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Transitive Closures of Binary Relations I.

VÁCLAV FLAŠKA, JAROSLAV JEŽEK, TOMÁŠ KEPKA AND JUHA KORTELAINEN

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Transitive closures of binary relations and relations α with the property that any two α-sequences connecting two given elements are of the same length are investigated.
Vyšetřují se tranzitivní uzávěry binárních relací a relací α s vlastností, že každé dvě α-posloupnosti spojující dané dva prvky mají stejnou délku.

The present short note collect a few elementary observations concerning the transitive closures of binary relations. All the formulated results are fairly basic and of folklore character to much extent. Henceforth, we shall not attribute them to any particular source!

1. Preliminaries

Let S be a set, $\mathbf{id}_S = \{(a,a) \mid a \in S\}$ and $\mathbf{ir}_S = (S \times S) - \mathbf{id}_S$.

Let α be a binary relation defined in S (i.e., $\alpha \subseteq S \times S$). We put $\mathbf{i}(\alpha) = \alpha \cap \mathbf{ir}_S$ and $\mathbf{r}(\alpha) = \alpha \cup \mathbf{id}_S$. The relation α is called

– irreflexive if $\alpha \subseteq \mathbf{ir}_S$ (equivalently, $\alpha \cap \mathbf{id}_S = \emptyset$ or $\mathbf{i}(\alpha) = \alpha$);
– reflexive if $\mathbf{id}_S \subseteq \alpha$ (or $\mathbf{r}(\alpha) = \alpha$);

Department of Algebra, MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic

Department of Information Processing Science, University of Oulu, P.O. Box 3000 FIN 90014, Oulu, Finland

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E-mail address: flaska@karlin.mff.cuni.cz
E-mail address: jezek@karlin.mff.cuni.cz
E-mail address: kepka@karlin.mff.cuni.cz
E-mail address: juha.kortelainen@oulu.fi

- strictly (or sharply) antisymmetric if $(a, b) \in \alpha$ implies $(b, a) \notin \alpha$;
- antisymmetric if $a = b$ whenever $(a, b) \in \alpha$ and $(b, a) \in \alpha$;
- symmetric if $(a, b) \in \alpha$ implies $(b, a) \in \alpha$;
- transitive if $(a, c) \in \alpha$ whenever $(a, b) \in \alpha$ and $(b, c) \in \alpha$;
- a quasiordering if α is reflexive and transitive;
- a strict (or sharp) ordering if α is irreflexive and transitive;
- a near-ordering if α is antisymmetric and transitive;
- a (reflexive) ordering if α is reflexive, antisymmetric and transitive;
- a tolerance if α is reflexive and symmetric;
- an equivalence if α is reflexive, symmetric and transitive.

1.1 Lemma. *Let α be a binary relation on a set S.*
(i) *α is both irreflexive and reflexive iff $\alpha = \emptyset = S$.*
(ii) *α is strictly antisymmetric iff α is irreflexive and antisymmetric.*
(iii) *α is both antisymmetric and symmetric iff $\alpha \subseteq \mathbf{id}_S$.*
(iv) *If α is transitive then α is irreflexive iff α is strictly antisymmetric.*
(v) *If α is irreflexive, symmetric and transitive then $\alpha = \emptyset$.*
(vi) *If α is symmetric, transitive and if for every $a \in S$ there is at least one $b \in S$ with either $(a, b) \in \alpha$ or $(b, a) \in \alpha$ then α is an equivalence.*

Proof. It is obvious. □

1.2 Lemma. *Let α be a binary relation on a set S.*
(i) $\mathbf{i}(\alpha)$ *is the irreflexive core of α (i.e., the largest irreflexive relation contained in α).*
(ii) $\mathbf{r}(\alpha)$ *is the reflexive closure of α (i.e., the smallest reflexive relation containing α).*
(iii) $\mathbf{i}(\alpha) = \mathbf{ir}(\alpha) \subseteq \mathbf{r}(\alpha) = \mathbf{ri}(\alpha)$.
(iv) *If α is antisymmetric then $\mathbf{i}(\alpha)$ is strictly antisymmetric and $\mathbf{r}(\alpha)$ is antisymmetric.*
(v) *If α is symmetric then $\mathbf{i}(\alpha)$ and $\mathbf{r}(\alpha)$ are symetric.*
(vi) *If α is transitive then $\mathbf{r}(\alpha)$ is a quasiordering.*
(vii) *If α is a near-ordering then $\mathbf{i}(\alpha)$ is a strict ordering and $\mathbf{r}(\alpha)$ is an ordering.*

Proof. It is obvious. □

2. Isolated elements

Let α be a binary relation on a set S. For every element $a \in S$ put $\mathbf{R}(a, \alpha) = \{b \mid (a, b) \in \alpha\}$ and $\mathbf{L}(a, \alpha) = \{b \mid (b, a) \in \alpha\}$.

2.1 Lemma. *The following conditions are equivalent for a binary relation α on a set S:*

(i) *α is irreflexive (reflexive, resp.);*
(ii) $a \notin \mathbf{R}(a,\alpha)\,(a \in \mathbf{R}(a,\alpha)$, *resp.) for every* $a \in S$;
(iii) $a \notin \mathbf{L}(a,\alpha)\,(a \in \mathbf{L}(a,\alpha)$, *resp.) for every* $\in S$.

Proof. It is obvious. □

2.2 Lemma. *Let α be a binary relation on a set S.*
(i) *α is strictly antisymmetric iff* $\mathbf{R}(a,\alpha) \cap \mathbf{L}(a,\alpha) = \emptyset$ *for every* $a \in S$.
(ii) *α is antisymmetric iff* $\mathbf{R}(a,\alpha) \cap \mathbf{L}(a,\alpha) \subseteq \{a\}$ *for every* $a \in S$.

Proof. It is obvious. □

2.3 Lemma. *The following conditions are equivalent for a binary relation α on a set S:*
(i) *α is symmetric;*
(ii) $\mathbf{R}(a,\alpha) \subseteq \mathbf{L}(a,\alpha)$ *for every* $a \in S$;
(iii) $\mathbf{L}(a,\alpha) \subseteq \mathbf{R}(a,\alpha)$ *for every* $a \in S$;
(iv) $\mathbf{R}(a,\alpha) = \mathbf{L}(a,\alpha)$ *for every* $a \in S$.

Proof. It is obvious. □

2.4 Lemma. *The following conditions are equivalent for a binary relation α on a set S:*
(i) *α is transitive;*
(ii) $\mathbf{R}(b,\alpha) \subseteq \mathbf{R}(a,\alpha)$ *for all* $a \in S$ *and* $b \in \mathbf{R}(a,\alpha)$;
(iii) $\mathbf{L}(b,\alpha) \subseteq \mathbf{L}(a,\alpha)$ *for all* $a \in S$ *and* $b \in \mathbf{L}(a,\alpha)$.

Proof. It is obvious. □

An element $a \in S$ is called
– right (or upwards) strictly α-isolated if $\mathbf{R}(a,\alpha) = \emptyset$;
– right (or upwards) α-isolated if $\mathbf{R}(a,\alpha) \subseteq \{a\}$;
– right (or upwards) α-pseudoisolated if $\mathbf{R}(a,\alpha) \subseteq \mathbf{L}(a,\alpha)$.
Left (or downwards) (strictly) α-(pseudo)isolated elements are defined dually.

2.5 Lemma. *Let α be a binary relation on a set S.*
(i) *If α is irreflexive then every right isolated element is right strictly isolated.*
(ii) *If α is reflexive then no element is right strictly isolated.*
(iii) *If α is strictly antisymmetric then every right pseudoisolated element is right strictly isolated.*
(iv) *If α is antisymmetric then every right pseudoisolated element is right isolated.*
(v) *If α is symmetric then an element is right (strictly) isolated iff it is left (strictly) isolated.*

Proof. It is obvious. □

2.6 Lemma. *The following conditions are equivalent for a binary relation α on a set S:*

(i) *α is symmetric;*

(ii) *Every element is right pseudoisolated;*

(iii) *Every element is left pseudoisolated.*

Proof. It is obvious. □

2.7 Lemma. *Let α be a binary relation on a set S. If α is transitive, $a \in S$ is a right pseudoisolated element and $(a, b) \in \alpha$ then b is right pseudoisolated.*

Proof. It is obvious. □

2.8 Lemma. *Let α be a binary relation on a set S.*

(i) *An element is right strictly $\mathbf{i}(\alpha)$-isolated iff it is right $\mathbf{i}(\alpha)$-isolated and iff it is right α-isolated.*

(ii) *An element is right $\mathbf{i}(\alpha)$-pseudoisolated iff it is right α-pseudoisolated.*

(iii) *An element is right $\mathbf{r}(\alpha($-isolated iff it is right α-isolated.*

(iv) *An element is right $\mathbf{r}(\alpha)$-pseudoisolated iff it is right α-pseudoisolated.*

Proof. It is obvious. □

3. Finite α-sequences

Let α be a binary relation on a set S. A finite sequence $(a_0, a_1, \ldots, a_m)$, $m \geq 1$, of elements of S is an α-sequence if $(a_i, a_{i+1}) \in \alpha$ for every $0 \leq i \leq m-1$. The positive integer m is the length of the sequence. Furthermore, we say that an α-sequence $(b_0, \ldots, b_n)$ is a reduct of the α-sequence $(a_0, \ldots, a_m)$ if there are integers $0 = j_0 < j_1 < \ldots < j_{n-1} = m$ such that $b_i = a_{j_i}$ for every $0 \leq i \leq n$.

An α-sequence $(a_0, \ldots, a_m)$ is called

- weakly pseudoirreducible if $a_i \neq a_{i+1}$ whenever $2 \leq m$ and $0 \leq i < m$;
- pseudoirreducible if $a_i \neq a_j$ whenever $0 \leq i < j \leq m$ and $(i,j) \neq (0,m)$;
- irreducible if $(a_i, a_j) \notin \alpha$ whenever $2 \leq m$ and $0 \leq i < i+2 \leq j \leq m$.

3.1 Lemma. *Let $(a_0, \ldots, a_m)$ be an α-sequence.*

(i) *If the sequence is pseudoirreducible then it is weakly pseudoirreducible.*

(ii) *If the sequence is irreducible then it is pseudoirreducible.*

(iii) *If the sequence is pseudoirreducible and $a_0 \neq a_m$ ($a_0 = a_m$, resp.) then the elements $a_0, \ldots, a_m$ ($a_0, \ldots, a_{m-1}$, resp.) are pairwise different.*

(iv) *If $m = 1$ then the sequence is irreducible.*

Proof. (ii) Let $(a_0, \ldots, a_m)$ be an irreducible α-sequence. Suppose that $a_i = a_j$ where $0 \leq i < j \leq m$. If $j < m$ then $(a_i, a_{j+1}) \in \alpha$, a contradiction. If $0 < i$ then $(a_{i-1}, a_j) \in \alpha$, a contradiction. Thus $(i,j) = (0,m)$.

The other items are easy to see. □

3.2 Proposition. *Every finite α-sequence has at least one irreducible reduct.*

Proof. Let $(a_0, \ldots, a_m)$ be an α-sequence such that $m \geq 2$ and $(a_i, a_j) \in \alpha$, where $0 \leq i < i + 2 \leq j \leq m$. Then $(a_0, \ldots, a_i, a_j, \ldots, a_m)$ is an α-sequence of length at most $m - 1$. Consequently, we can proceed by induction on m. □

3.3 Corollary. *An α-sequence is irreducible if and only if it has no proper reduct.*

3.4 Lemma. *Every α-sequence is weakly pseudoirreducible if and only if α is irreflexive.*

Proof. It is obvious. □

The relation α is called superirreflexive if every α-sequence is pseudoirreducible.

3.5 Lemma. *The following conditions are equivalent for a relation α:*
(i) *α is superirreflexive;*
(ii) *$a_0 \neq a_m$ for every α-sequence $(a_0, \ldots, a_m)$;*
(iii) *The elements $a_0, \ldots, a_m$ are pairwise different for every α-sequence $(a_0, \ldots, a_m)$.*

Proof. (i) implies (ii). If $a_0 = a_m$ then $(a_0, \ldots, a_m, a_1)$ is a non-pseudoirreducible α-sequence.
(ii) implies (iii). Let $(a_0, \ldots, a_m)$ be an α-sequence with $a_i = a_j$, $0 \leq i < j \leq m$. Then $(a_i, a_{i+1}, \ldots, a_j)$ is an α-sequence contradicting (ii).
(iii) implies (i). This is trivial. □

3.6 Lemma. *If α is superirreflexive then α is irreflexive and strictly antisymmetric.*

Proof. It is easy. □

The relation α is called
- (totally) antitransitive if every α-sequence is irreducible;
- regular if $m = n$ whenever $(a_0, \ldots, a_m)$ and $(b_0, \ldots, b_n)$ are α-sequences such that $a_0 = b_0$ and $a_m = b_n$;
- weakly regular if $m = n$ whenever $(a_0, \ldots, a_m)$ and $(b_0, \ldots, b_n)$ are irreducible α-sequences such that $a_0 = b_0$ and $a_m = b_n$.

3.7 Lemma. *Let α be a relation.*
(i) *If α is regular then α is antritransitive.*
(ii) *If α is antitransitive and weakly regular then α is regular.*

Proof. It is obvious. □

3.8 Example. Put $S = \{0,1,2,3,4\}$.
(i) $\alpha_1 = \{(0,1),(1,0)\}$ is irreflexive, weakly regular, but not superirreflexive.
(ii) $\alpha_2 = \{(0,1),(1,2),(0,2)\}$ is superirreflexive, but not antitransitive.

(iii) $\alpha_3 = \{(0,1),(0,3),(1.2),(2,4),(3,4)\}$ is antitransitive, but not weakly regular (and thus not regular).
(iv) $\alpha_4 = \{(0,1\})$ is regular.

4. Transitive closures

Let α be a binary relation on a set S. We define a relation $\gamma = \mathbf{t}(\alpha)$ on S by $(a,b) \in \gamma$ if and only if there exists at least one finite α-sequence $(a_0, a_1, \dots, a_m)$ such that $a_0 = a$ and $a_m = b$.

4.1 Proposition. *Let α be a binary relation on a set S; put $\gamma = \mathbf{t}(\alpha)$.*
(i) *γ is the transitive closure of α, i.e., γ is the smallest transitive relation containing α.*
(ii) *$(a,b) \in \gamma$ if and only if there exists an irreducible α-sequence $(a_0, \dots, a_m)$ such that $a_0 = a$ and $a_m = b$.*
(iii) *If α is reflexive (symmetric, resp.) then γ is reflexive (symmetric, resp.).*
(iv) *If α is a tolerance then γ is an equivalence.*

Proof. It is easy (use 3.2). □

4.2 Proposition. *Let α be a binary relation on a set S.*
(i) *The relation $\delta = \mathbf{rt}(\alpha) = \mathbf{tr}(\alpha)$ is the quasiorder closure of α, i.e., the smallest quasiordering containing α.*
(ii) *If α is symmetric then δ is an equivalence.*
(iii) $\mathbf{it}(\alpha) \subseteq \mathbf{ti}(\alpha) \subseteq \mathbf{t}(\alpha)$.

Proof. It is easy. □

4.3 Proposition. *Let α be a binary relation on a set S; put $\gamma = \mathbf{t}(\alpha)$. The following conditions are equivalent:*
(i) *γ is irreflexive;*
(ii) *γ is strictly antisymmetric;*
(iii) *γ is a strict ordering;*
(iv) *α is superirreflexive.*

Proof. It is easy (use 3.5). □

4.4 Proposition. *Let α be a binary relation on a set S and let δ be the quasiorder closure of α. The following conditions are equivalent:*
(i) *δ is antisymmetric;*
(ii) *δ is an ordering;*
(iii) *$a_0 = a_1 = \dots = a_m$ whenever $(a_0, a_1, \dots, a_m)$ is an α-sequence with $a_0 = a_m$;*
(iv) *$\mathbf{i}(\alpha)$ is superirreflexive.*

Proof. It is easy. □

4.5 Proposition. *Let α be a binary relation on a set S, put $\gamma = \mathbf{t}(\alpha)$.*

(i) *An element of S is right strictly γ-isolated if and only if it is right strictly α-isolated.*

(ii) *An element of S is right γ-isolated if and only if it is right α-isolated.*

Proof. It is easy. □

4.6 Proposition. *Let α be a binary relation on a set S and let δ be the quasiorder closure of α. An element of S is right δ-isolated if and only if it is right α-isolated.*

Proof. It is easy. □

4.7 Lemma. *If S is finite and α is superirreflexive then for every $a \in S$ there exists at least one right strictly α-isolated element $b \in S$ with $(a, b) \in \delta$, where δ is the quasiorder closure of α.*

Proof. It is easy. □

5. Infinite α-sequences

Let α be a binary relation on a set S, $\gamma = \mathbf{t}(\alpha)$ and $\delta = \mathbf{rt}(\alpha)$. An infinite (right or upwards directed) sequence $(a_0, a_1, a_2, \ldots)$ of elements of S *is an* α-sequence if $(a_i, a_{i+1}) \in \alpha$ for every $i \geq 0$.

An (infinite) α-sequence $(a_0, a_1, a_2, \ldots)$ is called

– weakly pseudoirreducible if $a_i \neq a_{i+1}$ for every $i \geq 0$;

– pseudoirreducible if $a_i \neq a_j$ for all $0 \leq i < j$ (i.e., if the elements $a_0, a_1, a_2, \ldots$ are pairwise different);

– irreducible if $(a_i, a_j) \notin \alpha$ for al $0 \leq i < i + 2 \leq j$.

5.1 Lemma. *Let α be a binary relation on a set S.*

(i) *Every irreducible infinite α-sequence is pseudoirreducible.*

(ii) *Every pseudoirreducible infinite α-sequence is weakly pseudoirreducible.*

Proof. It is obvious. □

5.2 Lemma. *The following conditions are equivalent for a binary relation α on S:*

(i) *Every finite α-sequence can be extended to an infinite one;*

(ii) *Every right strictly α-isolated element is left strictly α-isolated.*

Proof. It is easy. □

5.3 Lemma. *The following conditions are equivalent for a binary relation α on S:*

(i) *Every weakly pseudoirreducible finite α-sequence can be extended to a weakly pseudoirreducible infinite one;*

(ii) *Every right α-isolated element is left α-isolated.*

Proof. It is easy. □

5.4 Proposition. *Let α be a binary relation on a set S and assume that there exist no weakly pseudoirreducible infinite α-sequences. Then:*

(i) *α is antisymmetric;*

(ii) *$\mathbf{i}(\alpha)$ is superirreflexive;*

(iii) *For every $a \in S$ there exists at least one right α-isolated element $b \in S$ with $(a,b) \in \delta$.*

Proof. It is easy (for (ii) see 3.5 and 4.4). □

5.5 Proposition. *Let α be a binary relation on a set S and assume that there exist no infinite α-sequences. Then:*

(i) *α is strictly antisymmetric and superirreflexive;*

(ii) *For every $a \in S$ there exists at least one right strictly α-isolated element $b \in S$ with $(a,b) \in \delta$.*

Proof. It is easy. □

5.6 Lemma. *Let α be a binary relation on a set S.*

(i) *Every infinite α-sequence is weakly pseudoirreducible if and only if α is irreflexive.*

(ii) *Every infinite α-sequence is pseudoirreducible if and only if α is superirreflexive.*

(iii) *If α is antitransitive then every infinite α-sequence is irreducible.*

(iv) *If α is a near-ordering then every weakly pseudoirreducible infinite α-sequence is pseudoirreducible.*

Proof. It is easy (for (i) use 3.4 and for (ii) use 3.5). □

5.7 Example. Consider the relation α_2 from 3.8(ii). This relation is not antitransitive. Clearly, there exist no infinite α_2-sequences and hence every infinite α_2-sequence is irreducible.

5.8 Lemma. *Let $(a_0, a_1, a_2, \dots)$ be an infinite γ-sequence with $(a_{i+1}, a_i) \notin \gamma$ for every $i \geq 0$. Then $(a_k, a_j) \notin \delta$ for every $0 \leq j < k$ (in particular, the sequence is pseudoirreducible).*

Proof. If $k = j + 1$ then $(a_k, a_j) = (a_{j+1}, a_j) \notin \gamma$ and, if $a_k = a_j$ then $(a_k, a_j) = (a_j, a_j) = (a_j, {}_{j+1}) \in \gamma$, a contradiction. If $j + 2 \leq k$ and $(a_k, a_j) \in \gamma$ then $(a_j, a_{k-1}) \in \gamma$ and $(a_k, a_j) \in \gamma$ yields $(a_k, a_{k-1}) \in \gamma$, a contradiction again. Finally, if $j + 2 \leq k$ and $a_k = a_j$ then $(a_j, a_{k-1}) \in \gamma$ and $(a_j, a_{k-1}) = (a_k, a_{k-1}) \notin \gamma$, a contradiction. □

6. Confluent relations

A binary relation α on a set S is said to be

– right (or upwards) strictly confluent if for all $a, b, c, \in S$ such that $(a,b) \in \alpha$ and $(a,c) \in \alpha$ there exists at least one $d \in S$ with $(b,d) \in \alpha$ and $(c,d) \in \alpha$;

– right (or upwards) confluent if for all $a, b, c \in S$ such that $(a, b) \in \alpha$, $(a, c) \in \alpha$ and $b \neq c$ there exists at least one $d \in S$ with $(b, d) \in \alpha$ and $(c, d) \in \alpha$;
– right (or upwards) almost confluent if for all $a, b, c \in S$ such that $(a, b) \in \alpha$, $(a, c) \in \alpha$, $(b, c) \notin \alpha$, $(c, b) \notin \alpha$ and $b \neq c$ there exists at least one $d \in S$ with $(b, d) \in \alpha$ and $(c, d) \in \alpha$ (then $b \neq d \neq c$ and $b \neq a \neq c$).

Left (or downwards) confluent relations are defined dually.

6.1 Lemma. *Let α be a binary relation on S.*
(i) *If α is right almost confluent then $\mathbf{i}(\alpha)$ is right almost confluent.*
(ii) *If α is right almost confluent then $\mathbf{r}(\alpha)$ is right strictly confluent.*

Proof. It is obvious. □

6.2 Lemma. *Let α be a right almost confluent relation on S and let $a, b, c \in S$ be such that $(a, b) \in \alpha$ and $(a, c) \in \alpha$.*
(i) *If b is right pseudoisolated then either $(c, b) \in \mathbf{r}(\alpha)$ or there exists an element $d \in S$ such that $(c, d) \in \alpha$, $(d, b) \in \alpha$ and $(b, d) \in \alpha$ (then $(c, b) \in \alpha$ provided that α is transitive).*
(ii) *If b is right isolated then $(c, b) \in \mathbf{r}(\alpha)$.*

Proof. It is obvious. □

6.3 Lemma. *Let α be a right almost confluent relation on S. Then for every $a \in S$ there exists at most one right isolated element $b \in S$ such that $(a, b) \in \alpha$.*

Proof. Use 6.2. □

6.4 Lemma. *A binary relation α on S is right strictly confluent if and only if α is right confluent and every right strictly isolated element is left strictly isolated.*

Proof. It is obvious. □

6.5 Lemma. *A binary relation α on S is right confluent if and only if α is right almost confluent and the following two conditions are satisfied:*
(a) *if $a, b, c \in S$ are pairwise different elements such that $(a, b) \in \alpha$, $(a, c) \in \alpha$ and $(b, c) \in \alpha$ then $(b, d) \in \alpha$ and $(c, d) \in \alpha$ for at least one $d \in S$;*
(b) *if $a, b \in S$ are such that $a \neq b$, $(a, a) \in \alpha$ and $(a, b) \in \alpha$ then $(a, e) \in \alpha$ and $(b, e) \in \alpha$ for at least one $e \in S$.*

Proof. It is obvious. □

6.6 Lemma. *Let α be an irreflexive relation. Then α is right confluent if and only if α is right almost confluent and satisfies* 6.5(a).

Proof. Use 6.5. □

6.7 Lemma. *Let α be a transitive relation. Then α is right strictly confluent if and only if α is right almost confluent and every right strictly isolated element is left strictly isolated.*

Proof. It is obvious. □

6.8 Proposition. *Let α be a right almost confluent relation on S. Then:*
(i) *The transitive closure $\gamma = \mathbf{t}(\alpha)$ is right almost confluent.*
(ii) *The quasiorder closure $\delta = \mathbf{rt}(\alpha)$ is right strictly confluent.*

Proof. By 6.1(ii), the relation $\beta = \mathbf{r}(\alpha)$ is right strictly confluent. We are going to show that $\delta = \mathbf{t}(\beta)$ is also right strictly confluent. Let $(a_0, \dots, a_m)$ and $(b_0, \dots, b_n)$ be δ-sequences such that $a_0 = b_0$.

Consider first the case $m = 1$. Using induction, we find elements $c_1, \dots, c_n \in S$ in the following way: Since β is right strictly confluent, there is an element $c_1 \in S$ with $(a_1, c_1) \in \beta$ and $(b_1, c_1) \in \beta$. Now, if $1 \leq j < n$ and $c_1, \dots, c_j$ are found such that the sequence $(a_1, c_1, c_2, \dots, c_j)$ is a β-sequence and all the pairs (b_1, c_1), $(b_2, c_2), \dots, (b_j, c_j)$ are in β then $(c_j, c_{j+1}) \in \beta$ and $(b_{j+1}, c_{j+1}) \in \beta$ for some $c_{j+1} \in S$. Consequently, by induction, $(b_n, c_n) \in \beta$ and $(a_1, c_1, \dots, c_n)$ is a β-sequence. Thus $(a_m, c_n) \in \delta$ and $(b_n, c_n) \in \delta$.

In the general case we proceed by induction on $m + n$. In view of the preceding part of the proof, assume that $m \geq 2$. Then there is a $c \in S$ with $(a_{m-1}, c) \in \delta$ and $(b_n, c) \in \delta$. Furthermore, $(a_{m-1}, a_m) \in \beta$ and hence $(a_m, d) \in \delta$ and $(c, d) \in \delta$ and $(c, d) \in \delta$ for at least one $d \in S$. Consequently, $(a_m, d) \in \delta$ and $(b_n, d) \in \delta$.

We have proved that δ is right strictly confluent and the fact that γ is rigth almost confluent follows easily. □

6.9 Proposition. *Let α be a right almost confluent relation on S such that every right strictly α-isolated element is left strictly α-isolated. Then $\mathbf{t}(\alpha)$ is right strictly confluent.*

Proof. By 6.8(i), $\mathbf{t}(\alpha)$ is right almost confluent. The rest follows from 6.7. □

6.10 Proposition. *Let α be a right almost confluent relation on S. The following conditions are equivalent for an element $a \in S$:*
(i) *a is right γ-pseudoisolated, where $\gamma = \mathbf{t}(\alpha)$;*
(ii) *a is right δ-pseudoisolated, where $\delta = \mathbf{rt}(\alpha)$;*
(iii) *$(b, a) \in \gamma$ for every $b \in \mathbf{R}(a, \alpha)$;*
(iv) *$(b, a) \in \delta$ for every $b \in \mathbf{R}(a, \beta)$, where $\beta = \mathbf{r}(\alpha)$.*

Proof. Clearly, (i) is equivalent to (ii), (iii) is equivalent to (iv) and (i) implies (iii). It remains to show that (iii) implies (i).

Let $(a, b) \in \gamma$. Then $a = a_0$ and $b = a_m$ for an α-sequence $(a_0, \dots, a_m)$. We are going to prove $(b, a) \in \gamma$ by induction on m. We can assume that $a \neq b$. The case $m = 1$ is clear. Let $m \geq 2$. We have $(a_{m-1}, a) \in \gamma$ by induction and we have $(a_{m-1}, b) \in \alpha$. Proceeding similarly as in the proof of 6.8, we find an element $c \in S$

such that $(a,c) \in \beta$ and $(b,c) \in \delta$. Then $(c,a) \in \delta$ and hence $(b,a) \in \delta$, so that $(b,a) \in \gamma$. □

6.11 Lemma. *Let α be a right confluent relation on S. If $a,b,c \in S$ are such that $(a,b) \in \alpha$, $(a,c) \in \mathbf{t}(\alpha)$ and $(b,c) \notin \mathbf{rt}(\alpha)$ then there is $d \in S$ with $(c,d) \in \alpha$ and $(b,d) \in \mathbf{d}(\alpha)$.*

Proof. Since $(a,c) \in \mathbf{t}(\alpha)$, there is a finite α-sequence $(a_0, \dots, a_m)$, $m \geq 1$, such that $a_0 = a$ and $a_m = c$. If $b = a_1$, then $(b,c) = \mathbf{rt}(\alpha)$, a contradiction. Thus $b \neq a_1$ and, since α is right confluent, there is $b_1 \in S$ with $(b,b_1) \in \alpha$ and $(a_1,b_1) \in \alpha$. Now, if $b_0 = b$, b_1, ..., b_n, $n < m$, are such that $(b_{i-1},b_i) \in \alpha$, $1 \leq i \leq n$, $(a_j,b_j) \in \alpha$, $0 \leq j \leq \ \leq n$, then $b_n \neq a_{n+1}$ (otherwise $(b,c) \in \mathbf{rt}(\alpha)$) and we find $b_{n+1} \in S$ with $(b_n,b_{n+1}) \in \alpha$ and $(a_{n+1},b_{n+1}) \in \alpha$. Proceeding by induction we obtain $b_m \in S$ such that $(b_{m-1},b_m) \in \alpha$ and $(a_m,b_m) \in \alpha$, and hence $(c,b_m) \in \alpha$ and $(b,b_m) \in \mathbf{t}(\alpha)$ and we put $d = b_m$, which completes the proof. □

7. Free confluent extensions

Let α be a binary relation on a set S.

Denote by $\varrho(S,\alpha)$ the set of two-element subsets $\{b,c\} \subseteq S$ such that there exists at least one $a \in S$ with $(a,b) \in \alpha$, $(a,c) \in \alpha$, but no $d \in S$ with $(b,d) \in \alpha$, $(c,d) \in \alpha$ (notice that $\varrho(S,\alpha)$ is empty if and only if α is right confluent). Let f be a bijection of $\varrho(S,\alpha)$ onto a set T disjoint with S. Put $\sigma(S) = S \cup T$ and $\sigma(\alpha) = \alpha \cup \{(b,f(\{b,c\})), (c,f(\{b,c\})) \mid \{b,c\} \in \varrho(S,\alpha)\}$.

7.1 Lemma. *Let α be a binary relation on a set S.*

(i) *$S \subseteq \sigma(S)$, $\alpha \subseteq \sigma(\alpha)$ and $\alpha = \sigma(\alpha) \restriction S$.*
(ii) *$f(\{b,c\})$ is right strictly $\sigma(\alpha)$-isolated for every $\{b,c\} \in \varrho(S,\alpha)$.*
(iii) *If $\{x,y\} \in \varrho(\sigma(S), \sigma(\alpha))$ then either $x \notin S$ or $y \notin S$.*
(iv) *If α is antitransitive then $\sigma(\alpha)$ is so.*

Proof. It is easy. □

Consider the infinite sequence $\alpha = \sigma^0(\alpha) \subseteq \sigma^1(\alpha) \subseteq \sigma^2(\alpha) \subseteq \dots$ where $\sigma^{i+1}(\alpha) = \sigma(\sigma^i(\alpha))$ *for* $i \geq 0$. Put $\tau(\alpha) = \bigcup_{i=0}^{\infty} \sigma^i(\alpha)$.

7.2 Proposition. *Let α be a binary relation on a set S.*

(i) *$\alpha \subseteq \tau(\alpha)$.*
(ii) *$\tau(\alpha)$ is right confluent.*
(iii) *If α is antitransitive then $\tau(\alpha)$ is so.*

Proof. Easy to check (use 7.1). □

Denote by $\kappa(\alpha)$ the set of right stictly α-isolated elements $a \in S$ such that a is not left strictly α-isolated. Let g be a bijection of $\kappa(\alpha)$ onto a set R disjoint with S. Put $\lambda(S) = S \cup R$ and $\lambda(\alpha) = \alpha \cup \{(a,g(a)) \mid a \in \kappa(\alpha)\}$.

7.3 Lemma. *Let α be a binary relation on a set S.*

(i) $S \subseteq \lambda(S)$, $\alpha \subseteq \lambda(\alpha)$ *and* $\alpha = \lambda(\alpha) \restriction S$.
(ii) $g(a)$ *is right strictly* $\lambda(\alpha)$*-isolated for every* $a \in \kappa(\alpha)$.
(iii) $\kappa(\lambda(\alpha)) \cap S = \emptyset$.
(iv) *If* α *is antitransitive then* $\lambda(a)$ *is so.*
(v) *If* α *is right confluent then* $\lambda(a)$ *is so.*

Proof. It is easy. □

Consider the infinite sequence $\alpha = \lambda^0(\alpha) \subseteq \lambda^1(\alpha) \subseteq \lambda^2(\alpha) \subseteq \dots$ where $\lambda^{i+1}(\alpha) = \lambda(\lambda^i(\alpha))$ for $i \geq 0$. Put $\mu(\alpha) = \bigcup_{i=0}^{\infty} \lambda^i(\alpha)$ and $\vartheta(\alpha) = \mu\tau(\alpha)$.

7.4 Proposition. *Let α be a binary relation on a set S.*

(i) $\alpha \subseteq \mu(\alpha)$.
(ii) *Every right strictly* $\mu(\alpha)$*-isolated element is left strictly* $\mu(\alpha)$*-isolated.*
(iii) *If* α *is antitransitive then* $\mu(\alpha)$ *is so.*
(iv) *If* α *is right confluent then* $\mu(\alpha)$ *is so.*

Proof. It is easy (use 7.3). □

7.5 Proposition. *Let α be a binary relation on a set S.*

(i) $\alpha \subseteq \vartheta(\alpha)$.
(ii) $\vartheta(\alpha)$ *is right strictly confluent.*
(iii) *If* α *is antitransitive then* $\vartheta(\alpha)$ *is so.*

Proof. Combine 7.2 and 7.4. □

8. Regular relations

8.1 Proposition. *Let α be a right confluent relation on a set S. If $(a_0, \dots, a_m)$ and $(b_0, \dots, b_n)$ are α-sequences such that $a_0 = b_0$, $a_m = b_n$ and a_m is right strictly α-isolated then $m = n$.*

Proof. We will proceed by induction on $m + n$. We have $m + n \geq 2$ and, if $m + n = 2$, then $m = 1 = n$. Henceforth, assume that $1 \leq n \leq m$ and $2 \leq m$.

If $a_1 = b_1$ then $n \geq 2$, since $a_m = b_n$ is right strictly α-isolated. Consequently, $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$ are α-sequences of length $m - 1$ and $n - 1$, resp. Now, $m - 1 = n - 1$ by induction and then $m = n$.

It remains to consider the case $a_1 \neq b_1$. Since α is right confluent, there is an element $c_1 = S$ with $(a, c_1) \in \alpha$ and $(b_1, c_1) \in \alpha$. Since b_n is right strictly α-isolated, we have $n \geq 2$, proceeding similarly, we find an index $1 \leq k < n$ and elements $c_1, \dots, c_k$ such that $(c_i, c_{i+1}) \in \alpha$ for every $1 \leq i < k$, $(b_j, c_j) \in \alpha$ for every $1 \leq j \leq k$ and $c_k = b_{k+1}$ (use again the fact that b_n is right strictly α-isolated). Clearly, $(a_1, c_1, \dots, c_{k-1}, b_{k+1}, \dots, b_n)$ and $(a_1, a_2, \dots, a_m)$ are α-sequences of length

$n-1$ and $m-1$, resp. Thus $n-1=m-1$ by induction and we get $m=n$ again. □

8.2 Proposition. *Let α be a right confluent relation on a set S. If $(a_0, \ldots, a_m)$ and $(b_0, \ldots, b_n)$ are α-sequences such that $a_0 = b_0$, $a_m = b_n$ and $(a_m, a) \in \mathbf{rt}(\alpha)$ for at least one right strictly α-isolated element a, then $m = n$.*

Proof. If a_m is right strictly α-isolated then the equality $m = n$ is proved in 8.1. If a_m is not right strictly α-isolated then there is an α-sequence $(a_m, a_{m+1}, \ldots, a_{m+k})$, $k \geq 1$, such that $a_{m+k} = a$. Then $(a_0, \ldots, a_{m+k})$ and $(b_0, \ldots, b_n, a_{m+k}, \ldots, a_{m+k})$ are α-sequences of length $m+k$ and $n+k$, resp. Thus $m+k=n$ by 8.1 and hence $m=n$. □

8.3. Corollary. *Let α be a right confluent relation on a set S such that for every $a \in S$ there exists at least one right strictly α-isolated element $b \in S$ with $(a,b) \in \mathbf{rt}(\alpha)$. Then α is regular (and hence it is strictly antisymmetric and antitransitive).*

8.4 Proposition. *Let α be a right almost confluent relation on a set S such that for every $a \in S$ there exists at least one right α-isolated element $b \in S$ with $(a,b) \in \mathbf{rt}(\alpha)$. Then $\mathbf{i}(\alpha)$ is regular if and only if $(a,c) \notin \alpha$ whenever $a,b,c \in S$ are such that $a \neq b \neq c \neq a$, $(a,b) \in \alpha$ and $(b,c) \in \alpha$.*

Proof. Only the converse implication needs a proof. By 6.1(i), $\mathbf{i}(\alpha)$ is right almost confluent. Hence $\mathbf{i}(\alpha)$ is right confluent by 6.6. Further, $\mathbf{rt}(\alpha) = \mathbf{rti}(\alpha)$ and our result follows from 2.8(i) and 8.3. □

8.5 Proposition. *Let α be a right almost confluent relation on a finite set S. Then α is regular if and only if α is superirreflexive.*

Proof. Combine 4.7 and 8.3. □

8.6 Example. Put $S = \{0,1,2,3\}$.

(i) $\alpha_1 = \{(0,1), (1,2), (2,0), (2,3), (3,1)\}$ is both right and left strictly confluent, strictly antisymmetric, but not regular.

(ii) $\alpha_2 = \{(0,1), (0,2), (1,0), (1,2), (2,1)\}$ is both right and left strictly confluent, irreflexive, symmetric, transitive, but not regular.

8.7 Example. Put $S = \{0,1,2,\ldots\}$ and $\alpha = \{(i,i+1), (i,i+2) \mid i = S\}$. Then α is right strictly confluent and superirreflexive. On the other hand, α is not weakly regular.

8.8 Example. Consider the relation α_3 from 3.8(iii) and put $\varepsilon = \vartheta(\alpha)$ (see Section 7). Then ε is antitransitive and right strictly confluent. On the other hand, ε is not weakly regular.

8.9 Example. The relation $\{(0,1), (2,1), (2,3)\}$ on $\{0,1,2,3\}$ is an example of a regular which is neither right nor left almost confluent.

9. Roots of near-orderings

Let α be a near-ordering on a set S. The root (or the covering relation) $\zeta = \sqrt{\alpha}$ of α is the binary relation defined by $(a,b) \in \zeta$ if and only if $(a,b) \in \mathbf{i}(\alpha)$ and $c \in \{a,b\}$ whenever $(a,c) \in \alpha$ and $(c,b) \in \alpha$.

9.1 Lemma. *Let α be a near-ordering on a set S.*

(i) $\sqrt{\alpha} \subseteq \alpha$ *and* $\sqrt{\alpha}$ *is antitransitive.*

(ii) $\sqrt{\alpha} = \sqrt{\mathbf{i}(\alpha)} = \sqrt{\mathbf{r}(\alpha)} = \mathbf{i}(\sqrt{\alpha})$.

(iii) $\sqrt{\alpha} \subseteq \mathbf{t}(\sqrt{\alpha}) \subseteq \mathbf{i}(\alpha) \subseteq \alpha \subseteq \mathbf{r}(\alpha)$.

(iv) *Both* $\mathbf{t}(\sqrt{\alpha})$ *and* $\mathbf{i}(\alpha)$ *are strict orderings on S.*

Proof. It is obvious. □

9.2 Remark. Let α be a near-ordering on S.

(i) $\sqrt{\alpha} = \emptyset$ if and only if α is dense. That is, for all $(a,b) \in \mathbf{i}(\alpha)$ there exists at least one $c \in S$ with $(a,c) \in \mathbf{i}(\alpha)$ and $(c,b) \in \mathbf{i}(\alpha)$. If this condition is satisfied then either $\alpha \subseteq \mathbf{id}_S$ or S is infinite.

(ii) $\sqrt{\alpha} = \alpha$ if and only α is irreflexive and every finite α-sequence has length 1.

A near-ordering will be called resuscitable if $\alpha \subseteq \mathbf{rt}(\sqrt{\alpha})$.

9.3 Lemma. *Let α be a resuscitable near-ordering.*

(i) $\sqrt{\alpha} \subseteq \mathbf{t}(\sqrt{\alpha}) = \mathbf{i}(\alpha) \subseteq \alpha \subseteq \mathbf{r}(\alpha) \subseteq \mathbf{rt}(\sqrt{\alpha})$.

(ii) *If α is reflexive then* $\alpha = \mathbf{rt}(\sqrt{\alpha})$.

(iii) *If α is irreflexive then* $\alpha = \mathbf{t}(\sqrt{\alpha})$.

(iv) *Both* $\mathbf{i}(\alpha)$ *and* $\mathbf{r}(\alpha)$ *are resuscitable.*

Proof. It is obvious. □

9.4 Lemma. *Every near-ordering on a finite set is resuscitable.*

Proof. It is obvious. □

9.5 Lemma. *Let α be a resuscitable near-ordering on a set S such that $\beta = \sqrt{\alpha}$ is right confluent. If $a,b,c \in S$ are such that $(a,b) \in \beta$ and $d = \sup_{\mathbf{r}(\alpha)}(a,c)$, $e = \sup_{\mathbf{r}(\alpha)}(b,c)$ exist in S then either $d = e$ or $(d,e) \in \beta$.*

Proof. If $a = d$ then $(c,a) \in \mathbf{r}(\alpha)$, and hence $(c,b) \in \mathbf{r}(\alpha)$, $e = b$ and $(d,e) = (a,b) \in \beta$. Now, we can assume that $a \neq d$. Then $(a,d) \in \mathbf{i}(\alpha) = \mathbf{t}(\beta)$. Moreover, if $(b,d) \in \mathbf{r}(\alpha)$, then $d = e$, and so we can also assume that $d \neq e$ and $(b,d) \notin \mathbf{r}(\alpha)$. Then $(b,d) \notin \mathbf{rt}(\beta)$ and, by 6.11, there is $f \in S$ with $(d,f) \in \beta$ and $(b,f) \in \mathbf{t}(\beta) \subseteq \alpha$. From this $(e,f) \in \mathbf{r}(\alpha)$ and, since $d \neq e$, we get $e = f$. □

9.6 Lemma. *Let β be a binary relation satisfying the equivalent conditions of 4.4. Put* $\alpha = \mathbf{t}(\beta)$.

(i) α *is a near-ordering.*

(ii) $\sqrt{\alpha} \subseteq \beta \subseteq \alpha$ *and* $\mathbf{t}(\sqrt{\alpha}) \subseteq \alpha$.

(iii) *If* $\mathbf{i}(\beta)$ *is antitransitive, then* $\mathbf{i}(\beta) = \sqrt{\mathbf{i}(\alpha)}$, $\mathbf{i}(\alpha) = \mathbf{it}(\beta) \subseteq \mathbf{ti}(\beta) = = \mathbf{t}(\sqrt{\mathbf{i}(\alpha)}) \subseteq \mathbf{t}(\sqrt{\alpha})$ *and* $\alpha \subseteq \mathbf{ri}(\alpha) \subseteq \mathbf{rt}(\sqrt{\alpha})$ *(in particular,* α *is resuscitable).*

9.7. Corollary. *Let* β *be an antitransitive binary relation satisfying the equivalent conditions of 4.4. Then* $\beta = \sqrt{\mathbf{t}(\beta)}$ *and* $\mathbf{t}(\beta)$ *is resuscitable.*

9.8 Lemma. *Let* β *be a binary relation on a finite set such that* β *satisfies the equivalent conditions of 4.4. Put* $\alpha = \mathbf{t}(\beta)$. *Then* $\mathbf{t}(\sqrt{\alpha}) = \mathbf{i}(\alpha) \subseteq \alpha \subseteq \mathbf{rt}(\sqrt{\alpha}) \subseteq \subseteq \mathbf{r}(\alpha)$.

Proof. Combine 9.4 and 9.6. □