/1** (1965), 15–42.

[3] Hájek, P.: *Berichtigung zu meine Arbeit "Die Szászschen Gruppoide"*, Matem.-fyz. časopis SAV **15/4** (1965), 331.

[4] Kepka, T., Trch, M.: *Groupoids and the associative law I. (Associative triples)*, Acta Univ. Carolinae Math. Phys. **33/1** (1991), 69–86.

[5] Kepka, T., Trch, M.: *Groupoids and the as sociative law II. (Groupoids with small semigroup distance)*, Acta Univ. Carolinae Math. Phys. **34/1** (1993), 67–83.

[6] Kepka, T., Trch, M.: *Groupoids and the as sociative law III. (Szász-Hájek groupoids)*, Acta Univ. Carolinae Math. Phys. **36/1** (1995), 69–86.

[7] Kepka, T., Trch, M.: *Groupoids and the as sociative law IV. (Szász-Hájek groupoids of type (a, b, a))*, Acta Univ. Carolinae Math. Phys. **35/1** (1994), 31–42.

[8] Kepka, T., Trch, M.: *Groupoids and the as sociative law V. (Szász-Hájek groupoids of type (a, a, b))*, Acta Univ. Carolinae Math. Phys. **36/1** (1995), 31–44.

[9] Kepka, T., Trch, M.: *Groupoids and the as sociative law VI. (Szász-Hájek groupoids of type (a, b, c))*, Acta Univ. Carolinae Math. Phys. **38/1** (1997), 13–21.

[10] Szász, G.: *Unabhängigkeit der Assoziativitätsbedingungen*, Acta Sci. Math. Szeged **15** (1953–4), 20–28.

[11] Szász, G.: *Über Unbhängigkeit der Assoziativitätsbedingungen kommutativer multiplikativer Strukturen*, Acta Sci. Math. Szeged **15** (1953–4), 130–142.

[12] Trch, M.: *Groupoids and the Associative Law VII. (Semigroup Distances of SH-groupoids)*, Acta Univ. Carolinae Math. Phys. **47/1** (2006), 57–63.

[13] Trch, M.: *Groupoids and the Associative Law VIIA. (SH-Groupoids of type (A,B,A) and their semigroup distances)*, Acta Univ. Carolinae Math. Phys. **48/1** (2007), 43–54.